Flattening the Sky with HEALPix

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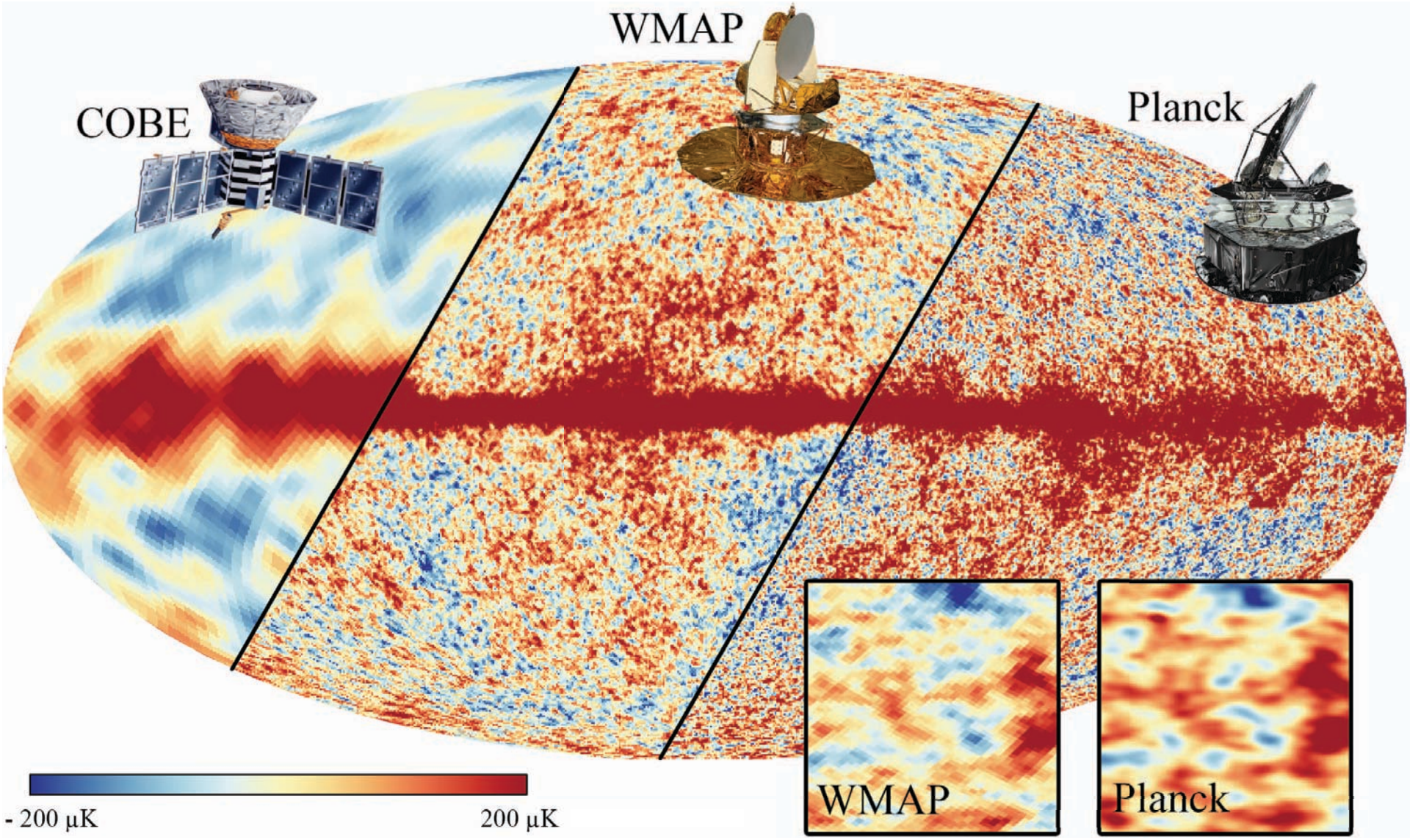
HEALPix

