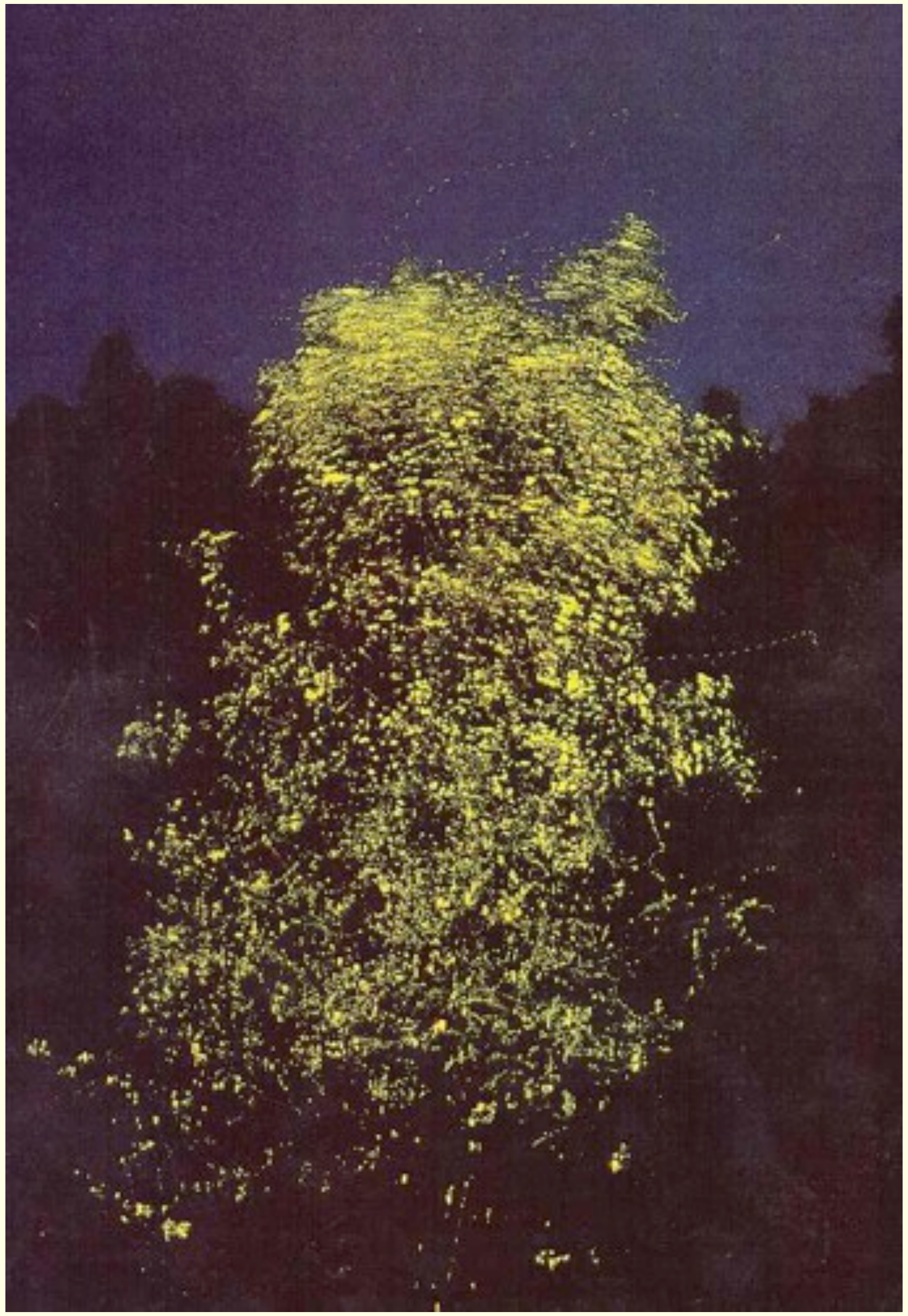
Collective dynamics in large oscillator ensembles: The Kuramoto model and beyond

Michael Rosenblum

Institute of Physics and Astronomy, Potsdam University, Germany

URL: *www.stat.physik.uni-potsdam.de/~mros*



Fireflies synchrony

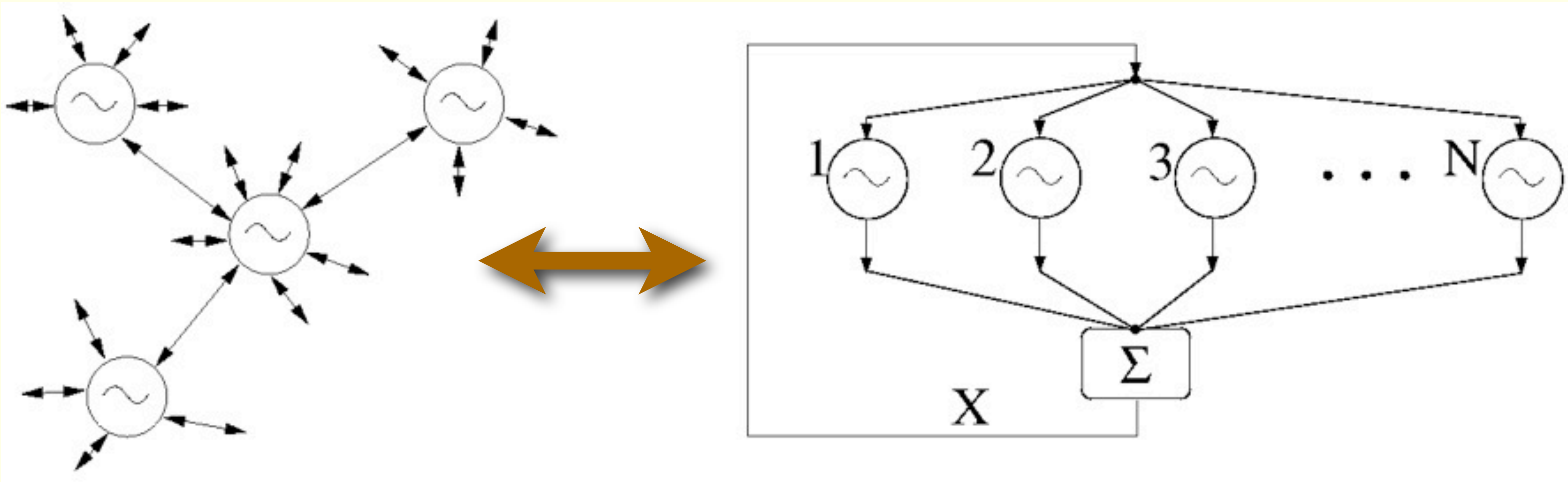
Engelbert Kaempfer, 1680

Norbert Wiener:

same mechanism is responsible for generation of alpha-rhythms

Globally coupled oscillators

Global coupling = all-to-all coupling = mean field coupling



Assumption: equal coupling strength for each pair

Oscillators: active, self-sustained

Some natural phenomena modeled by globally coupled systems

- Collective dynamics in arrays of Josephson junctions or lasers
- Collective dynamics of oscillators coupled via a common load
- Synchronous emission of light pulses by fireflies or of acoustic pulses by crickets
- Coordinated firing of cardiac pacemaker cells
- Synchronization of glycolytic oscillations of yeast cells
- Rhythmical applause in a large audience
- Pedestrian synchrony on footbridges
- Brain rhythms, etc

