Introduction to Stochastic Dynamical Systems

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Project FAPESP-DFG

Project 5:

Random and stochastic partial differential equations and stochastic flows.
Research Team: P. Imkeller (Humboldt), P. Ruffino (UNICAMP)

Project 18:

Models interpreted by non-Gaussian dynamical systems and their metastable behavior
Research Team: P. Catuogno (UNICAMP), M. Högele (Potsdam), P. Imkeller (Humboldt), P. Ruffino (UNICAMP)

Sobre dinâmica estocástica

Um termo vasto, muitas áreas e subáreas. Por exemplo:

1) Dinâmica discreta

(no tempo ou no espaço). Atirar moedas, dados, sequências de saídas de geradores aleatórios, medidas no clima, etc. Dinâmica de partículas, modelos de Ising, termodinâmica. Dadas probabilidades de saltos, qual a configuração estacionária (medida invariante), etc.

2) Equações aleatórias:

$$\dot{x}_t = f(\omega, t, x_t).$$

As soluções tem trajetórias diferenciáveis.
Uma única aleatoriedade/sorteio determina toda a trajetória!

Sobre dinâmica estocástica

3) Equações diferenciais estocásticas:

O ruído é introduzido permanentemente no sistema. Comparar com sistema de controle:

$$\dot{x}_t = A_0(x_t) + A_1(x_t)u_t^1$$

Equações estocásticas de Itô correspondem a controles aleatórios (movimentos brownianos)

Sobre esse mini-curso

Parte I:

Introdução - básica. Concentrando $\pm$ 3 cursos de mestrado nessas 3 horas: Teoria da medida, processos estocásticos e equações diferenciais estocásticas.
Referências: Muitas! Ver, por exemplo.
"Uma iniciação aos Sistemas Dinâmicos Estocásticos". 3a. ed., IMPA, 2010.
www.ime.unicamp.br/∼ruffino

Parte II:

Motivação de um problema específico em dinâmica estocástica.

Parte II:

Problema específico.

Princípio da média para difusões em variedades folheadas:

Colaboradores: I. Gonzales-Gargate, P.H. da Costa, Michael Högele (Potsdam, Alemanha), Jan Gairing (Humboldt University, Berlin).

Introduction

Averaging on submanifolds: System with two variables

$$\begin{cases} x_t \in L & \text{with invariant measure } \nu \text{ in } L \\ y_t \in V & \text{the `main' observable motion, `perturbed' by } x_t \end{cases}$$

Such that

$$\dot{y}_t = F(x_t, y_t)$$

Main idea: approximation of F by its average in x:

Precisely: how close is y_t to the solution of

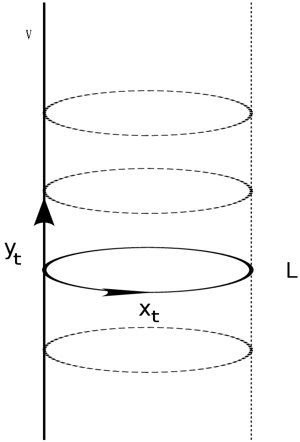
$$\dot{\bar{y}}_t = \bar{F}(\bar{y}_t)$$

where

$$\bar{F}(y) = \int_L F(y, x)\, d\nu(x) \quad ?$$

"Averaging principle" $=$ introducing a parameter dependence on ϵ (in few minutes) then y_t^ϵ converges to $\bar{y}_t$.

In a simplified picture:



Previous and related results (among others):

1) Diffusions on foliated space is interesting by itself: especially after the boost given by Lucy Garnett's paper: 'Foliations, the ergodic theorem and Brownian motion'. J. Funct. Anal. (1983).

2) M. Freidlin and A. Wentzell - *Random Perturbations of Dynamical Systems*.

3) Xue-Mei Li – An averaging principle for a completely integrable stochastic Hamiltonian system. *Nonlinearity*, 21 (2008) 803-822.

4) I. Gonzales-Gargate and R (Gaussian noise):
An averaging principle for diffusions in foliated spaces
Arxiv:1212.1587v4, (2013).

5) M. Högele and R:
Averaging along Lévy diffusions on foliated spaces
http://www.math.uni-potsdam.de/Jpreprint/2013.html

6) P.H. da Costa and R:
Dynamics generated by Le Jan-Raimond flows of measurable mappings.
Arxiv:1306.0123, (2013).