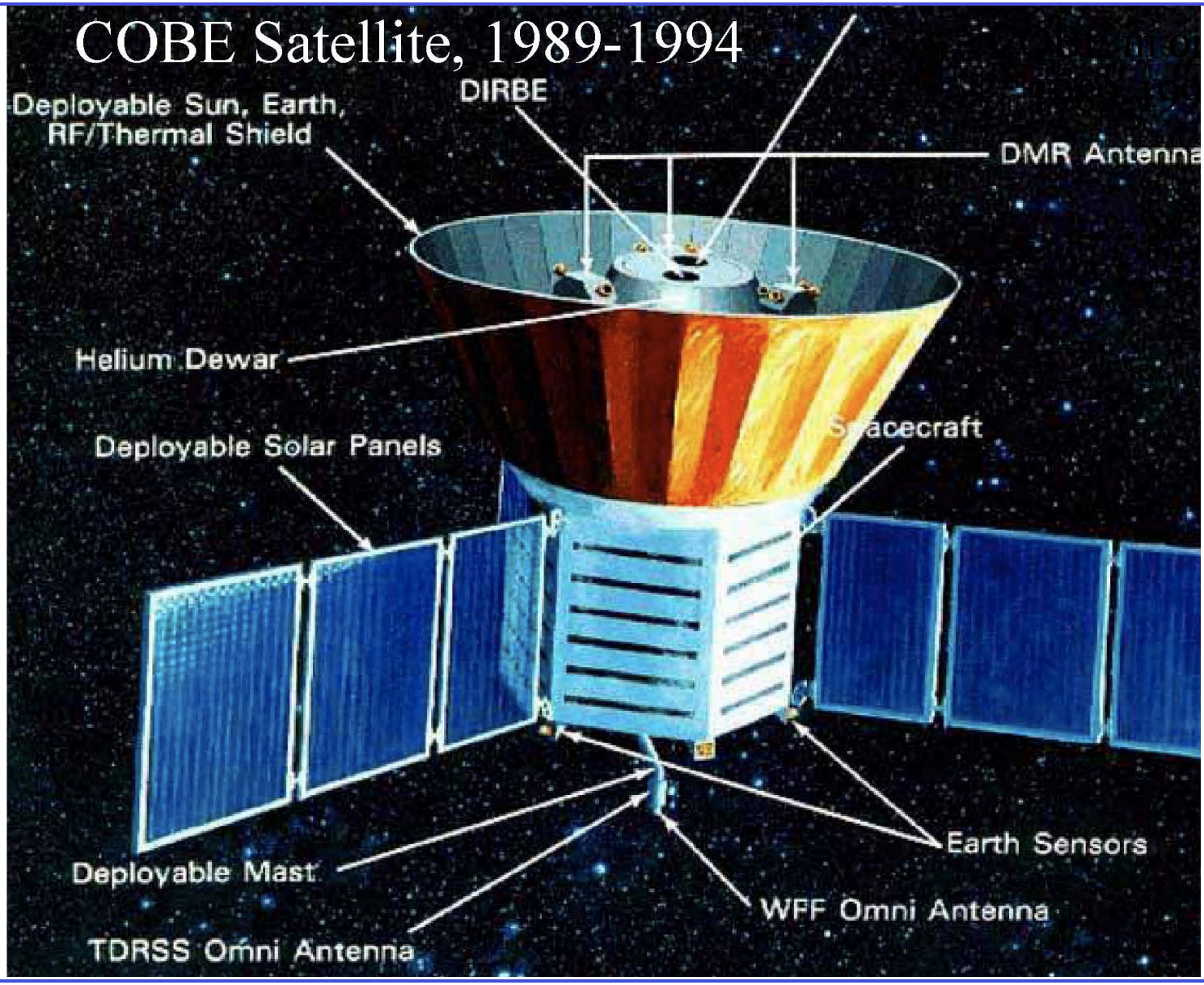
- Hierarchical Equal Area iso-Latitude Pixelization
- HEALPix was constructed to support:
  - Hierarchical structure of the data base for data objects distributed on the sphere
  - Exactly equal area of pixels - resolution elements of the tessellation
  - Iso-latitude distribution of pixel centers, which constitute the point-set for numerical quadrature on the sphere to enable fast spherical harmonic transforms [$P_{lm}(\theta)$ only need to be evaluated once per latitude, hence overall complexity of harmonic transform evaluations is ~$N_{pix}^{3/2}$, where N_{pix} is the total number of pixels on the sky]
- HEALPix was invented (~16 years ago!) by K.M. Górski (then in Copenhagen), and developed and supported by KMG (now at JPL/Caltech), Eric Hivon (IAP, France), A.J. Banday (MPA Garching, Germany), B.D. Wandelt (UIUC, IL), M. Reinecke (MPA Garching, Germany)
- HEALPix software is a suite of Fortran90, IDL, C, C++, Python, and Java programs for data digitization, manipulation, and mathematical analysis, including fast spherical harmonic synthesis/analysis transforms, with documentation and user manuals
- HEALPix is the data structure for WMAP, and Planck; it is an adopted data standard in FITS, etc.
- Distributed through the web-site http://healpix.sourceforge.net (http://healpix.jpl.nasa.gov)
- Contact: krzysztof.m.gorski@jpl.nasa.gov & healpix-support@lists.sourceforge.net