Pedestrian synchrony: London Millennium bridge

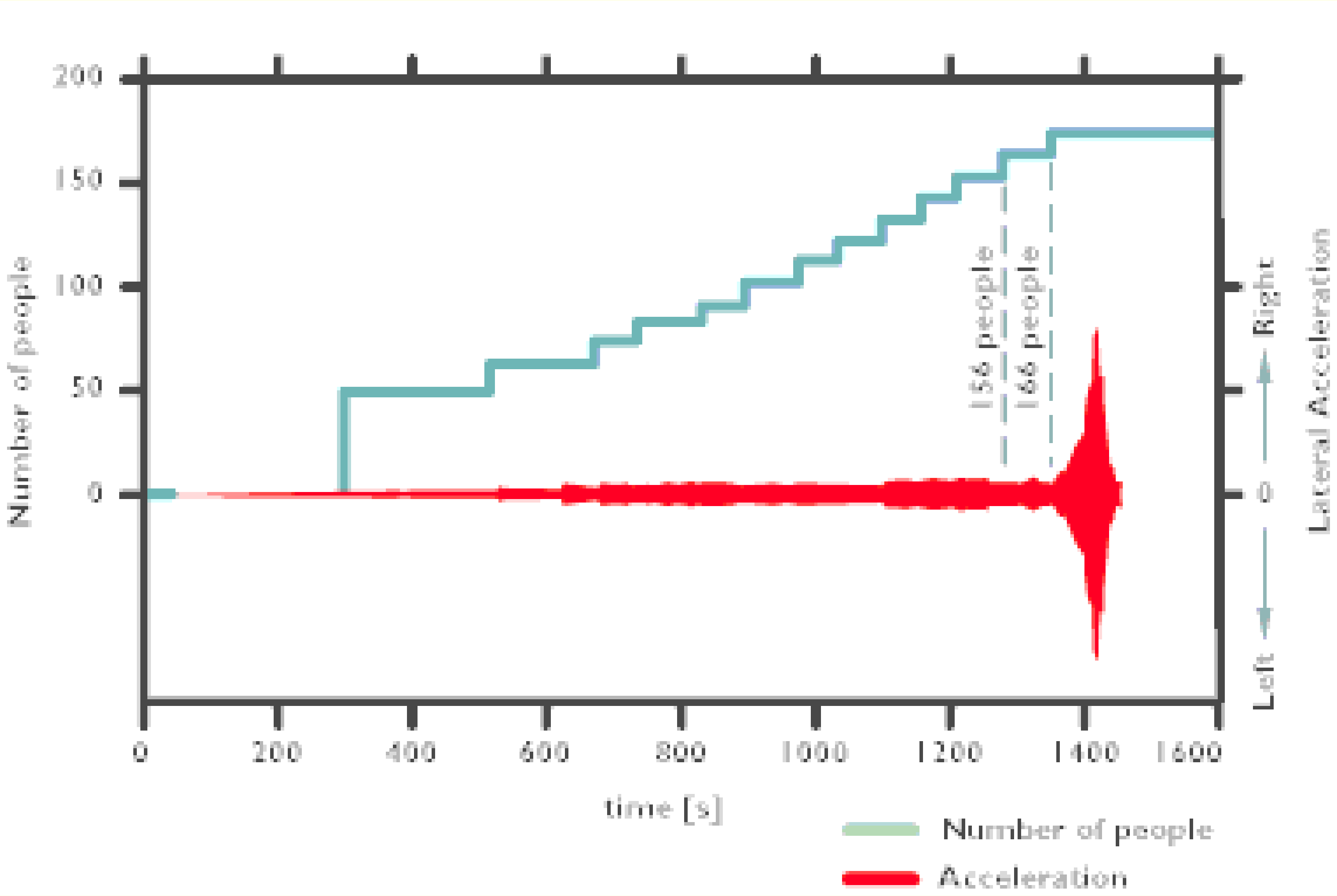
Millennium bridge: opening day

Millennium bridge: opening day

Millennium bridge: testing

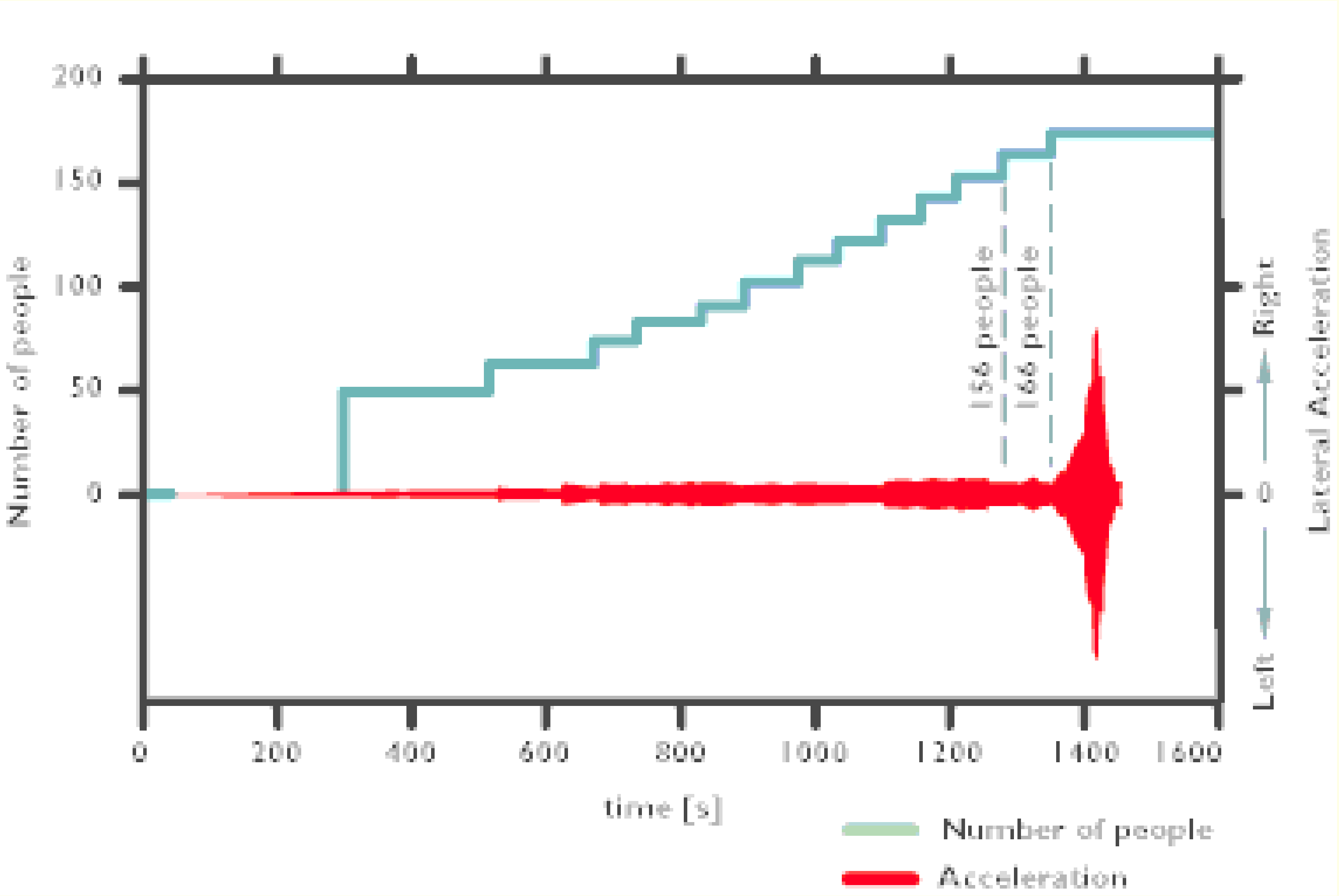
Millennium bridge: testing

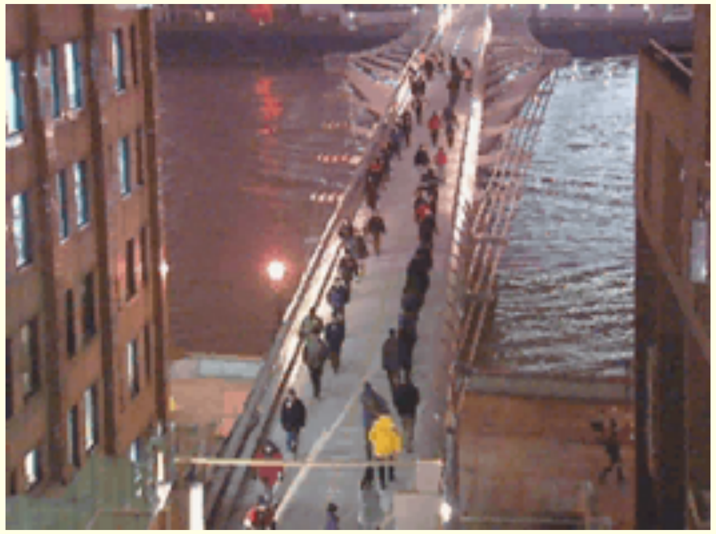
Pedestrian synchrony: London Millennium bridge



Threshold effect!

Pedestrian synchrony: London Millennium bridge





Threshold effect!

Globally coupled oscillators: qualitative discussion

We know: forced oscillator can be entrained by the force

Suppose (some part) of the population synchronizes

→ there exist a non-zero oscillatory mean field

→ oscillators are forced → they synchronize!

Suppose the oscillators do not synchronize →

→ there exist no mean field → oscillators are not forced

→ they remain asynchronous!

We are going to make this self-consistent consideration qualitative!

Forced oscillator, sine-coupling

$$\dot{\varphi} = \omega + \varepsilon \sin(\varphi_{ext} - \varphi)$$

Amplitude of the force

Phase of the force, $\varphi_{ext} = \nu t$

$$\boxed{\psi = \varphi - \varphi_{ext}}$$

$$\dot{\varphi} = \omega + \varepsilon \sin(\varphi_{ext} - \varphi) \quad (1)$$

$$\dot{\varphi}_{ext} = \nu \quad (2)$$

$$(1) - (2) \quad \Longrightarrow \quad \boxed{\dot{\psi} = \omega - \nu - \varepsilon \sin \psi}$$

Synchronous solution $\dot{\psi} = 0 \quad \Longrightarrow \quad \boxed{\psi = \arcsin \frac{\omega - \nu}{\varepsilon}}$

N all-to-all coupled oscillators: The Kuramoto model

Oscillator, forced by another one: $\dot{\varphi} = \omega + \varepsilon \sin(\varphi_{ext} - \varphi)$

Oscillator, equally forced by $N-1$ oscillators:

$$\dot{\varphi}_k = \omega_k + \tilde{\varepsilon} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k) \ , \quad k = 1, \ldots, N$$

It is convenient to set: $\tilde{\varepsilon} = \varepsilon / N$

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k)$$

Yoshiki Kuramoto, *1975, 1984*

The Kuramoto model: mean field

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N}\mathrm{Im}\left(\sum_j e^{i(\varphi_j-\varphi_k)}\right)$$

$$\dot{\varphi}_k = \omega_k + \varepsilon\mathrm{Im}\left(e^{-i\varphi_k}\boxed{\frac{1}{N}\sum_j e^{i\varphi_j}}\right)$$

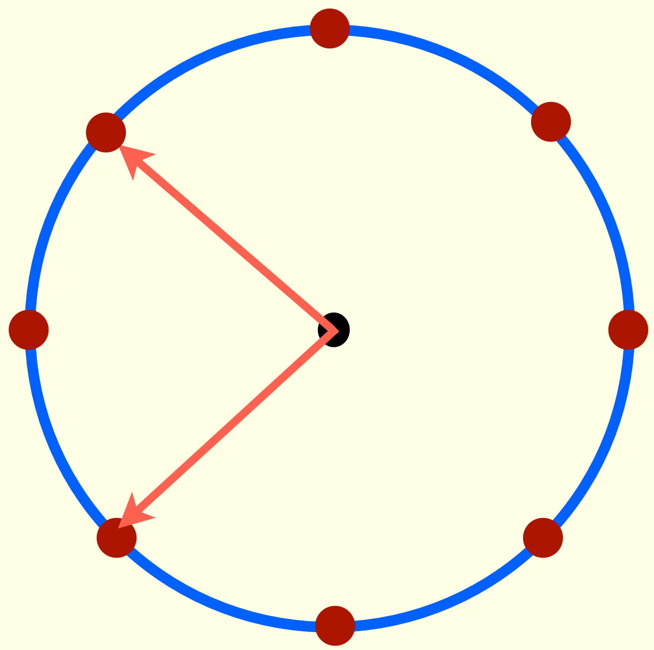
Mean field $Y = N^{-1}\sum_j e^{i\varphi_j} = Re^{i\Theta}$

$$\boxed{\dot{\varphi}_k = \omega_k + \varepsilon R\sin(\Theta-\varphi_k)\;, \quad k=1,\ldots,N}$$

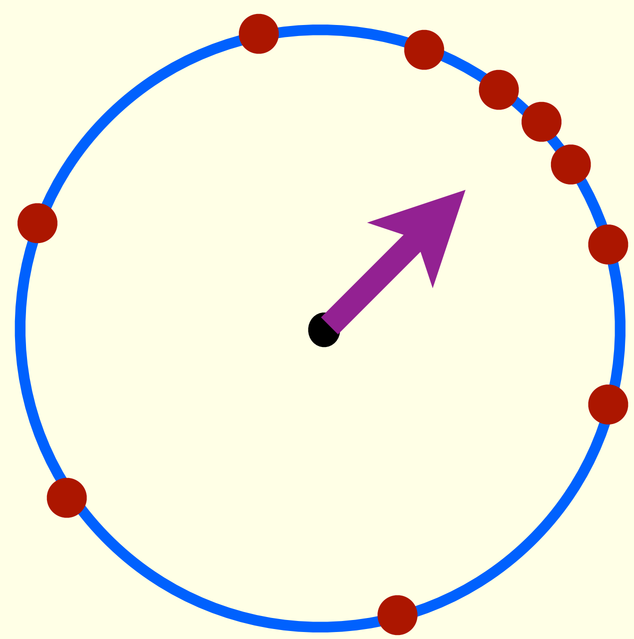
The Kuramoto model: mean field $Re^{i\Theta} = N^{-1}\sum_j e^{i\varphi_j}$

Physical meaning: Θ and R are the phase and the amplitude of the collective mode

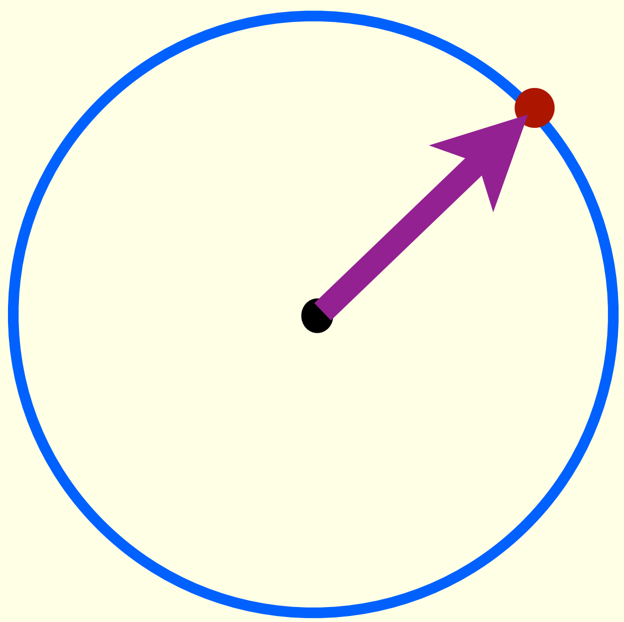
Snapshot of phases



asynchrony
$R = 0$



(partial) synchrony
$0 < R < 1$



full synchrony
$R = 1$

R is called the **order parameter**

The Kuramoto model: analysis (Y. Kuramoto 1975, 1984)

We look for the harmonic solution $\boldsymbol{R} = \text{const}\ , \boldsymbol{\Theta} = \bar{\boldsymbol{\omega}} \boldsymbol{t}$

$$\dot{\boldsymbol{\varphi}}_{\boldsymbol{k}} = \boldsymbol{\omega}_{\boldsymbol{k}} + \boldsymbol{\varepsilon} \boldsymbol{R} \sin(\boldsymbol{\Theta} - \boldsymbol{\varphi}_{\boldsymbol{k}}) \qquad (1)$$

$$\dot{\boldsymbol{\Theta}} = \bar{\boldsymbol{\omega}} \qquad (2)$$

Phase difference $\boldsymbol{\psi}_{\boldsymbol{k}} = \boldsymbol{\varphi}_{\boldsymbol{k}} - \boldsymbol{\Theta}$

$$(1) - (2) \quad \Longrightarrow \quad \dot{\boldsymbol{\psi}}_{\boldsymbol{k}} = \boldsymbol{\omega}_{\boldsymbol{k}} - \bar{\boldsymbol{\omega}} - \boldsymbol{\varepsilon} \boldsymbol{R} \sin \boldsymbol{\psi}_{\boldsymbol{k}}$$

Cf. the equation for the forced oscillator:

$$\dot{\boldsymbol{\psi}}_{\boldsymbol{k}} = \boldsymbol{\omega}_{\boldsymbol{k}} - \boldsymbol{\nu} - \boldsymbol{\varepsilon} \sin \boldsymbol{\psi}_{\boldsymbol{k}}$$

The Kuramoto model: analysis II

Solution of the equation $\dot{\psi}_k = \omega_k - \bar{\omega} - \varepsilon R \sin \psi_k$

- Synchronous solution $\dot{\psi}_k = 0 \, , \psi_k = \arcsin \frac{\omega_k - \bar{\omega}}{\varepsilon R}$

  exists if $|\omega_k - \bar{\omega}| \leq \varepsilon R$

- Asynchronous solution $\dot{\psi}_k \neq 0$

  is found if $|\omega_k - \bar{\omega}| > \varepsilon R$

Generally, there are two groups of oscillators:

synchronous and asynchronous ones

The Kuramoto model: Assumption 1

- We consider the limit $N \to \infty$

→ We describe oscillators frequencies by a distribution $g(\omega)$

The mean field equation $Re^{i\Theta} = N^{-1} \sum_j e^{i\varphi_j}$ becomes:

$$Re^{i\bar{\omega}t} = \int_{-\pi}^{\pi} e^{i(\bar{\omega}t+\psi)} \rho(\psi) d\psi$$

(using $\Theta = \bar{\omega}t$, $\varphi = \bar{\omega}t + \psi$)

Since we have two groups: $\rho(\psi) = \rho_s(\psi) + \rho_{as}(\psi)$

Finally: $$\boxed{R = \int_{-\pi}^{\pi} e^{i\psi} [\rho_s(\psi) + \rho_{as}(\psi)] d\psi}$$

The main idea of the analysis

Assume that $\boldsymbol{R}$ and $\bar{\boldsymbol{\omega}}$ are known

→ find distribution of phases $\boldsymbol{\rho(\psi; R, \bar{\omega})}$ for given forcing

Equation

$$\boldsymbol{R = \int_{-\pi}^{\pi} e^{i\psi}[\rho_s(\psi; R, \bar{\omega}) + \rho_{as}(\psi; R, \bar{\omega})]d\psi}$$