Foliated spaces:

Let M be a differentiable manifold.
An n-dimensional foliation in M provides a partition of M into immersed complete connected submanifolds of dimension n:

$$M = \bigcup_{x \in M} L_x$$

where L_x is the submanifold passing thorough x.

The equivalent classes L_x with $x \in M$ are called the leaves of the foliation.

Examples

1. (Trivial) Spherical foliation of $\mathbf{R}^n \setminus \{0\}$:

$$\mathbf{R}^n \setminus \{0\} = \bigcup_{r>0} S(0,r)$$

   $(n-1)$-dimension compact foliation. Codimension d=1.

2. In general, (locally) coordinate system generates a foliation when one of the parameters is fixed.
3. Liouville torus in Hamiltonian/simplectic structure (as in X.-M. Li).
4. Circular foliation in $M = \mathbb{R}^3 \setminus \{(0,0,z), z \in \mathbb{R}\}$
   Leaf through a point $q_0 = (x_0, y_0, z_0)$, $x_0 > 0$, is the horizontal circle

$$L_{q_0} = \{(\sqrt{x_0^2 + y_0^2}\cos\theta, \sqrt{x_0^2 + y_0^2}\sin\theta, z_0), \quad \theta \in [0, 2\pi)\}.$$

5. Irrational flow in the torus. Leaves are dense (hence not embedded).

Local coordinates

Our hypotheses on the geometrical framework:

Leaves are compact. There exists a neighbourhood U of L_{x_0} and a diffeomorphism $\xi : U \to L_{x_0} \times V$, with $0 \in V \subset \mathbf{R}^d$ a connected open set.

Denote by $\pi : U \to V$ the second coordinate of ξ.
Write $\pi = (\pi_1, \ldots, \pi_d)$.

The unperturbed SDE:

$$dx_t = X_0(x_t)dt + \sum_{i=1}^{r} X_i(x_t) \circ dB_t^i$$

where each $X_i(x) \in T_x L_x$. The support theorem guarantees that x_t lays in the leaf L_{x_0}. We assume that it generates a unique ergodic invariant measures μ_x in the leaf L_x.

The perturbed SDE:

$$dx_t^\epsilon = X_0(x_t^\epsilon)dt + \sum_{i=1}^{r} X_i(x_t^\epsilon) \circ dB_t^i + \epsilon K(x_t^\epsilon)dt,$$

where K is a 'perturbing' vector field transversal to the leaves.

Averaging on the leaves

Given a continuous function $g : M \to \mathbb{R}$. The average of g on each leaf L_x is given by $Q^g : V \subset \mathbf{R}^d \to \mathbf{R}$. Precisely, for each vertical coordinate $v \in V$,

$$Q^g(v) = \int_{L_x} g(y)\, d\mu_x(y)$$

where μ_x is an ergodic invariant measure in L_x for the unperturbed system.

In particular, we consider the average of the vertical components of the perturbing vector field $\pi_*(K)$. So, for each $i = 1, \ldots, d$,

$$Q^{d\pi_i(K)}(v) = \int_{L_x} d\pi_i K(y)\, d\mu_x(y).$$

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$$Q^{d\pi_i(K)}(v) = \int_{L_x} d\pi_i K(y)\, d\mu_x(y).$$

Denote by $\gamma(t)$ the rate of convergence of the ergodic unperturbed dynamic in L_{x_0} such that

$$\left(\mathbb{E} \left| \frac{1}{t} \int_0^t \pi_i K(x_s^0)\, ds - Q^{d\pi_i(K)}(\pi(x_0^0)) \right|^p \right)^{\frac{1}{p}} \leq \gamma(t)$$

for each i, where $c(t) \searrow 0$ when t goes to infinity.

Example: (1) the foliated space:

- **Foliated space:**

$$M = \mathbb{R}^3 \setminus \{(0,0,z), z \in \mathbb{R}\}$$

- **leaf through a point** $q_0 = (x_0, y_0, z_0)$, $x_0 > 0$, is the **horizontal circle**

$$L_{q_0} = \{(\sqrt{x^2+y^2}\cos\theta, \sqrt{x^2+y^2}\sin\theta, z), \quad \theta \in [0, 2\pi)\}.$$

- Local foliated coordinates in $U = \mathbb{R}^3 \setminus \{(x,0,z); x \leqslant 0; z \in \mathbb{R}\}$ given by **cylindrical coordinates**.