Almost 50 years since the discovery of the CMB,
after scores of suborbital CMB experiments,
~~and two decades of 3 dedicated satellite missions –~~
we reached a REMARKABLE moment, but why are we doing this?

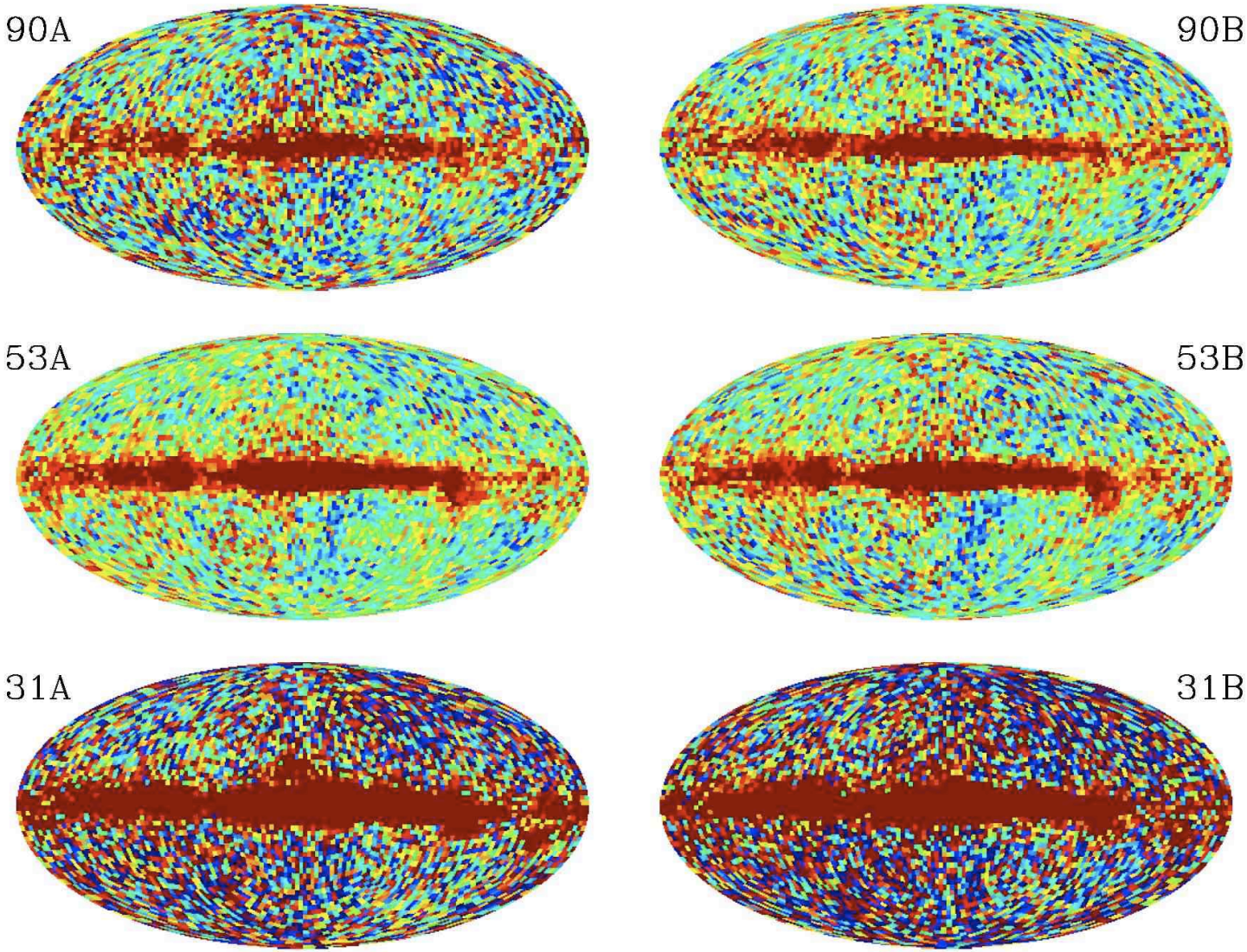


COBE

COsmic Background Explorer

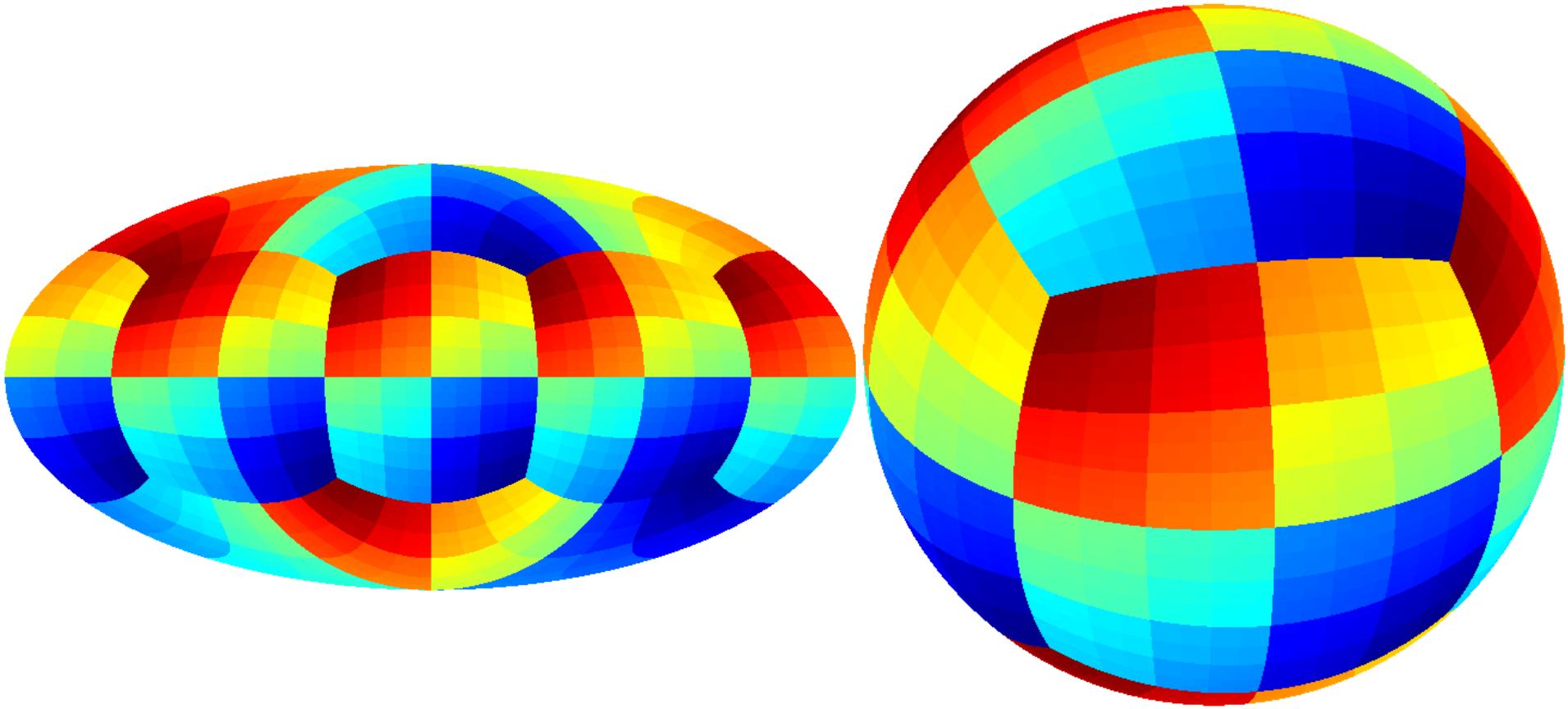


COBE-DMR Sky Maps

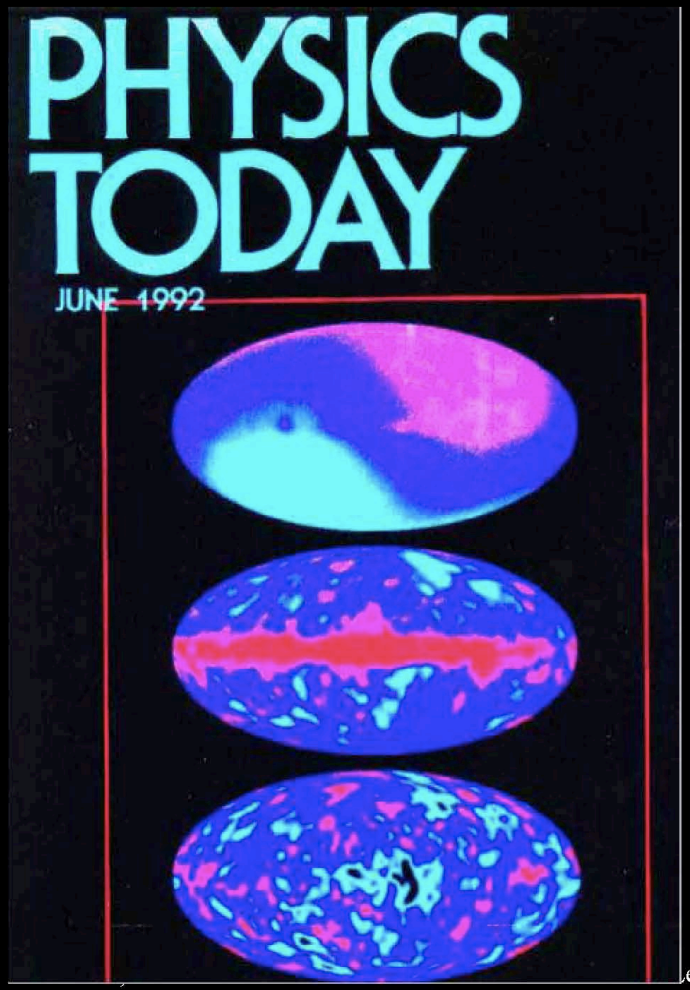