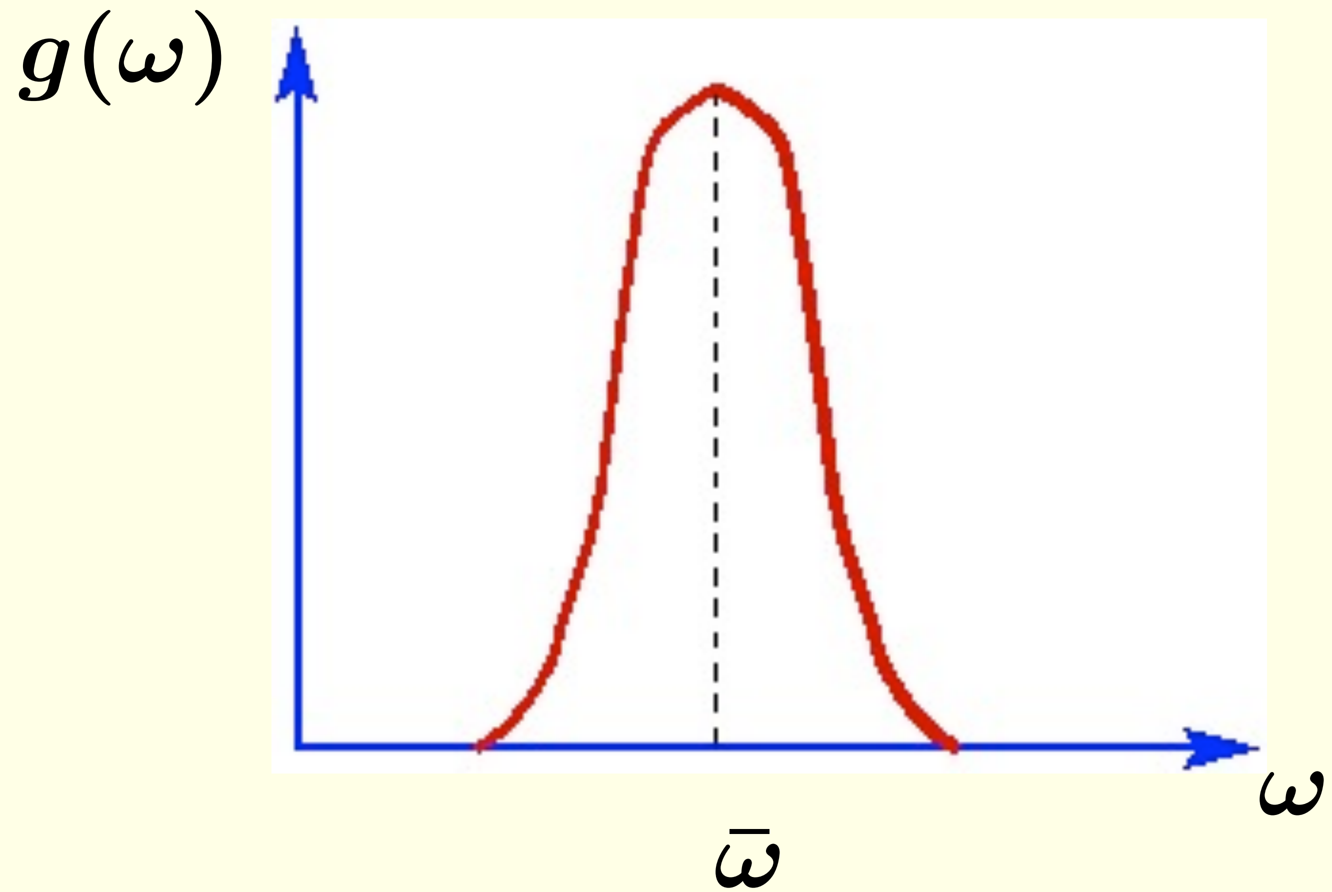
is then an integral complex equation for two quantities $\boldsymbol{R}$ and $\bar{\boldsymbol{\omega}}$

The Kuramoto model: Assumption 2

- We assume a symmetric unimodal distribution of frequencies

Symmetry:

$$g(\bar{\omega} + x) = g(\bar{\omega} - x)$$



- Natural guess: the mean field frequency is equal to $\bar{\omega}$

  We show later that it is true

The synchronous group

- Synchronous solution $\dot{\psi} = 0 \; , \psi = \arcsin \dfrac{\omega - \bar{\omega}}{\varepsilon R}$

  exists if $|\omega - \bar{\omega}| \leq \varepsilon R$

  Since $\psi = \psi(\omega)$, we have for the distributions:

$$\rho_s(\psi) = g(\omega) \left| \frac{d\omega}{d\psi} \right|$$

  Using $\omega = \bar{\omega} + \varepsilon R \sin \psi$, we obtain

$$\rho_s(\psi) = \begin{cases} g(\bar{\omega} + \varepsilon R \sin \psi) \varepsilon R \cos \psi \; , & \text{if} \;\; |\psi| \leq \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

The asynchronous group

- Asynchronous solution $\dot{\psi} \neq 0$ of the equation

$$\dot{\psi}_k = \omega_k - \bar{\omega} - \varepsilon R \sin \psi_k$$

is found if $|\omega - \bar{\omega}| > \varepsilon R$

Distribution for given frequency:

$$P(\psi, \omega) = \frac{C}{|\dot{\psi}|} = \frac{C}{|\omega - \bar{\omega} - \varepsilon R \sin \psi|}$$

Normalization yields

$$P(\psi, \omega) = \frac{\sqrt{(\omega - \bar{\omega})^2 - \varepsilon^2 R^2}}{2\pi |\omega - \bar{\omega} - \varepsilon R \sin \psi|}$$

The asynchronous group II

Averaging over frequencies:

$$\rho_{as}(\psi) = \int_{|\omega-\bar{\omega}|>\varepsilon R} g(\omega) P(\psi,\omega) d\omega$$

Let us denote $\boxed{\omega - \bar{\omega} = x}$ →

$$\rho_{as}(\psi) = \int_{|x|>\varepsilon R} \frac{g(\bar{\omega}+x)\sqrt{x^2-\varepsilon^2 R^2}}{2\pi|x-\varepsilon R\sin\psi|} dx$$

$$= \int_{\varepsilon R}^{\infty} \frac{g(\bar{\omega}+x)\sqrt{x^2-\varepsilon^2 R^2}}{2\pi(x-\varepsilon R\sin\psi)} dx$$

$$+ \int_{-\infty}^{-\varepsilon R} \frac{g(\bar{\omega}+x)\sqrt{x^2-\varepsilon^2 R^2}}{2\pi(-x+\varepsilon R\sin\psi)} dx$$

The asynchronous group III

$$\rho_{as}(\psi) = \int_{\varepsilon R}^{\infty} \frac{g(\bar{\omega}+x)\sqrt{x^2-\varepsilon^2 R^2}}{2\pi(x-\varepsilon R\sin\psi)}dx$$

$$+\int_{-\infty}^{-\varepsilon R} \frac{g(\bar{\omega}+x)\sqrt{x^2-\varepsilon^2 R^2}}{2\pi(-x+\varepsilon R\sin\psi)}dx$$

Substituting in the 2nd integral $x \to -x$

and using $g(\bar{\omega}+x) = g(\bar{\omega}-x)$

$$\rho_{as}(\psi) = \int_{\varepsilon R}^{\infty} \frac{g(\bar{\omega}+x)x\sqrt{x^2-\varepsilon^2 R^2}}{\pi(x^2-\varepsilon^2 R^2\sin^2\psi)}dx$$

Important is that

$$\boxed{\rho_{as}(\psi+\pi) = \rho_{as}(\psi)}$$

Final equations

$$\boxed{R = \int_{-\pi}^{\pi} e^{i\psi}[\rho_s(\psi) + \rho_{as}(\psi)]d\psi}$$

$$\boxed{\rho_s(\psi) = \begin{cases} g(\bar{\omega} + \varepsilon R \sin\psi)\varepsilon R \cos\psi \ , & \text{if } \ |\psi| \leq \pi/2 \\ 0 & \text{otherwise} \end{cases}}$$

Because of the periodicity $\boxed{\rho_{as}(\psi + \pi) = \rho_{as}(\psi)}$

$$\int_{-\pi}^{\pi} e^{i\psi}\rho_{as} d\psi = 0$$

We are left with

$$1 = \varepsilon \int_{-\pi/2}^{\pi/2} e^{i\psi} g(\bar{\omega} + \varepsilon R \sin\psi) \cos\psi d\psi$$

Final equations II

$$\boxed{1 = \varepsilon \int_{-\pi/2}^{\pi/2} e^{i\psi} g(\bar{\omega} + \varepsilon R \sin\psi) \cos\psi d\psi}$$

Real and imaginary parts:

$$1 = \varepsilon \int_{-\pi/2}^{\pi/2} g(\bar{\omega} + \varepsilon R \sin\psi) \cos^2\psi d\psi$$

$$0 = \varepsilon \int_{-\pi/2}^{\pi/2} g(\bar{\omega} + \varepsilon R \sin\psi) \cos\psi \sin\psi d\psi$$

Last equation is satisfied (integral of an odd function = 0); this confirms our guess that mean field frequency equals the maximal frequency: $\dot{\Theta} = \bar{\omega}$

Main result

Equation $$1 = \varepsilon \int_{-\pi/2}^{\pi/2} g(\bar{\omega} + \varepsilon R \sin\psi) \cos^2\psi \, d\psi$$

determines mean field amplitude as a function of the coupling ε

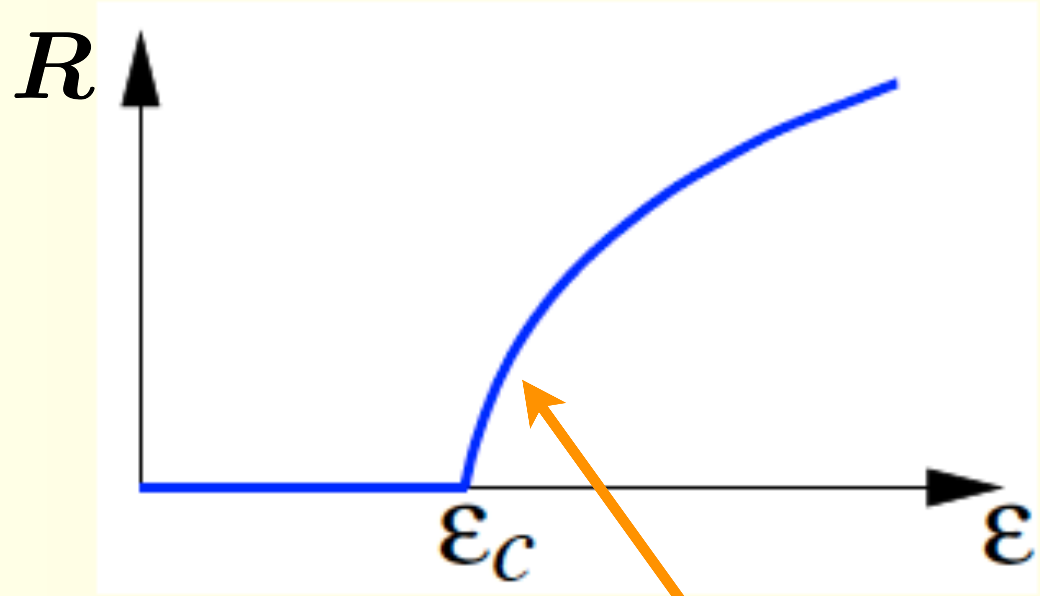
It can be solved for the Lorentzian distribution

$$g(\omega) = \frac{\gamma}{\pi[(\omega - \bar{\omega})^2 + \gamma^2]}$$

$$R = \sqrt{1 - \frac{2\gamma}{\varepsilon}}$$

Critical coupling:

$$\varepsilon_c = 2\gamma = \frac{2}{\pi g(\bar{\omega})}$$



$$R \sim \sqrt{\varepsilon - \varepsilon_c}$$

General unimodal distribution

For small R only oscillators with $\omega \cong \bar{\omega}$ synchronize

→ only small vicinity of the maximum is important

Equation $$1 = \varepsilon \int_{-\pi/2}^{\pi/2} g(\bar{\omega} + \varepsilon R \sin\psi) \cos^2\psi d\psi$$

Taylor expansion for small R:

$$g(\bar{\omega} + \varepsilon R \sin\psi) \approx g(\bar{\omega}) + \frac{g''}{2} \varepsilon^2 R^2 \sin^2\psi \rightarrow$$

$$1 = \varepsilon g(\bar{\omega}) \frac{\pi}{2} + \frac{g''}{16} \varepsilon^3 R^2 \pi \rightarrow \frac{|g''|\varepsilon^3}{8g(\bar{\omega})} R^2 = \varepsilon - \boxed{\frac{2}{\pi g(\bar{\omega})}}$$

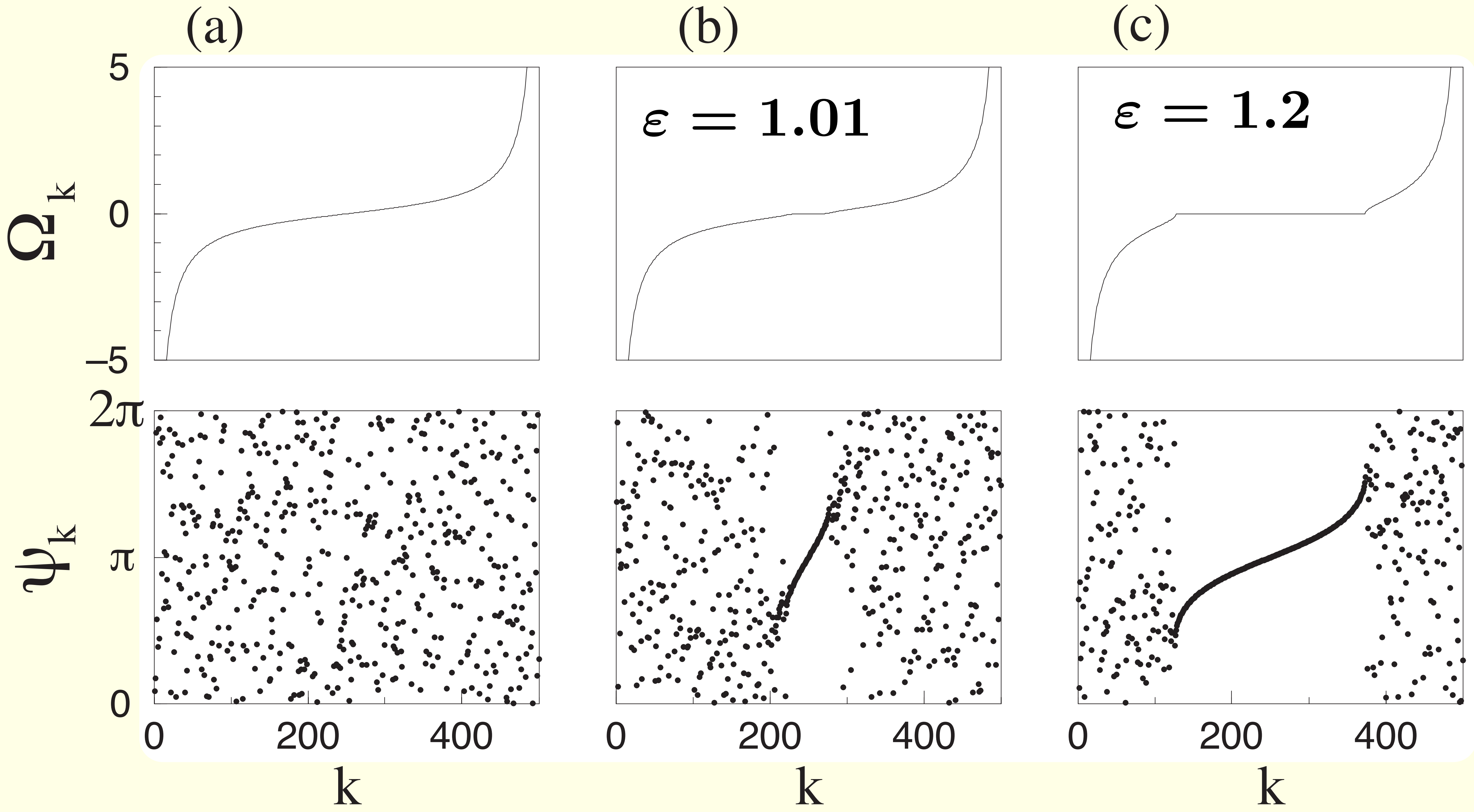
Critical coupling:

$$\boxed{R \sim \sqrt{\varepsilon - \varepsilon_c}}$$

Synchronization transition

- Unimodal frequency distribution: synchronization transition ~ 2nd order phase transition with the order parameter r
- Unimodal frequency distribution: synchronization transition ~ 1st order phase transition (D. Pazó, 2005)