- Hence, in this coordinates system, the **transversal projection** π

$$\pi_1(x,y,z) = r$$

and

$$\pi_2(x,y,z) = z$$

Example: (2) the dynamics:

Consider the foliated linear SDE on M consisting of random rotations:

$$dx_t = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_t \ (\lambda_1 \ dt + \lambda_2 \circ dB_t).$$

$(\lambda_1, \lambda_2 \neq 0.)$ The invariant probability measure μ_p in the leaf L_p passing through a point $p \in M$ is given by the normalized Lebesgue measures in $L_p \sim S^1$.

We investigate the effective behaviour of the transversal component for a small transversal perturbation of order ϵ:

$$dx_t^\epsilon = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_t^\epsilon \ (\lambda_1 \ dt + \lambda_2 \ dB_t) + \epsilon K(x_t^\epsilon) \ dt.$$

with initial condition $x_0 = (1, 0, 0)$.

Main result

Hypothesis on the invariant measures:
For any Lipschitz continuous function g on M, its corresponding average function Q^g on the transversal space V (which indexes the leaves) is also Lipschitz.

Example: nondegenerate diffusions on the leaves.

Consider the ODE in the transversal space V

$$\frac{dv}{dt} = \left(Q^{d\pi_1(K)}, \ldots, Q^{d\pi_d(K)} \right) (v(t))$$

with initial condition $v(0) = \pi(x_0) = 0$
(possible finite explosion time T_0).

Main theorem

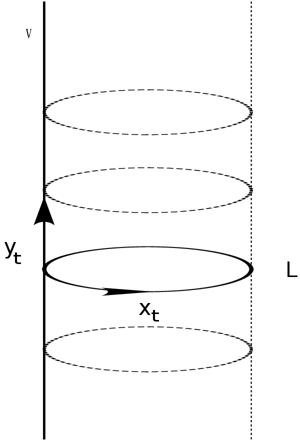
Theorem

There exists a $T_0 > 0$ *such that for all* $0 < t < T_0$, $\beta \in (0, 1/2)$ *and* $1 \leq p < \infty$,

$$\left[\mathbb{E} \left(\sup_{s \leq t} \left| \pi \left(x^{\epsilon}_{\frac{s}{\epsilon}} \right) - v(s) \right|^p \right) \right]^{\frac{1}{p}} \leq t\sqrt{\epsilon}\ h(t, \epsilon) + \gamma(t |\ln \epsilon|^{\frac{2\beta}{p}}) e^{Ct}.$$

where $h(t, \epsilon)$ *is continuous for* $t, \epsilon > 0$, *converges to zero when* $(t, \epsilon) \to 0$, $C > 0$ *and* $\gamma(\cdot)$ *is the rate of convergence of the ergodic theorem in the horizontal leaves.*

Interpretation of averaging:



Back to our example: Random rotation of horizontal circles in $M = \mathbb{R}^3 \setminus z$-axis

- The invariant probability measures μ_p in the leaves L_p passing through points $p \in M$ are given by normalized Lebesgue measures in the circle L_p. Hence they satisfy hypothesis.

- We investigate the effective behaviour of the transversal component for a small transversal perturbation of order ϵ:

$$dx_t^\epsilon = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_t^\epsilon \; (\lambda_1 \; dt + \lambda_2 \; \circ dB_t) + \epsilon K(x_t^\epsilon) \; dt.$$

with initial condition $x_0 = (1, 0, 0)$.

Example: horizontal perturbation

constant horizontal perturbation:

Consider $K = (k_1, k_2, 0)$ with respect to Euclidean coordinates in M.

Then $Q^{d\pi_1 K} = Q^{d\pi_2 K} = 0$. Hence the transversal component in the Theorem is constant $v(t) = (r(0), z(0))$ for all $t \geq 0$.

For comparison, the strong solution of the perturbed system is given by:

$$x^{\epsilon}_{\frac{t}{\epsilon}} = \begin{pmatrix} \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \\ \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} k_1 \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) + k_2 \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) - k_2 \\ -k_1 \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) + k_2 \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) - k_1 \\ 0 \end{pmatrix}$$

For $K = (1, 0, 0)$,

$$\begin{aligned} \pi_1(x^{\epsilon}_{\frac{t}{\epsilon}}) &= 1 + \epsilon^2 \left[2 + \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right)\right] \\ &\quad - \epsilon \left[\cos 2 \left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) + \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right)\right] \end{aligned}$$

Hence

$$|\pi_1(x^{\epsilon}_{\frac{t}{\epsilon}}) - 1| < h(t)\,\epsilon$$

according to the boundaries on the the rate of convergence stated in the theorem.