Data Structure of the COBE Sky Maps

Quadrilateralized Spherical Cube …



COBE-DMR Delivers …



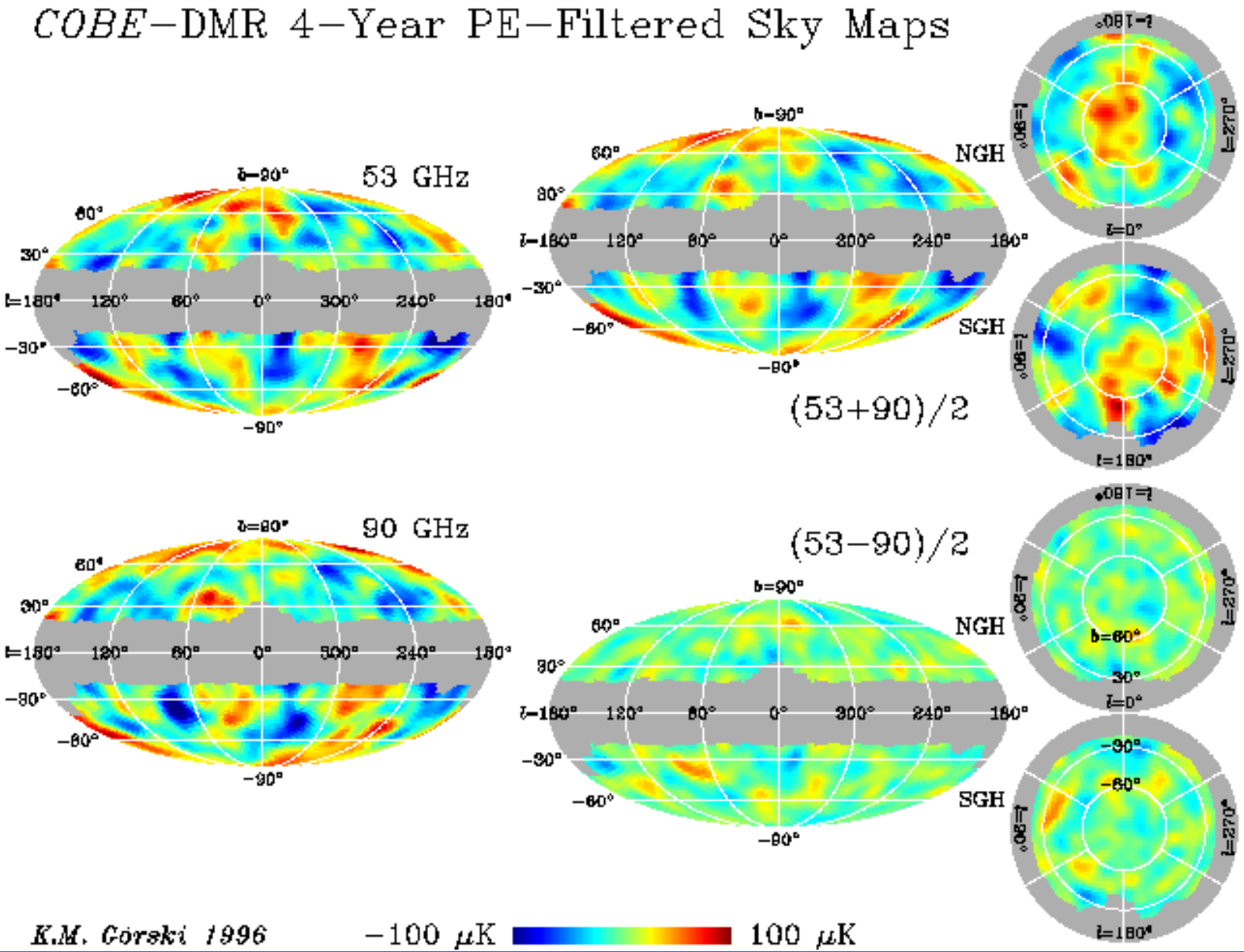
Sky map from DMR, 2.7 K +/- 0.003 K