Kuramoto model: numerical example



500 oscillators, Lorentzian distribution, $\varepsilon_c = 1$

The Kuramoto-Sakagichi model

Globally coupled Landau-Stuart oscillators, Nakagawa and Kuramoto, 1993, 1995

$$\frac{da_k}{dt} = (1+i\omega)a_k - (1+ic_2)|a_k|^2 a_k + \varepsilon(1+ic_1)(Y - a_k)$$

with $\quad Y = N^{-1}\sum_k^N a_k = Re^{i\Theta}\ , \quad k = 1, \dots, N$

Phase approximation yields the Kuramoto-Sakaguchi model

$$\boxed{\dot{\varphi}_k = \omega_k + \varepsilon R \sin(\Theta - \varphi_k + \beta)}$$

The Kuramoto-Sakagichi model

Additional parameter: phase shift β

Stability of the synchronous state $\varphi_1 = \ldots = \varphi_N = \Theta$

This state is stable, if

$$\left.\frac{\dot{\varphi}_k}{\varphi_k}\right|_{\varphi_k=\Theta} = -\varepsilon r \sin(\beta) < 0$$

coupling is attractive, if $|\beta| < \pi/2$

and repulsive, otherwise

Peculiar feature for non-identical oscillators:

$$\dot{\varphi}_k = \omega_k + \varepsilon r \sin(\Theta - \varphi_k + \beta)$$

non-monotonic synchronization transition for special distributions (Omel'chenko and Wolfrum, 2012)

General case of the phase approximation: the Daido model

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N}\sum_j Q_{k,j}(\varphi_j, \varphi_k)$$

Let $Q_{j,k} = Q$ $\Rightarrow$ $\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N}\sum_j Q(\varphi_j, \varphi_k)$

For weak coupling, we can average over $T \sim 2\pi/\omega$

$$\Rightarrow \quad \dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N}\sum_j h(\varphi_j - \varphi_k)$$

Fourier series $\quad h(x) = \sum_{n=-\infty}^{\infty} q_n e^{inx}$

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N}\sum_j \sum_n q_n e^{in(\varphi_j - \varphi_k)} =$$

$$\omega_k + \frac{\varepsilon}{N}\sum_n q_n e^{-in\varphi_k} \sum_j e^{in\varphi_j}$$

General case of the phase approximation: the Daido model

$$\varphi_k = \omega_k + \frac{\varepsilon}{N}\sum_j \sum_n q_n e^{in(\varphi_j - \varphi_k)} =$$

$$\omega_k + \varepsilon \sum_n q_n e^{-in\varphi_k} \boxed{\frac{1}{N}\sum_j e^{in\varphi_j}}$$

We define the generalized order parameters $Z_n = \dfrac{1}{N}\sum_j e^{in\varphi_j}$

Obviously Z_1 is the Kuramoto mean field

Finally,

$$\boxed{\varphi_k = \omega_k + \varepsilon \sum_n q_n Z_n e^{-in\varphi_k}}$$

New effects: clustering, multistability

Nonlinear coupling

Recall: globally and **linearly** coupled Landau-Stuart oscillators

$$\frac{da_k}{dt} = (1 + i\omega_k)a_k - (1 + ic_2)|a_k|^2 a_k + \varepsilon(1 + ic_1)(Y - a_k)$$

yield in phase approximation the Kuramoto-Sakaguchi model

$$\boxed{\dot{\varphi}_k = \omega_k + \varepsilon R \sin(\Theta - \varphi_k + \beta)}$$

Nonlinear coupling $(\varepsilon_1 + i\varepsilon_2)Y - (\eta_1 + i\eta_2)|Y|^2 Y$

yields $$\boxed{\dot{\varphi}_k = \omega_k + \varepsilon R \sin(\Theta - \varphi_k + \beta_0 + \beta_1 \varepsilon^2 R^2)}$$

General nonlinear model

$$\boxed{\dot{\varphi}_k = \omega_k(R, \varepsilon) + A(R, \varepsilon) \sin[\Theta - \varphi_k + \beta(R, \varepsilon)]}$$

Rosenblum and Pikovsky 2007, Pikovsky and Rosenblum 2011

Analytical tools

The model: sine-coupled phase oscillators

- The Watanabe-Strogatz theory for identical oscillators (Watanabe and Strogatz, PRL 1994, Physica D 1995
- The Ott-Antonsen theory (Ott and Antonsen, Chaos 2008, 2009)
- Extension of the WS theory to the case on nonidentical oscillators, its link with the OA theory (Pikovsky and Rosenblum 2008, 2011)

The Watanabe-Strogatz theory

Dynamical description in terms of collective variables

(Watanabe and Strogatz, PRL 1994, Physica D 1995):

The model: sine-coupled **identical** phase oscillators

$$\dot{\varphi}_k = \omega(t) + \mathrm{Im}\left[H(t)e^{-i\varphi_k}\right] \;, \quad k = 1, \ldots, N$$

$\omega = \mathrm{const}, H = \varepsilon e^{i\beta} Y$ → the Kuramoto-Sakaguchi model

$H = A(|Y|, \varepsilon) e^{i\beta(|Y|,\varepsilon)} Y$ → the generalized nonlinear model

Main result: for **any** functions $\omega(t)$, $H(t)$, the N-dimensional system is completely described by 3 global variables plus $N-3$ constants of motion

- global amplitude ρ , $0 \leq \rho \leq 1$
- global phases Ψ , Φ
- constants of motion ψ_k

The Watanabe-Strogatz equations

(in our notation, Pikovsky and Rosenblum, 1998)

$$\frac{d\rho}{dt} = \frac{1-\rho^2}{2}\mathrm{Re}(He^{-i\Phi})$$

$$\frac{d\Phi}{dt} = \omega + \frac{1+\rho^2}{2\rho}\mathrm{Im}(He^{-i\Phi})$$

$$\frac{d\Psi}{dt} = \frac{1-\rho^2}{2\rho}\mathrm{Im}(He^{-i\Phi})$$

Remark: the global variables used here differ from original WS variables:

$$\tilde{\rho} = \frac{2\rho}{1+\rho^2}\,, \quad \tilde{\Psi} = \Psi + \pi\,, \quad \tilde{\Phi} = \Phi + \pi$$

The WS equations: equivalent form

We introduce $z = \rho e^{i\Phi}\ , \quad \alpha = \Phi - \Psi$

and re-write the equations in the form:

$$\frac{dz}{dt} = i\omega z + \frac{1}{2}H - \frac{z^2}{2}H^*$$

$$\frac{d\alpha}{dt} = \omega + \mathrm{Im}(z^* H)$$

Relation to original variables

Time-dependent transformation (WS 1994, 1995)

$$\tan\left(\frac{\varphi_k - \Phi}{2}\right) = \frac{1-\rho}{1+\rho}\tan\left(\frac{\psi_k - \Psi}{2}\right)$$

Here: $\psi_k \; , k = 1, \ldots, N$ are constants of motion.

The transformation is overdetermined. To make it unambiguous, one should introduce 3 constraints, e.g.

$$\sum_{k=1}^{N} e^{i\psi_k} = 0 \; , \quad \sum_{k=1}^{N} \cos(2\psi_k) = 0$$

Equivalent form of the transformation (Pikovsky and Rosenblum, 2008, 2011):

$$\boxed{e^{i\varphi_k} = e^{i\Phi}\frac{\rho + e^{i(\psi_k - \Psi)}}{\rho e^{i(\psi_k - \Psi)} + 1}}$$

Meaning of the WS variables (qualitative)

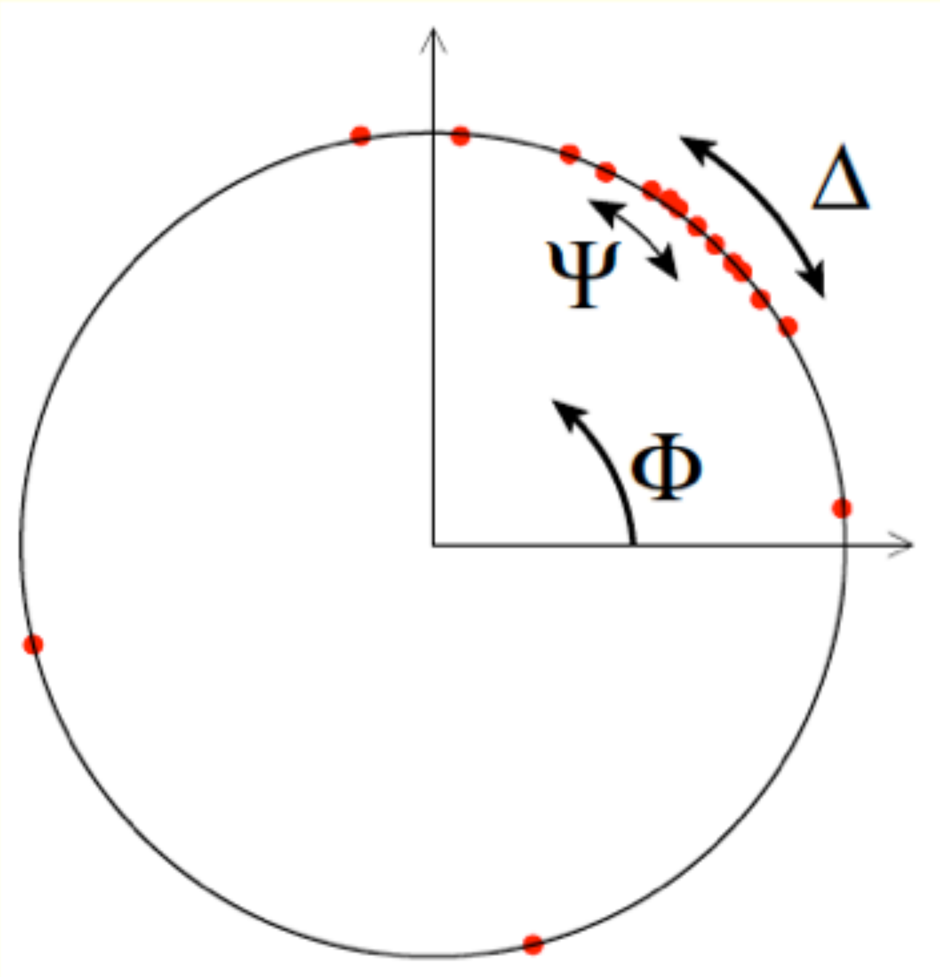
- Amplitude ρ is related to the width Δ of the bunch

  Easy to check: if $\rho = 0$ then $\boldsymbol{R} = 0$

  $\rho = 1$ then $\boldsymbol{R} = 1$

  Hence,

  amplitude $\rho \sim$ mean field amplitude $\boldsymbol{R}$
- Phase $\Phi \sim$ mean field phase Θ
- Phase Ψ: shift of individual phases with respect to Φ



Meaning of the WS variables (quantitative)

It can be shown that for the uniform distribution of constant of motion ψ_k and large N

$$\boxed{\rho = R\,, \Phi = \Theta}$$

Recall the equivalent variables $z = \rho e^{i\Phi}\,, \quad \alpha = \Phi - \Psi$

The WS equations for nonidentical oscillators (Pikovsky and Rosenblum, 2011, following WS, 1994)

- Continuous limit $N \to \infty$
- Let oscillators be described by a distribution density $W(\varphi, \omega, t)$
- It is convenient to write $W(\varphi, \omega, t) = g(\omega) w(\varphi, \omega, t)$

  where $g(\omega)$ is the distribution of frequencies
- Normalization

$$\int g(\omega) d\omega = 1, \quad \int_0^{2\pi} w(\varphi, \omega, t) d\varphi = 1$$

