Introduction to Stochastic Dynamical Systems (3^{rd} part)

Paulo Ruffino

Mathematics Institute
University of Campinas, Brazil

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Fórmula de Itô

X_t^1 e X_t^2 processos estocásticos em $\mathbf{R}$.
$f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ de classe C^2.
Então:

$$\begin{aligned} f(X_t^1, X_t^2) &= f(X_0^1, X_0^2) + \int_0^t \frac{\partial f}{\partial x_1}(X_s^1, X_s^2) dX_s^1 + \int_0^t \frac{\partial f}{\partial x_2}(X_s^1, X_s^2) dX_s^2 \\ &+ \frac{1}{2} \sum_{i,j=1}^{2} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s^1, X_s^2)[dX^1, dX^2]. \end{aligned}$$

Lembrar que $[B_t, B_t] = t$ ou

$$[dB_t, dB_t] = dt.$$

Além disso $[t, B_t] = 0$ e $[B_t^1, B_t^2] = 0$ se B_t^1 e B_t^2 forem independentes.

Parte II:

Problema específico.

Princípio da média para difusões em variedades folheadas:

Colaboradores: I. Gonzales-Gargate, P.H. da Costa, Michael Högele (Potsdam, Alemanha), Jan Gairing (Humboldt University, Berlin).

Introduction

Averaging on submanifolds: System with two variables

$$\begin{cases} x_t \in L & \text{with invariant measure } \nu \text{ in } L \\ y_t \in V & \text{the 'main' observable motion, 'perturbed' by } x_t \end{cases}$$

Such that

$$\dot{y}_t = F(x_t, y_t)$$

Main idea: approximation of F by its average in x:

Precisely: how close is y_t to the solution of

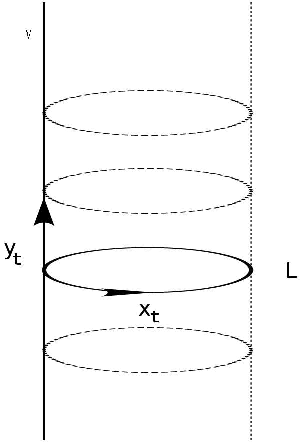
$$\dot{\bar{y}}_t = \bar{F}(\bar{y}_t)$$

where

$$\bar{F}(y) = \int_L F(y, x)\, d\nu(x) \quad ?$$

"Averaging principle" = introducing a parameter dependence on ϵ (in few minutes) then y_t^ϵ converges to $\bar{y}_t$.

In a simplified picture:



The random dynamics

The unperturbed SDE:

$$dx_t = X_0(x_t)dt + \sum_{i=1}^{r} X_i(x_t) \circ dB_t^i$$

where each $X_i(x) \in T_x L_x$. The support theorem guarantees that x_t lays in the leaf L_{x_0}. We assume that it generates a unique ergodic invariant measures μ_x in the leaf L_x.

The perturbed SDE:

$$dx_t^\epsilon = X_0(x_t^\epsilon)dt + \sum_{i=1}^{r} X_i(x_t^\epsilon) \circ dB_t^i + \epsilon K(x_t^\epsilon)dt,$$

where K is a 'perturbing' vector field transversal to the leaves.

Main result

A média do campo transversal (perturbação) K é dada por

$$Q^K = (Q^{K_1}, \ldots, Q^{K_d})$$

Hypothesis on the invariant measures:
Q^K is Lipschitz on the transversal space V (which indexes the leaves).

Example: any nondegenerate diffusions on the leaves.

Consider the ODE in the transversal space V

$$\frac{dv}{dt} = \left(Q^{K_1}, \ldots, Q^{K_d}\right)(v(t))$$

with initial condition $v(0) = \pi(x_0) = 0$
(possible finite explosion time T_0).

Main theorem

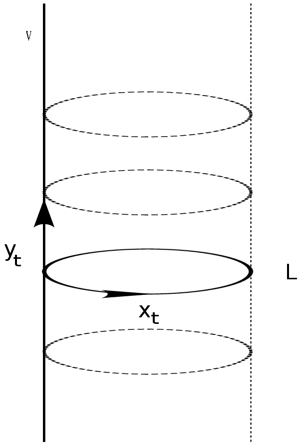
Theorem

There exists a $T_0 > 0$ *such that for all* $0 < t < T_0$, $\beta \in (0, 1/2)$ *and* $1 \leq p < \infty$,

$$\left[\mathbb{E} \left(\sup_{s \leq t} \left| \pi \left(x^{\epsilon}_{\frac{s}{\epsilon}} \right) - v(s) \right|^{p} \right) \right]^{\frac{1}{p}} \leq t\sqrt{\epsilon}\; h(t,\epsilon) + \gamma(t|\ln \epsilon|^{\frac{2\beta}{p}}) e^{Ct}.$$

where $h(t,\epsilon)$ *is continuous for* $t, \epsilon > 0$, *converges to zero when* $(t,\epsilon) \to 0$, $C > 0$ *and* $\gamma(\cdot)$ *is the rate of convergence of the ergodic theorem in the horizontal leaves.*