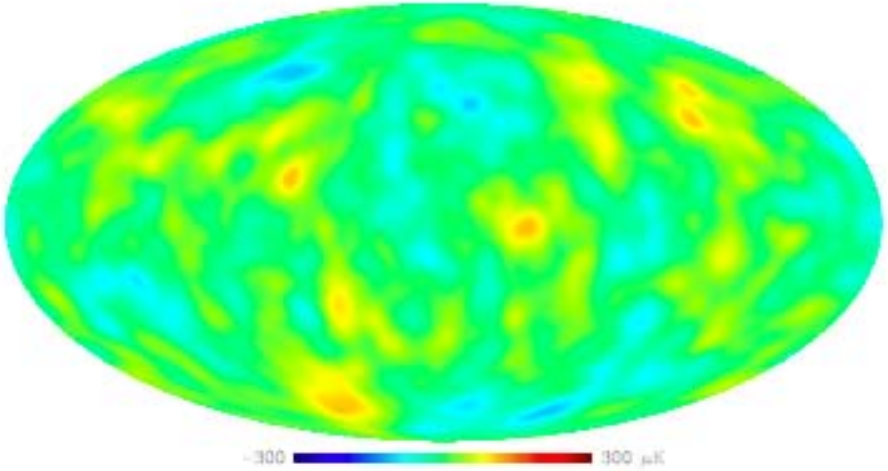
Doppler Effect of Earth's motion removed (v/c = 0.001)

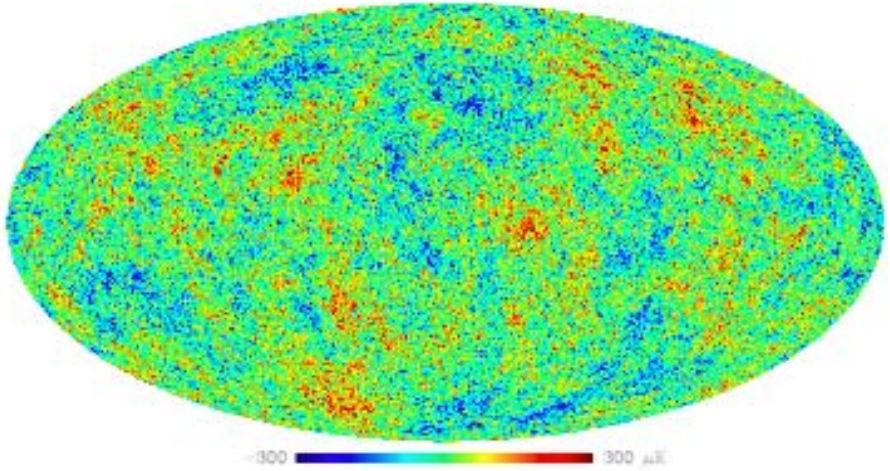
Cosmic temperature/density variations at 389,000 years, +/- 0.00003 K