The WS equations for nonidentical oscillators II

- Local Kuramoto mean field

$$Z(\omega, t) = \int_0^{2\pi} w(\omega, \varphi, t) e^{i\varphi} d\varphi$$

to be distinguished from the global mean field

$$Y(t) = \int_{-\infty}^{\infty} g(\omega) Z(\omega, t) d\omega$$

The WS equations for nonidentical oscillators III

- Continuity equation (conservation of the number of oscillators):

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \varphi}(w\dot{\varphi}) = 0$$

- Variable substitution in the continuity equation

$$t, \varphi, \omega \quad \rightarrow \quad \tau = t, \psi = \psi(\omega, \varphi, t), \nu = \omega$$

using

$$\boxed{e^{i\varphi_k} = e^{i\Phi} \frac{\rho + e^{i(\psi_k - \Psi)}}{\rho e^{i(\psi_k - \Psi)} + 1}}$$

distribution is transformed $w(\omega, \varphi, t) \quad \rightarrow \quad \sigma(\nu, \psi, \tau)$

The WS equations for nonidentical oscillators IV

distribution is transformed $w(\omega, \varphi, t) \quad \to \quad \sigma(\nu, \psi, \tau)$

New density distribution is **stationary**, provided the WS variables obey

$$\frac{\partial z(\omega, t)}{dt} = i\omega z + \frac{1}{2} H(\omega, t) - \frac{z^2}{2} H^*(\omega, t)$$

$$\frac{\partial \alpha(\omega, t)}{\partial t} = \omega + \mathrm{Im}[z^* H(\omega, t)]$$

Reduction of the WS equation system

It can be shown that for the uniform distribution of constant of motion ψ_k and large N

$$\rho = R\,,\ \Phi = \Theta \quad \longrightarrow \quad \boxed{z = \rho e^{i\Phi} = r e^{i\Theta} = Z}$$

Recall: z is local Kuramoto mean field

$\longrightarrow$ The WS equations become equations for Z

$$\frac{\partial Z(\omega,t)}{dt} = i\omega Z + \frac{1}{2}H(\omega,t) - \frac{Z^2}{2}H^*(\omega,t)$$

$$\frac{\partial \alpha(\omega,t)}{\partial t} = \omega + \mathrm{Im}[Z^* H(\omega,t)]$$

Most common case: mean field coupling

$$H(t) = EY = E\int g(\omega)Z(\omega,t)d\omega$$

Reduction of the WS equation system II

The WS equations become equations for Z

$$\boxed{\begin{aligned}\frac{\partial Z(\omega,t)}{dt} &= i\omega Z + \frac{1}{2}H(\omega,t) - \frac{Z^2}{2}H^*(\omega,t)\\ \frac{\partial \alpha(\omega,t)}{\partial t} &= \omega + \mathrm{Im}[Z^* H(\omega,t)]\end{aligned}}$$

Most common case: mean field coupling

$$H(t) = EY = E\int g(\omega)Z(\omega,t)d\omega$$

H is independent of α → 2nd equation becomes irrelevant

→ we are left with the Ott-Antonsen equation (Chaos, 2008)

The Ott-Antonsen theory (2008,2009)

Starting point:

$$\dot{\varphi}_k = \omega_k + \mathrm{Im}\left[H(t)e^{-i\varphi_k}\right] = \omega_k + \frac{He^{-i\varphi_k} - H^* e^{i\varphi_k}}{2i}$$

Limit $N \to \infty$, distribution density $W(\omega,\varphi,t) = g(\omega)w(\omega,\varphi,t)$

Continuity equation (conservation of the number of oscillators):

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \varphi}(w\dot{\varphi}) = 0$$

Generalized (local!) Daido order parameters:

$$Z_m(\omega,t) = \int_0^{2\pi} w(\omega,\varphi,t)e^{im\varphi}d\varphi$$

$$\rightarrow \quad \dot{Z}_m = \int_0^{2\pi} \frac{\partial w}{\partial t} e^{im\varphi} d\varphi = -\int_0^{2\pi} \frac{\partial}{\partial t}(w\dot{\varphi})e^{im\varphi}d\varphi$$

The Ott-Antonsen theory II

$$Z_m(\omega,t) = \int_0^{2\pi} w(\omega,\varphi,t)e^{im\varphi}d\varphi$$

$$\dot{Z}_m = \int_0^{2\pi} \frac{\partial w}{\partial t} e^{im\varphi}d\varphi = -\int_0^{2\pi} \frac{\partial}{\partial t}(w\dot{\varphi})e^{im\varphi}d\varphi$$

Integration by parts yields: $\dot{Z}_m = im\int_0^{2\pi} w\dot{\varphi}e^{im\varphi}d\varphi$

Substituting $\dot{\varphi} = \omega + \dfrac{He^{-i\varphi} - H^*e^{i\varphi}}{2i}$

we obtain an infinite-dimensional system of ODEs:

$$\dot{Z}_m = im\omega Z_m + \frac{m}{2}(HZ_{m-1} - H^*Z_{m+1})$$

The Ott-Antonsen theory III

we obtain an infinite-dimensional system of ODEs:

$$\dot{Z}_m = im\omega Z_m + \frac{m}{2}(HZ_{m-1} - H^* Z_{m+1})$$

A particular solution $Z_m = Z_1^m = Z^m$ yields a single equation

$$\boxed{\frac{\partial Z(\omega,t)}{dt} = i\omega Z + \frac{1}{2}H(\omega,t) - \frac{Z^2}{2}H^*(\omega,t)}$$

Most common case: Kuramoto-Sakaguchi mean field coupling

$$H(t) = EY = \varepsilon e^{i\beta}\int g(\omega)Z(\omega,t)d\omega$$

→ Closed equation for the mean field dynamics

For continuous frequency distribution the particular solution

$Z_m = Z_1^m = Z^m$ is the only attractor! (Ott and Antonsen, 2009)

The OA theory: an example
Kuramoto-Sakaguchi ensemble

Example: Lorentzian frequency distribution:

$$g(\omega) = [\pi(\omega^2 + 1)]^{-1}$$

Under assumption that $Z(\omega)$ is analytic in the upper half-plane

$$\int g(\omega) Z(\omega, t) d\omega = Z(i)$$

Substituting it along with $\omega = i$ we obtain the OA equation

$$\boxed{\dot{Y} = \left(\frac{\varepsilon}{2} e^{i\beta} - 1\right) - \frac{\varepsilon e^{-i\beta}}{2} Y^2 Y^*}$$

for the time evolution of the mean field.

WS theory: an example
Chimera state in two coupled oscillator populations

Model by Abrams et al (2008): two identical populations of the same size, different coupling within a population and in between

$$\dot{\varphi}^{(1,2)} = \omega + \mathrm{Im}\left[(\mu Z_{1,2} + \nu Z_{2,1}) e^{i(\beta - \varphi^{(1,2)})}\right]$$

Rotating coordinate frame: $\omega = 0$

The WS equations:

$$\dot{z}_{1,2} = \frac{1}{2} H_{1,2} - \frac{z_{1,2}^2}{2} H_{1,2}^*$$

$$\dot{\alpha}_{1,2} = \mathrm{Im}(z_{1,2}^* H_{1,2})$$

$$H_{1,2} = e^{i\beta}(\mu Z_{1,2} + \nu Z_{2,1})$$

Uniform distribution of constants of motion ψ $\longrightarrow$ $z = Z$

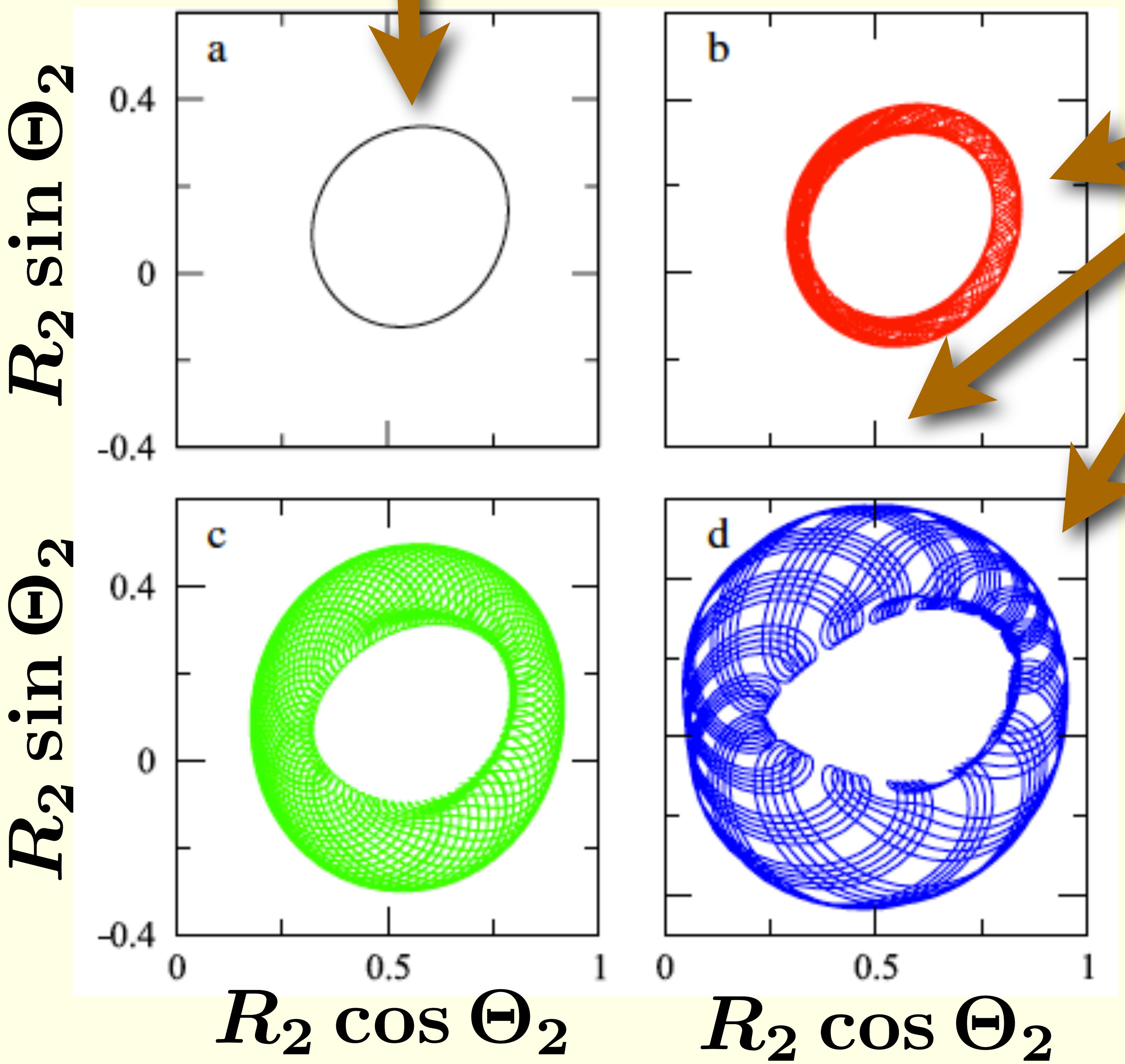
OA particular solution, 4-dimensional system

Otherwise: 6-dimensional system, $Z = z\gamma(z, \alpha; \psi)$

Chimera state in two coupled oscillator populations II

OA particular solution, 4-d system

Full 6-d system



Chimera state:

Symmetry breaking: one population synchronizes and another does not:

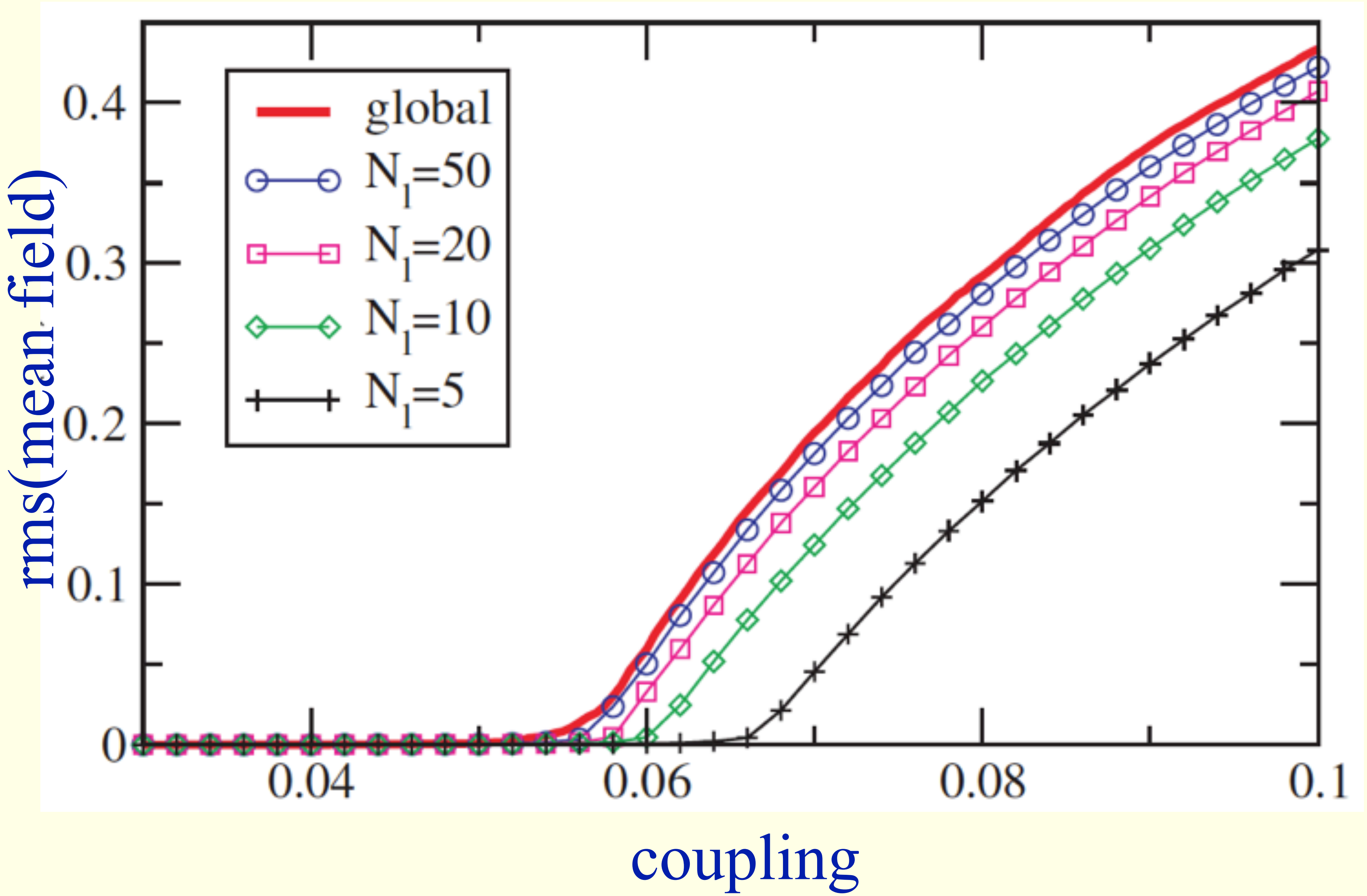
$$R_1 = 1, R_2 < 1$$

Different distributions of ψ yield another periodicity

→ **quasiperiodic chimera states!**

Control of collective dynamics

Highly interconnected neuronal network



10000 bursting neurons

(Rulkov map model)

Global coupling is a reasonable model of collective neuronal dynamics

Feedback control: How to suppress neuronal synchrony?