Interpretation of averaging:



Example: (1) the geometry

- **Foliated space:**

$$M = \mathbb{R}^3 \setminus \{(0,0,z), z \in \mathbb{R}\}$$

- **leaf through a point** $q_0 = (x_0, y_0, z_0)$, $x_0 > 0$, is the **horizontal circle**

$$L_{q_0} = \{(\sqrt{x^2+y^2}\cos\theta, \sqrt{x^2+y^2}\sin\theta, z), \quad \theta \in [0, 2\pi)\}.$$

- Local foliated coordinates in $U = \mathbb{R}^3 \setminus \{(x,0,z); x \leqslant 0; z \in \mathbb{R}\}$ given by **cylindrical coordinates**.

- Hence, (r in this coordinates system, the **transversal projection** π

$$\pi_1(x,y,z) = r$$

and

$$\pi_2(x,y,z) = z$$

Example: (2) the dynamics

Consider the foliated linear SDE on M consisting of random rotations:

$$dx_t = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_t \ (\lambda_1 \ dt + \lambda_2 \circ dB_t).$$

$(\lambda_1, \lambda_2 \neq 0.)$ The invariant probability measure μ_p in the leaf L_p passing through a point $p \in M$ is given by the normalized Lebesgue measures in $L_p \sim S^1$.

We investigate the effective behaviour of the transversal component for a small transversal perturbation of order ϵ:

$$dx_t^\epsilon = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_t^\epsilon \ (\lambda_1 \ dt + \lambda_2 \ dB_t) + \epsilon K(x_t^\epsilon) \ dt.$$

with initial condition $x_0 = (1, 0, 0)$.

Example: (3) the averaging

- The invariant probability measures μ_p in the leaves L_p passing through points $p \in M$ are given by normalized Lebesgue measures in the circle L_p. Hence they satisfy hypothesis.

- We investigate the effective behaviour of the transversal component for a small transversal perturbation of order ϵ:

$$dx_t^\epsilon = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_t^\epsilon \ (\lambda_1 \ dt + \lambda_2 \ \circ dB_t) + \epsilon K(x_t^\epsilon) \ dt.$$

with initial condition $x_0 = (1, 0, 0)$.

Example: horizontal perturbation

constant horizontal perturbation:

Consider $K = (k_1, k_2, 0)$ with respect to Euclidean coordinates in M.

Then $Q^{d\pi_1 K} = Q^{d\pi_2 K} = 0$. Hence the transversal component in the Theorem is constant $v(t) = (r(0), z(0))$ for all $t \geq 0$.

For comparison, the strong solution of the perturbed system is given by:

$$x^{\epsilon}_{\frac{t}{\epsilon}} = \begin{pmatrix} \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \\ \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} k_1 \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) + k_2 \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) - k_2 \\ -k_1 \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) + k_2 \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) - k_1 \\ 0 \end{pmatrix}$$

For $K = (1, 0, 0)$,

$$\begin{aligned} \pi_1(x^{\epsilon}_{\frac{t}{\epsilon}}) &= 1 + \epsilon^2 \left[2 + \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \right] \\ &\quad - \epsilon \left[\cos 2 \left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) + \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \right] \end{aligned}$$

Hence

$$|\pi_1(x^{\epsilon}_{\frac{t}{\epsilon}}) - 1| < h(t)\,\epsilon$$

according to the boundaries on the the rate of convergence stated in the theorem.

Vertical perturbation

Constant vertical perturbations.
$K = (0, 0, k_3)$, the radial average $Q^{K_1} = 0$ but $Q^{K_2} = k_3$ for every leave in M. Hence the averaged system $v(t) = (r(0), k_3 t)$.
For comparison, the solution of the perturbed system:

$$x^{\epsilon}_{\frac{t}{\epsilon}} = \begin{pmatrix} \cos\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \\ \\ \sin\left(\frac{\lambda_1 t}{\epsilon} + \lambda_2 B_{\frac{t}{\epsilon}}\right) \\ \\ k_3 t \end{pmatrix}$$

Hence, the comparison

$$\left|\pi_2\left(x^{\epsilon}_{\frac{s}{\epsilon}}\right) - v(t)\right| \equiv 0$$

for all $t \geq 0$ and the convergence of the theorem is trivially verified.

Linear perturbation.