Constant vertical perturbations.
$K = (0, 0, k_3)$, the radial average $Q^{d\pi_1 K} = 0$ but $Q^{d\pi_2 K} = k_3$ for every leave in M. Hence the averaged system $v(t) = (r(0), k_3 t)$.
For comparison, the solution of the perturbed system:

$$x^{\epsilon}_{\frac{t}{\epsilon}} = \begin{pmatrix} \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \\ \\ \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \\ \\ k_3 t \end{pmatrix}$$

Hence, the comparison

$$\left|\pi_2\left(x^{\epsilon}_{\frac{s}{\epsilon}}\right) - v(t)\right| \equiv 0$$

for all $t \geq 0$ and the convergence of the theorem is trivially verified.

Linear perturbation.

Linear perturbation of the form $K(x, y, z) = (x, 0, 0)$.
z-coordinate $Q^{d\pi_2 K} = 0$.
For the radial component, we have that $d\pi_1 K(u) = r_0 \cos^2 u$, where u is the angular coordinate of p whose distance to the z-axis (r-coordinate) is r_0. Hence the average with respect to the invariant measure on the leaves is given by $Q^{d\pi_1 K} = r/2$ for leaves with radius r.
The transversal system stated in the Theorem is then
$v(t) = (e^{\frac{t}{2}} r(0), z(0))$. Hence the main Theorem guarantees that the radial part of $x^{\epsilon}_{\frac{t}{\epsilon}}$ must have a behaviour close to the exponential $e^{\frac{t}{2}}$ in the sense that

$$\left[\mathbb{E} \left(\sup_{s \leq t} \left| \pi_1 \left(x^{\epsilon}_{\frac{t}{\epsilon}} \right) - e^{\frac{t}{2}} \right|^p \right) \right]^{\frac{1}{p}}$$

goes to zero when ϵ goes to zero.

Proof of the theorem (main ideas)

Remind: $v(t)$ is the solution of the ODE in the vertical space

$$\frac{dv}{dt} = \left(Q^{d\pi_1(K)}, \dots, Q^{d\pi_d(K)} \right) (v(t))$$

with initial condition $v(0) = \pi(x_0) = 0$.

Theorem

For all $0 < t < T_0$, $\beta \in (0, 1/2)$ *and* $1 \le p < \infty$,

$$\left[\mathbb{E} \left(\sup_{s \le t} \left| \pi\left(x^{\epsilon}_{\frac{s}{\epsilon}} \right) - v(s) \right|^p \right) \right]^{\frac{1}{p}} \le t\sqrt{\epsilon}\ h(t, \epsilon) + \gamma(t |\ln \epsilon|^{\frac{2\beta}{p}}) e^{Ct}.$$

Idea of the Proof

$$
\begin{aligned}
\left|\pi_i\left(x^{\epsilon}_{\frac{t}{\epsilon}}\right) - v_i(t)\right| &\leq \int_0^t \left| d\pi_i(K)(x^{\epsilon}_{\frac{s}{\epsilon}}) - Q^{d\pi_i(K)}(v(s))\right| ds \\
&\leq \int_0^t \left| Q^{d\pi_i(K)}\left(\pi\left(x^{\epsilon}_{\frac{s}{\epsilon}}\right)\right) - Q^{d\pi_i(K)}(v(s))\right| ds \\
&\qquad + \left| \int_0^t d\pi_i K(x^{\epsilon}_{\frac{r}{\epsilon}}) dr - \int_0^t Q^{d\pi_i(K)}(\pi(x^{\epsilon}_{\frac{r}{\epsilon}})) dr \right| \\
&\leq \quad C_i \int_0^t \left|\pi_i\left(x^{\epsilon}_{\frac{s}{\epsilon}}\right) - v_i(s)\right| ds + |\delta_i(\epsilon, t)|,
\end{aligned}
$$

Where

$$
\delta_i(\epsilon, t) = \int_0^t d\pi_i K(x^{\epsilon}_{\frac{r}{\epsilon}}) dr - \int_0^t Q^{d\pi_i(K)}(\pi(x^{\epsilon}_{\frac{r}{\epsilon}})) dr
$$

Idea of the Proof

By Gronwall's Lemma we have that

$$\left|\pi_i\left(x^{\epsilon}_{\frac{t}{\epsilon}}\right) - v_i(t)\right| \leq e^{Ct}|\delta_i(\epsilon, t)|.$$

All we have to do is to study the rate of convergence of $|\delta_i(\epsilon, t)|$ to zero.