Motivation: Deep Brain Stimulation

Hypotheses:

- The pathological rhythm emerges due to synchronization in a large neuronal population with high connectivity
- Local field potential reflects the mean field of the population
- Stimulation affects a large part of the population

Solution: feedback control

Feedback control: Theoretical studies

- Delayed feedback: Rosenblum & Pikovsky, Phys Rev Lett 2004, Phys Rev E, 2004
- Nonlinear feedback: Popovych, Hauptmann, & Tass, Phys Rev Lett, 2005
- Feedback without delay: Tukhlina, Rosenblum, Pikovsky, & Kurths, Phys Rev E, 2007
- Adaptive feedback: Montaseri, Javad Yazdanpanah, Pikovsky, & Rosenblum, Chaos, 2013

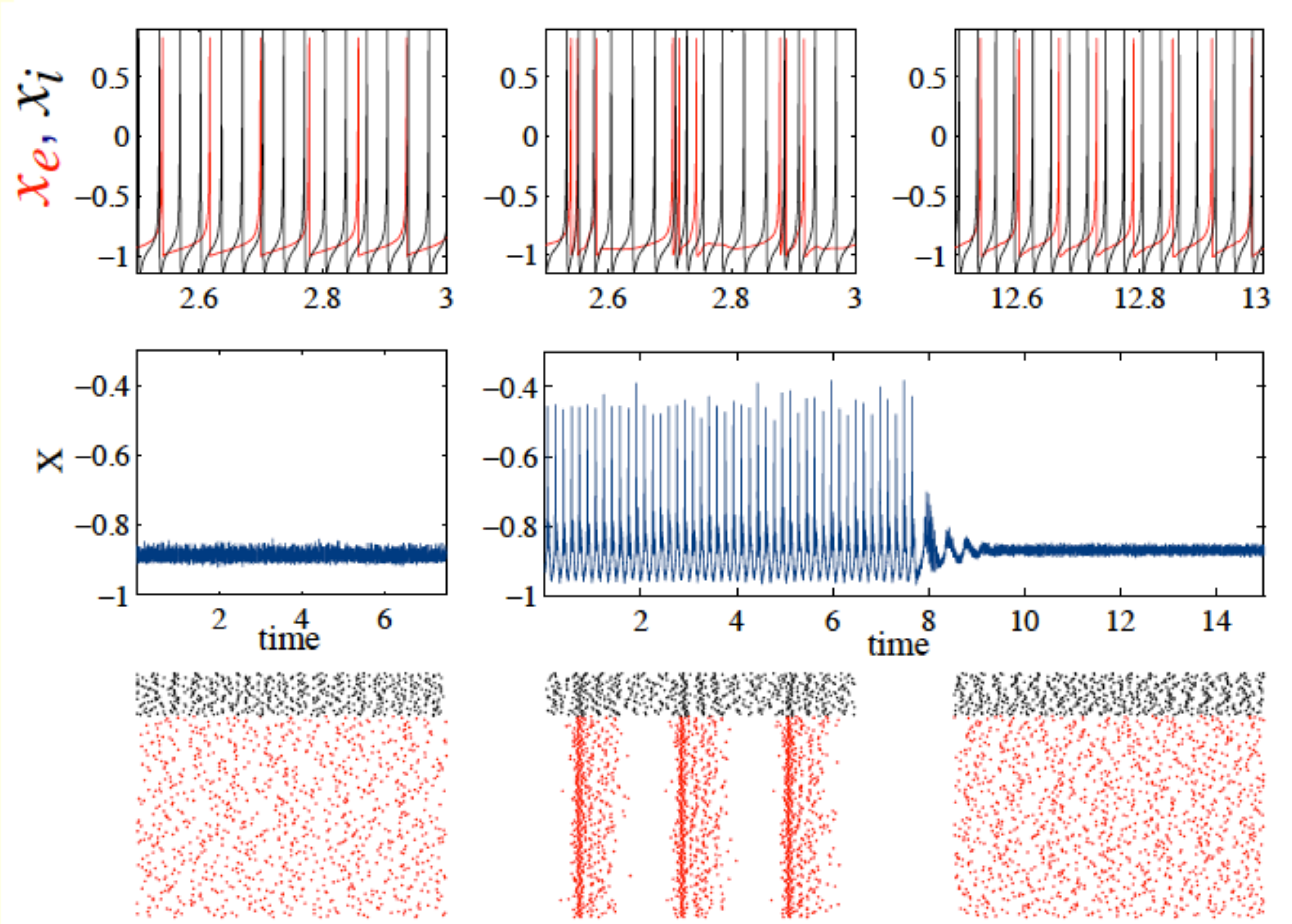
Delayed feedback control: An example

(Neuronal model: Rulkov et al., *J. Comp. Neurosci.* **17** 203 (2004))

- Inhibitory and excitatory neurons, synaptic connections
  - excitatory neurons: regular spiking type, $N_e = 1000$
  - inhibitory neurons: fast spiking type, $N_i = 200$
- Heterogeneity of the firing rates in subpopulations due to Gaussian distribution of external currents
  - excitatory neurons: average rate ≈ 11 Hz
  - inhibitory neurons: average rate ≈ 30 Hz
- Each unit interacts with all other units
  - excitatory connections: model of $AMPA$ mediated synapses
  - inhibitory connections: model of $GABA_A$ mediated synapses
  - excitatory synapses are faster than inhibitory, no synaptic delay

Delayed feedback control: An example

no connections connections connections+control

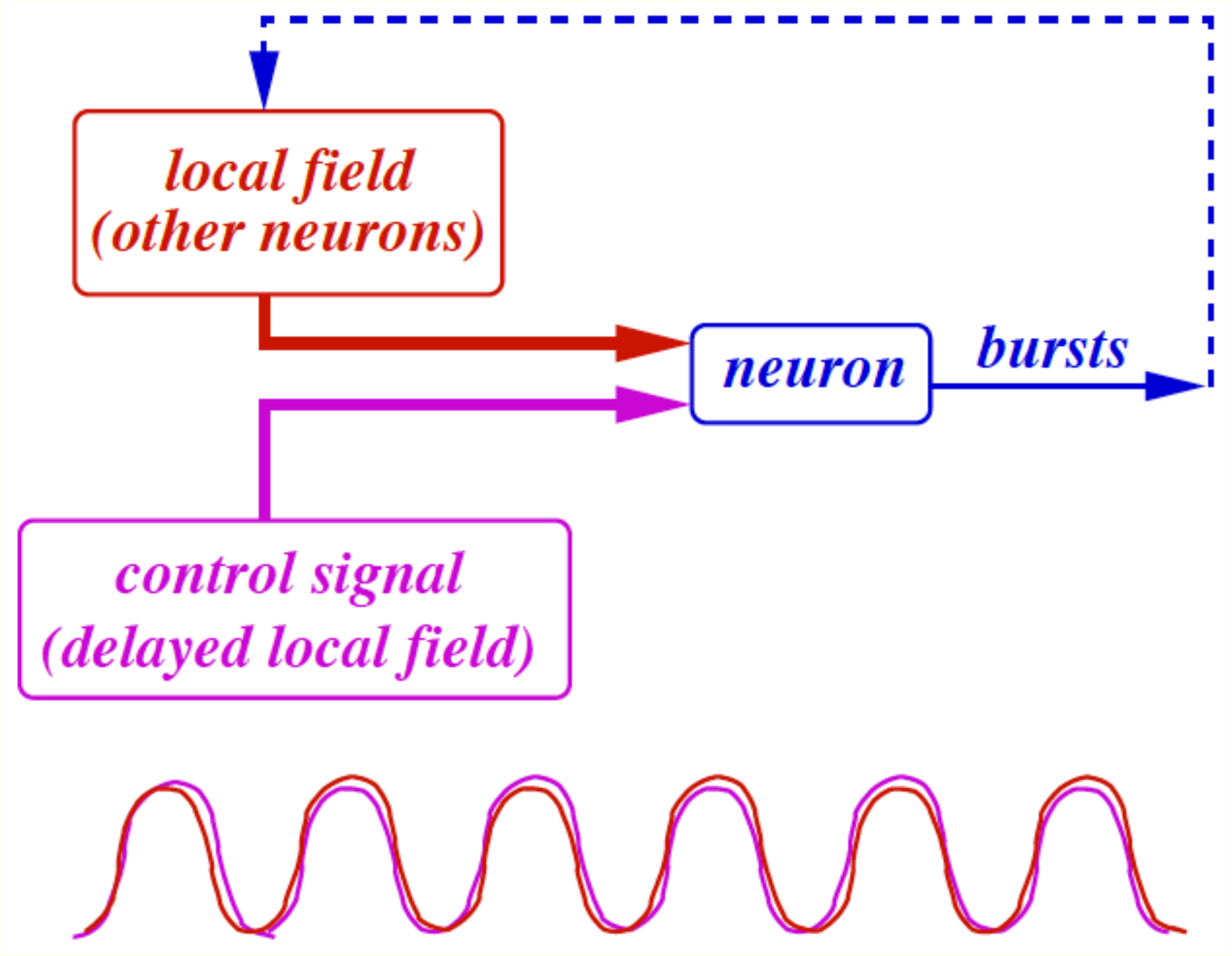


Input current to a neuron $= I_{ext} + \sum I_{synaptic} + I_{control}$

$$I_{control} \sim (\Phi(t - \mathcal{T}) - \Phi(t))$$

Why does it work?

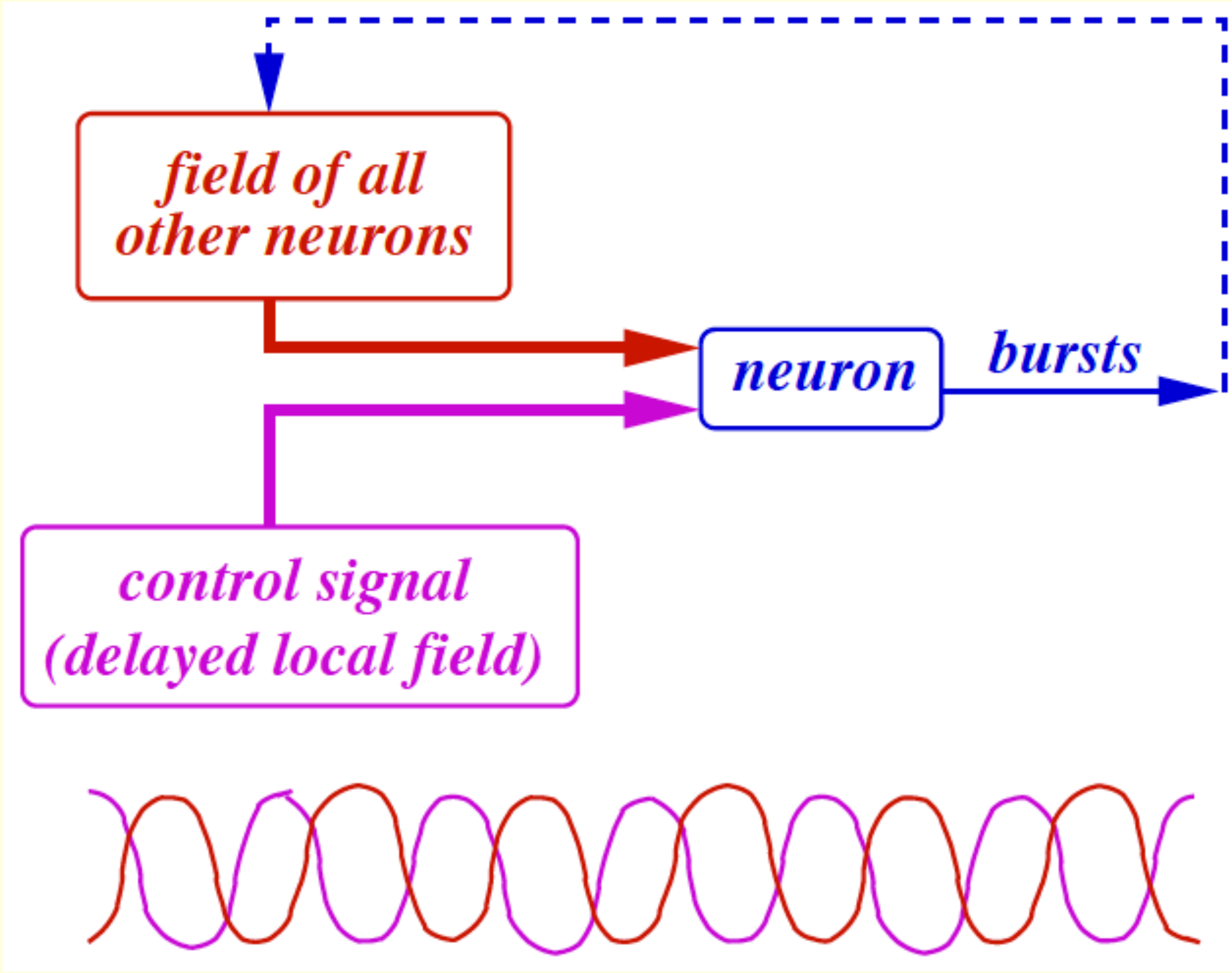
Positive feedback with delay ≈ average period of local field



$\Phi(t) + \Phi(t-T) \approx 2\Phi(t) \quad \Longrightarrow$ enhancement!