Linear perturbation of the form $K(x,y,z) = (x,0,0)$.
z-coordinate $Q^{d\pi_2 K} = 0$.
For the radial component, we have that $K_1(u) = r_0 \cos^2 u$, where u is the angular coordinate of p whose distance to the z-axis (r-coordinate) is r_0.
Hence the average with respect to the invariant measure on the leaves is given by $Q^{d\pi_1 K} = r/2$ for leaves with radius r.
The transversal system stated in the Theorem is then $v(t) = (e^{\frac{t}{2}} r(0), z(0))$. Hence the main Theorem guarantees that the radial part of $x^{\epsilon}_{\frac{t}{\epsilon}}$ must have a behaviour close to the exponential $e^{\frac{t}{2}}$ in the sense that

$$\left[\mathbb{E}\left(\sup_{s\leq t}\left|\pi_1\left(x^{\epsilon}_{\frac{t}{\epsilon}}\right) - e^{\frac{t}{2}}\right|^p\right)\right]^{\frac{1}{p}}$$

goes to zero when ϵ goes to zero.

Proof of the theorem (main ideas)

Remind: $v(t)$ is the solution of the ODE in the vertical space

$$\frac{dv}{dt} = \left(Q^{K_1}, \dots, Q^{K_d}\right)(v(t))$$

with initial condition $v(0) = \pi(x_0) = 0$.

Theorem

For all $0 < t < T_0$, $\beta \in (0, 1/2)$ *and* $1 \leq p < \infty$,

$$\left[\mathbb{E}\left(\sup_{s\leq t}\left|\pi\left(x^{\epsilon}_{\frac{s}{\epsilon}}\right) - v(s)\right|^p\right)\right]^{\frac{1}{p}} \leq t\sqrt{\epsilon}\ h(t,\epsilon) + \gamma(t|\ln\epsilon|^{\frac{2\beta}{p}})e^{Ct}.$$

Idea of the Proof

$$
\begin{aligned}
\left|\pi_i\left(x^{\epsilon}_{\frac{t}{\epsilon}}\right) - v_i(t)\right| &\le \int_0^t \left|K_i(x^{\epsilon}_{\frac{s}{\epsilon}}) - Q^{K_i}(v(s))\right| ds \\
&\le \int_0^t \left|Q^{K_i}\left(\pi\left(x^{\epsilon}_{\frac{s}{\epsilon}}\right)\right) - Q^{K_i}(v(s))\right| ds \\
&\quad + \left|\int_0^t K_i(x^{\epsilon}_{\frac{r}{\epsilon}})dr - \int_0^t Q^{K_i}(\pi(x^{\epsilon}_{\frac{r}{\epsilon}}))dr\right| \\
&\le \quad C_i \int_0^t \left|\pi_i\left(x^{\epsilon}_{\frac{s}{\epsilon}}\right) - v_i(s)\right| ds + |\delta_i(\epsilon, t)|,
\end{aligned}
$$

Where

$$
\delta_i(\epsilon, t) = \int_0^t K_i(x^{\epsilon}_{\frac{r}{\epsilon}})dr - \int_0^t Q^{K_i}(\pi(x^{\epsilon}_{\frac{r}{\epsilon}}))dr
$$

Idea of the Proof

By Gronwall's Lemma we have that

$$\left|\pi_i\left(x^{\epsilon}_{\frac{t}{\epsilon}}\right) - v_i(t)\right| \leq e^{Ct}|\delta_i(\epsilon, t)|.$$

All we have to do is to study the rate of convergence of $|\delta_i(\epsilon, t)|$ to zero.

$\square$

This follows by a proposition.

Open problems:

1) Change the model of perturbation: stochastic transversal perturbation (work in progress with J. Gairing, Humboldt University).

2) Application: For a compact manifold L embedded in R^n. Consider a tubular neighbourhood, and a Brownian motion in L perturbed in the normal direction proportional to the curvature. Average transversal behaviour depends on the Euler characteristic of L.

Sobre dinâmica estocástica

Um termo vasto, muitas áreas e subáreas. Por exemplo:

1) Dinâmica discreta

(no tempo ou no espaço). Atirar moedas, dados, sequências de saídas de geradores aleatórios, medidas no clima, etc. Dinâmica de partículas, modelos de Ising, termodinâmica. Dadas probabilidades de saltos, qual a configuração estacionária (medida invariante), etc.

2) Equações aleatórias:

$$\dot{x}_t = f(\omega, t, x_t).$$

As soluções tem trajetórias diferenciáveis.
Uma única aleatoriedade/sorteio determina toda a trajetória!

Sobre dinâmica estocástica

3) Equações diferenciais estocásticas:

O ruído é introduzido permanentemente no sistema. Comparar com sistema de controle:

$$\dot{x}_t = A_0(x_t) + A_1(x_t)u_t^1$$

Equações estocásticas de Itô correspondem a controles aleatórios (movimentos brownianos)

Resumo do que falamos nesse mini-curso

Parte I:

Introdução - linguagem básica.
Teoria da medida, processos estocásticos e equações diferenciais estocásticas.
Referências: Muitas! Ver, por exemplo.
"Uma iniciação aos Sistemas Dinâmicos Estocásticos". 3a. ed., IMPA, 2010.
www.ime.unicamp.br/∼ruffino

Parte II:

Motivação de um problema específico em dinâmica estocástica.