$\square$

This is the content of the next Proposition.

Proposition

Remind: For a Lipchitz continuous function g,

$$\delta(\epsilon, t) = \int_0^t g(x^{\epsilon}_{\frac{r}{\epsilon}})dr - \int_0^t Q^g(\pi(x^{\epsilon}_{\frac{r}{\epsilon}}))dr$$

Proposition

For $1 \leq p < \infty$

$$\left(\mathbb{E} \sup_{s \leq t} |\delta(\epsilon, s)|^p\right)^{\frac{1}{p}} \leq t\sqrt{\epsilon}\, h(t, \epsilon) + \gamma(t|\ln \epsilon|^{\frac{2\beta}{p}}).$$

where $h(t, \epsilon)$ *is continuous for* $t, \epsilon > 0$, *converges to zero when* $(t, \epsilon) \to 0$ *and* $\gamma(\cdot)$ *is the rate of convergence of the ergodic theorem for the unperturbed system in the horizontal leaves.*

Idea of the Proof of the Proposition

We rescale the time interval to $[0, \frac{t}{\epsilon})$ and consider a partition $t_n = n\Delta t$ such that

$$0 = t_0 < t_1 < \cdots < t_N \leq \frac{t}{\epsilon}.$$

with increments

$$\Delta t = \frac{t}{|\ln \epsilon|^{-\frac{2\beta}{p}}}.$$

and

$$N = N(\epsilon) = \lfloor \epsilon^{-1} |\ln \epsilon|^{-\frac{2\beta}{p}} \rfloor$$

Idea of the Proof

We rewrite the first integral in the definition of δ as

$$\epsilon \int_0^{\frac{t}{\epsilon}} g(x_r^\epsilon) dr = \epsilon \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} g(x_r^\epsilon) dr + \epsilon \int_{t_N}^{\frac{t}{\epsilon}} g(x_r^\epsilon) dr.$$

Denote by θ_t the canonical shift operator on the probability space. Let $F_t(\cdot, \omega)$ with $t \geq 0$ be the flow of the original unperturbed system in M. Triangular inequality splits our calculation into four parts

$$|\delta(\epsilon, t)| \leq |A_1| + |A_2| + |A_3| + |A_4|, \tag{1}$$

where:

$$
\begin{aligned}
A_1 &= \epsilon \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left[g(x_r^\epsilon) - g(F_{r-t_n}(x_{t_n}^\epsilon, \theta_{t_n}(\omega))) \right] dr, \\
A_2 &= \epsilon \sum_{n=0}^{N-1} \left[\int_{t_n}^{t_{n+1}} g(F_{r-t_n}(x_{t_n}^\epsilon, \theta_{t_n}(\omega))) dr - \Delta t Q^g(\pi(x_{t_n}^\epsilon)) \right], \\
A_3 &= \sum_{n=0}^{N-1} \epsilon \Delta t Q^g(\pi(x_{t_n}^\epsilon)) - \int_0^t Q^g(\pi(x_{\frac{r}{\epsilon}}^\epsilon)) dr, \\
A_4 &= \epsilon \int_{t_N}^{\frac{t}{\epsilon}} g(x_r^\epsilon) dr.
\end{aligned}
$$

Lemma

Let τ^ϵ be the exit time of the neighbourhood U of L_{x_0}. For any differentiable function $g : M \to \mathbf{R}$ and $1 \le p < \infty$ we have

$$\left[\mathbb{E}\left(\sup_{s\le t\wedge\tau^\epsilon} \left|g(x_s^\epsilon) - g(x_s^0)\right|^p\right)\right]^{\frac{1}{p}} \le K_1\,\varepsilon\,t\,e^{K_2 t^p}.$$

where $K_1, K_2 \ge 0$ are constants depending on upper bounds of the norms of the perturbing vector field K and of the derivatives of g, $X_0, X_1 \cdots, X_r$ with respect to the coordinate system.

Remark: Li's estimates is $K_1\epsilon(t+t^2)$.

Open problems:

1) Change the model of perturbation: stochastic transversal perturbation (work in progress with J. Gairing, Humboldt University).

2) Application: For a compact manifold L embedded in R^n. Consider a tubular neighbourhood, and a Brownian motion in L perturbed in the normal direction proportional to the curvature. Average transversal behaviour depends on the Euler characteristic of L.