Why does it work?

Positive feedback with delay $\approx$ half of average period of local field



$$\Phi(t)+\Phi(t-T/2)\approx 0 \quad \Longrightarrow$$ suppression!

What kind of feedback do we need?

- Vanishing-stimulation control
- Based on the noisy measurement, contaminated by other rhythms
- Adaptation: there are unknown parameters!

Model equation

We assume Hopf bifurcation for the mean field and write for the collective mode the normal form equation with the control term

$$\dot{Z} = (\xi + i\omega_0)Z - \zeta|Z|^2 Z + \varepsilon e^{i\beta} u$$

Desynchronization of the ensemble = stabilization of the equilibrium point $Z = 0$ of the model equation

Control input u shall be determined from the measurement Φ

Frequency ω can be measured, other parameters are unknown

We suggest a simple solution without delay

Vanishing-stimulation suppression by a feedback without delay

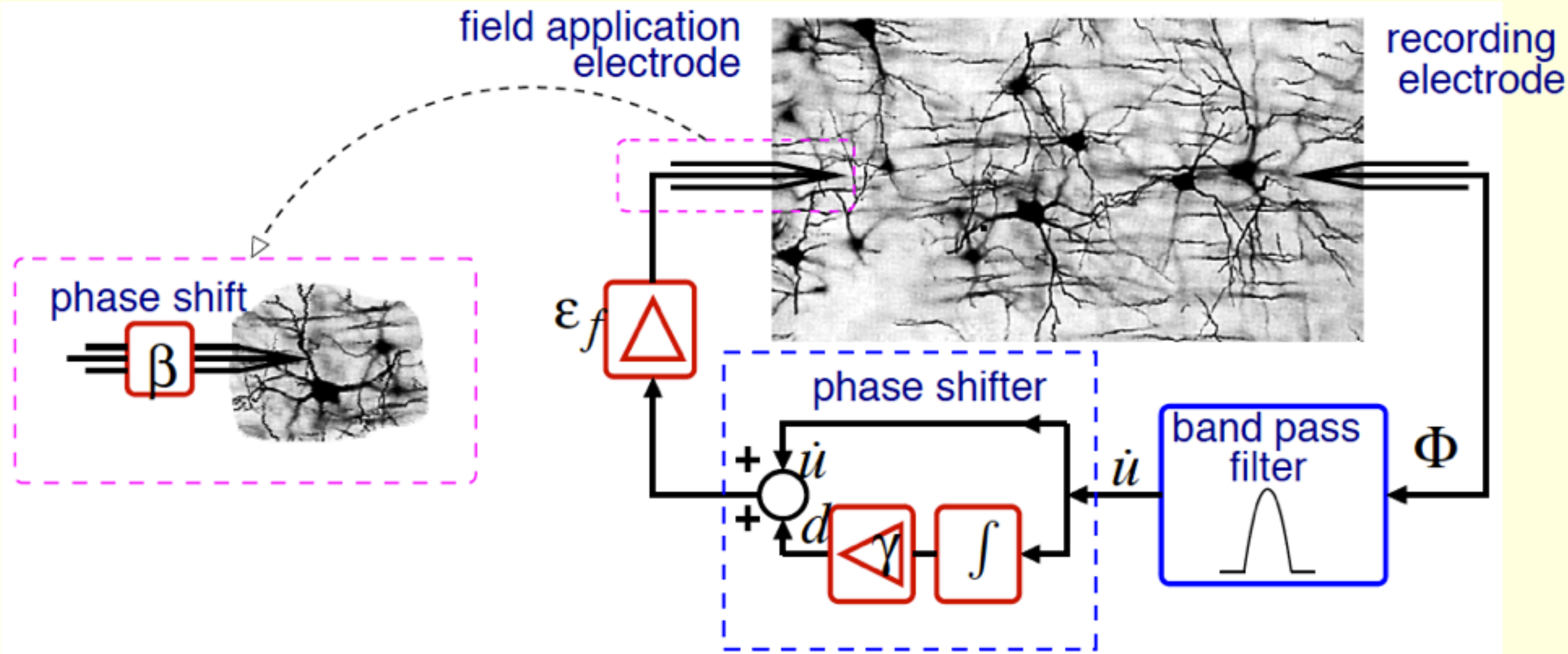
The role of the delay is to provide a certain phase shift: stimulation C should be in anti-phase with Φ to destroy the ensemble synchrony

Alternative approach: passive linear oscillator in the feedback loop

Tukhlina et al, Phys. Rev. E **75**, 011918 (2007)

- **Control theory viewpoint:** 2nd order filter in the feedback loop
- **Physics viewpoint:** interaction of active and passive media

Vanishing-stimulation suppression by a feedback without delay



Important: unknown phase shift β inherent to stimulation

Band pass filter: $\ddot{u} + \alpha\dot{u} + \omega_0^2 u = \Phi$

Integrator: $\mu\dot{d} + d = \dot{u}$

Vanishing-stimulation suppression by a feedback without delay

It works perfectly, but we have to tune two parameters:

- feedback gain γ
- phase shift α, which compensates the inherent phase shift β

$\longrightarrow$ We use an adaptive scheme

Adaptive controller

$$\ddot{u} + \delta\dot{u} + \omega_0^2 u = \Phi$$ filters the measured signal

$$\mu\dot{d} + d = \dot{u}$$ integrator

$$x = \delta\dot{u}\ , y = \delta\mu\omega_0 d$$ auxiliary variables, phase shift $\pi/2$

$$I = \sqrt{x^2 + y^2}$$ ~ oscillation amplitude

$$S = I[1 + \tanh(k_s(I - h_s))]$$ sigmoidal function of the amplitude

$$\dot{\alpha} = k_\alpha S\ , \qquad \alpha(0) = 0$$ adaptive phase shift

$$\dot{\gamma} = k_{\gamma 1} S \cosh^{-1}(k_{\gamma 2}\gamma/\omega_0)\ , \quad \gamma(0) = 0$$ adaptive gain

$$u = -\gamma(x\cos\alpha + y\sin\alpha)$$ control input

Why does it work?

Recall the model equation

$$\dot{Z} = (\xi + i\omega_0)Z - \zeta|Z|^2 Z + \varepsilon e^{i\beta} u$$

Suppose we measure $\Phi = Z$

Equilibrium can be stabilized with $u = e^{-\beta} Z$

and $\varepsilon = -\gamma \,, \gamma > \xi$

$u = \mathrm{Re}\left(e^{-\beta} Z\right)$ shall work as well

In our case $x \sim \mathrm{Re}\,(Z) \;, y \sim \mathrm{Im}\,(Z)$

$x \cos\alpha + y \sin\alpha$ describes rotation by $-\alpha$

Hence, $u = -\gamma(x\cos\alpha + y\sin\alpha)$ is what we need if $\alpha = \beta$!

Adaptive desynchronization: Example I

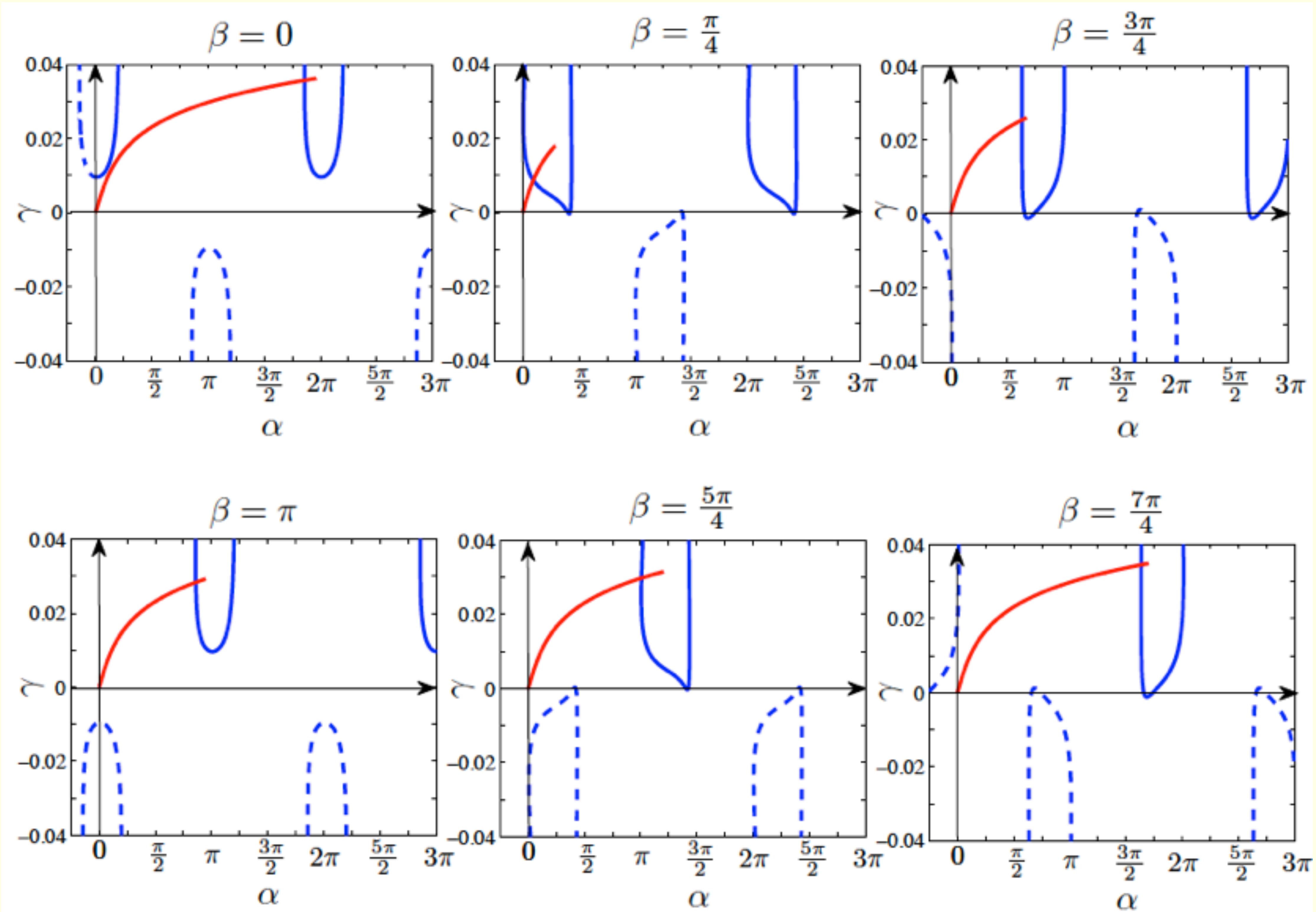
1000 non-identical Stuart-Landau oscillators

$$\dot{x}_i = ax_i - \omega_i y_i - x_i(x_i^2 + y_i^2) + CX + u\cos\beta,$$
$$\dot{y}_i = \omega_i x_i + ay_i - y_i(x_i^2 + y_i^2) + CY + u\sin\beta,$$

Parameters of the model equation can be found by fitting

$$\dot{\boldsymbol{Z}} = (\boldsymbol{\xi} + \boldsymbol{i\omega_0})\boldsymbol{Z} - \boldsymbol{\zeta}|\boldsymbol{Z}|^2\boldsymbol{Z} + \boldsymbol{\varepsilon e^{i\beta}u}$$

Adaptive desynchronization: Example I

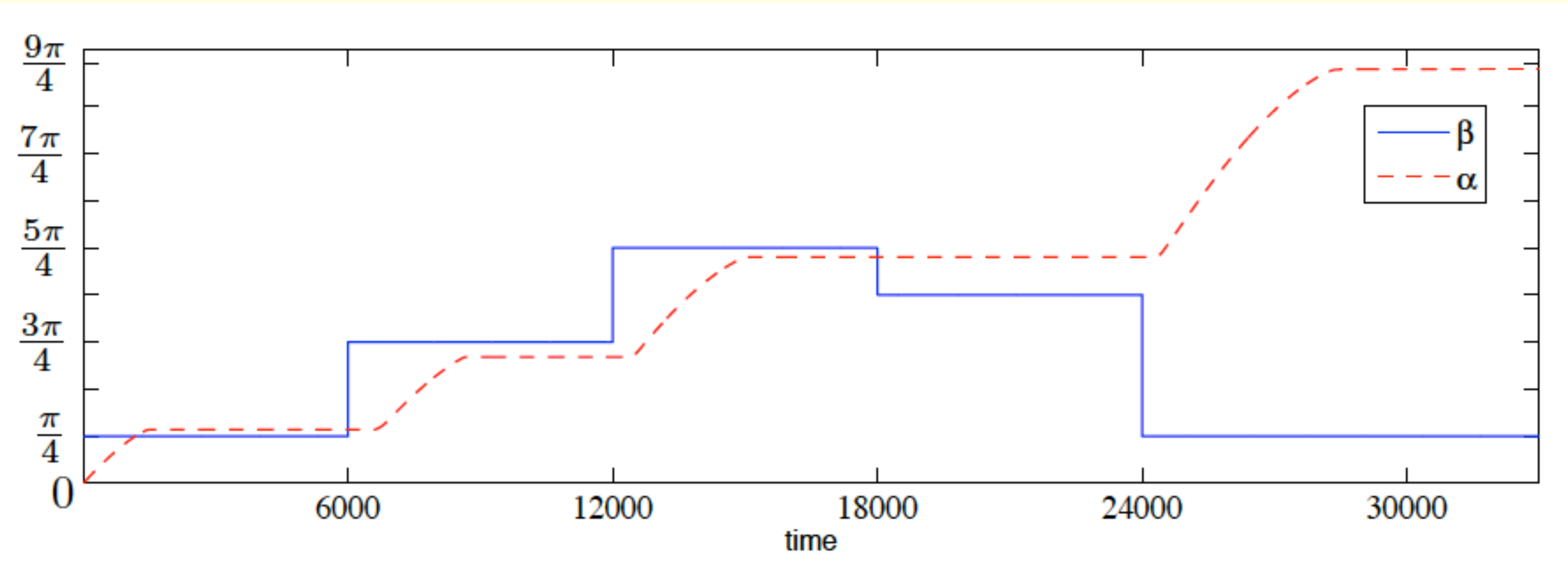


Blue: stability domains (theory)

Red: evolution of the adaptive parameters

Adaptive desynchronization: Example I

1000 non-identical Stuart-Landau oscillators



adaptation in case of time-varying β

Adaptive desynchronization: Example II

200 non-identical Hindmarsh-Rose neurons

$$\dot{x}_i = 3x_i^2 - x_i^3 + y_i - z_i + I_i - \frac{C}{N-1}(x_i + V_c)\sum_{j\neq i}^{N}\left[1 + e^{\left(\frac{x_i - x_0}{\eta}\right)}\right]^{-1} + u\ ,$$

$$\dot{y}_i = -5x_i^2 - y_i + 1\ ,$$

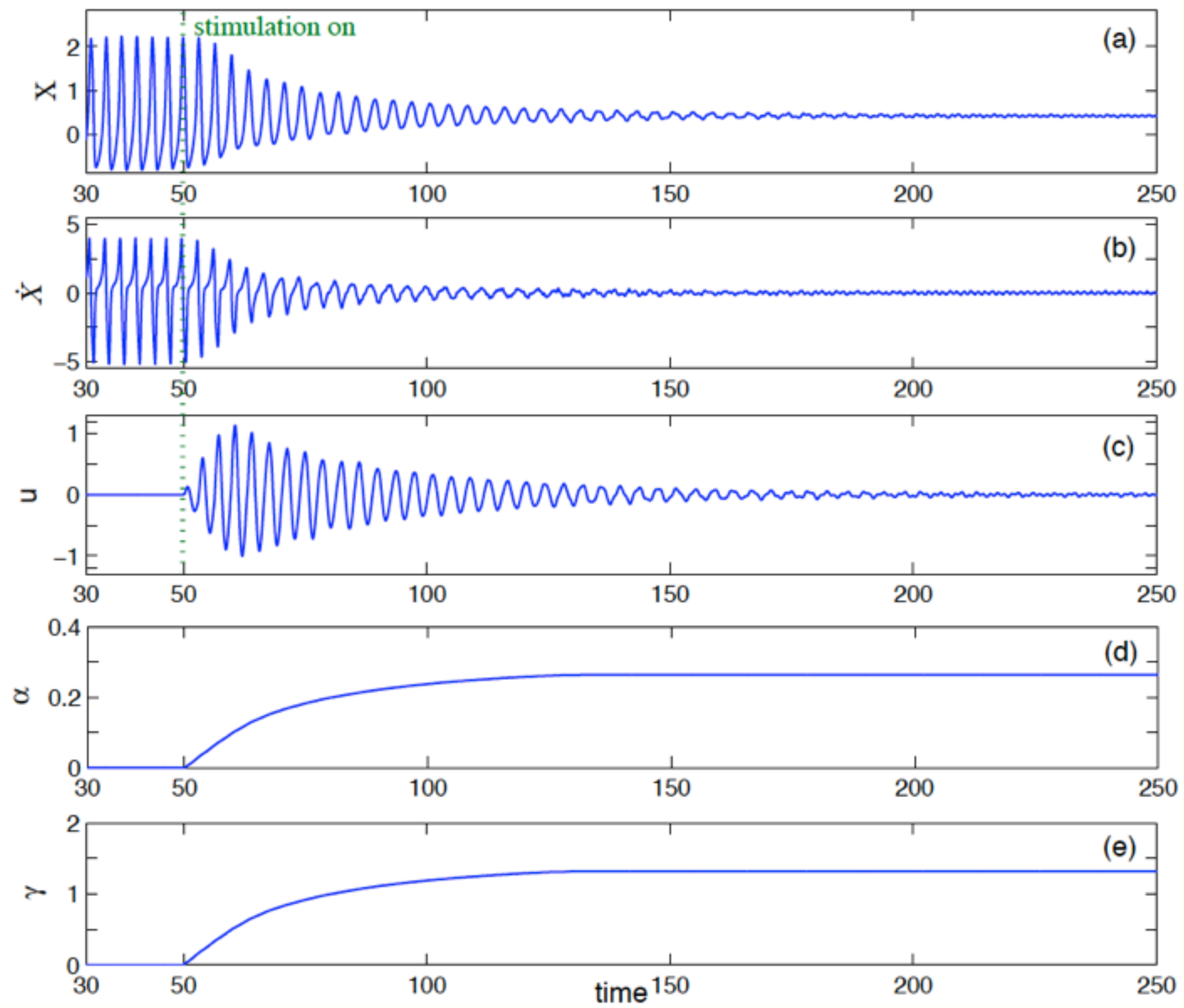
$$\dot{z}_i = r[\nu(x_i - \chi) - z_i]\ .$$

Synaptic coupling

Model of the local field potential $\boldsymbol{\Phi = \dot{X} = N^{-1}\sum_j \dot{x}_j}$

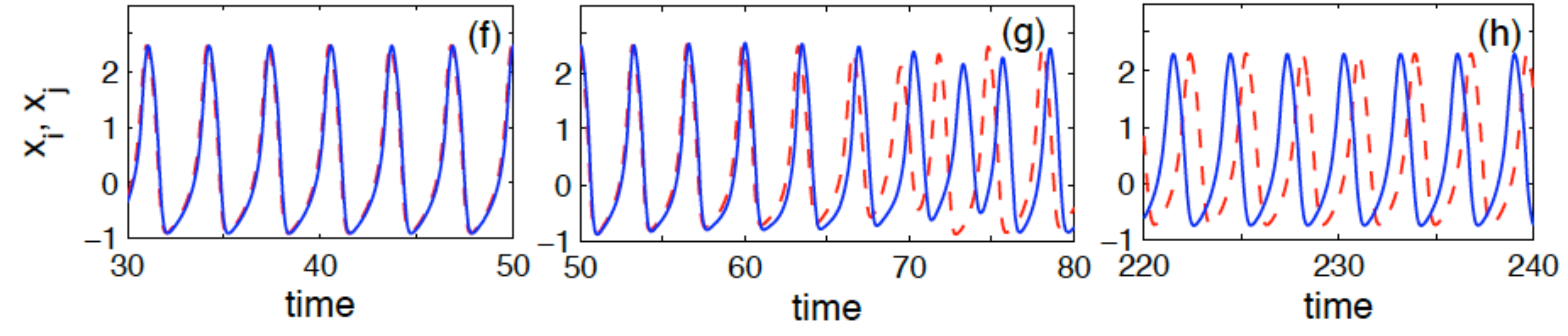
Adaptive desynchronization: Example II

Spiking Hindmarsh-Rose neurons



Adaptive desynchronization: Example II

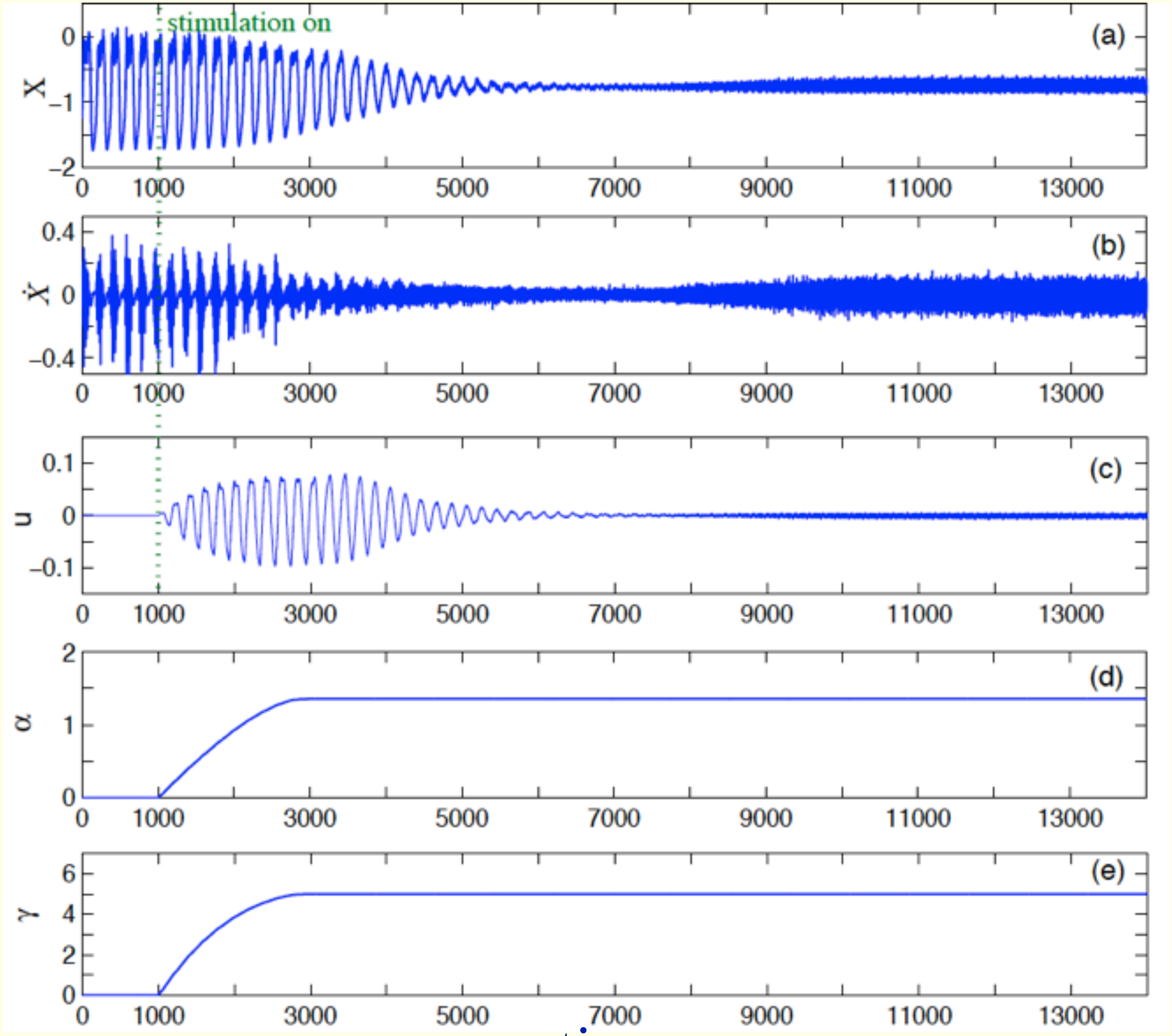
Spiking Hindmarsh-Rose neurons



Neurons are desynchronized, not suppressed!

Adaptive desynchronization: Example II

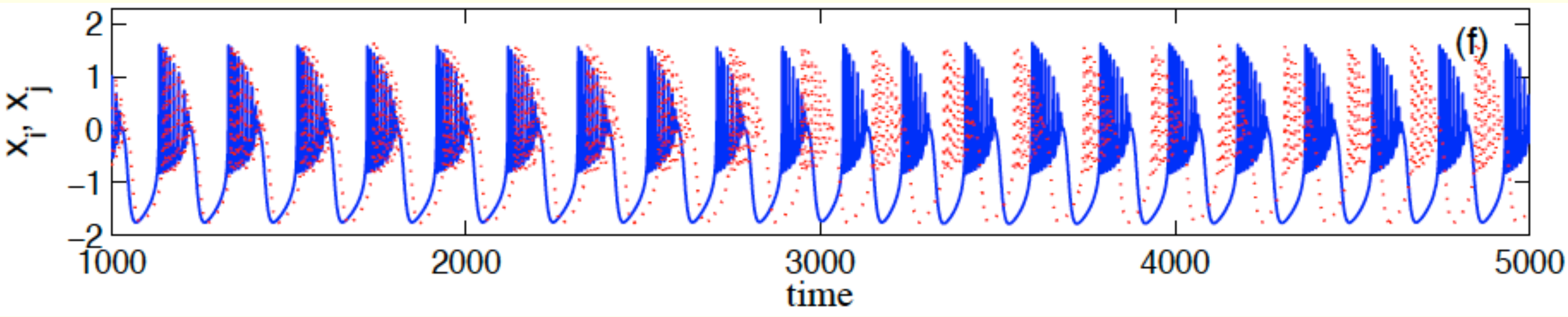
Bursting Hindmarsh-Rose neurons



time

Adaptive desynchronization: Example II

Bursting Hindmarsh-Rose neurons



Neurons are desynchronized, not suppressed!

Conclusions

- Tendency to synchrony is not monotonic
- Complex dynamics after synchrony breaking
- Reliable desynchronization of neuronal synchrony by adaptive feedback controller
- Vanishing-stimulation suppression of undesired rhythms

Thank you for your attention!