ecture 2006

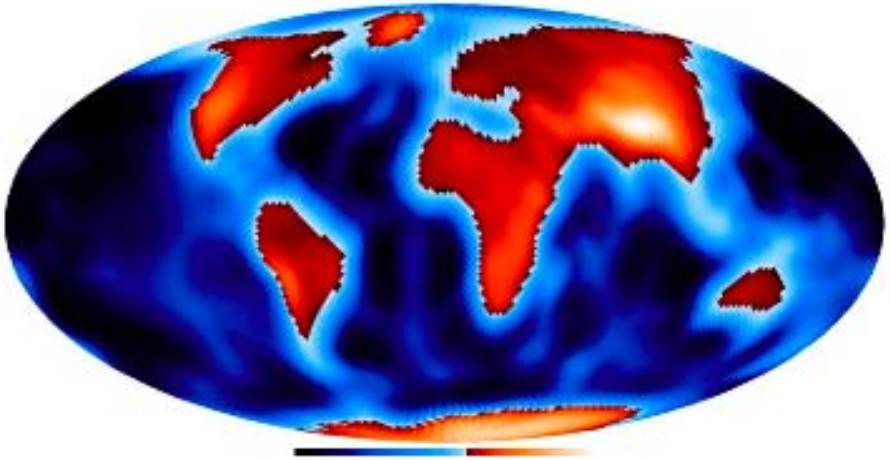
Last scattering surface according to COBE-DMR

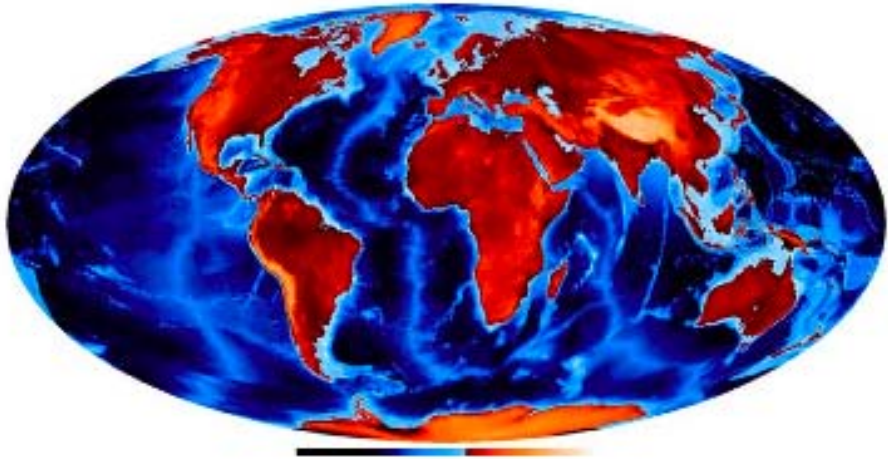




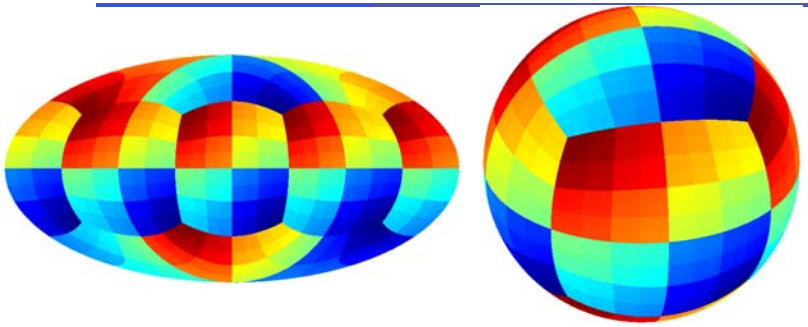


After COBE – Quickly increased apetite, we wanted more, much more …

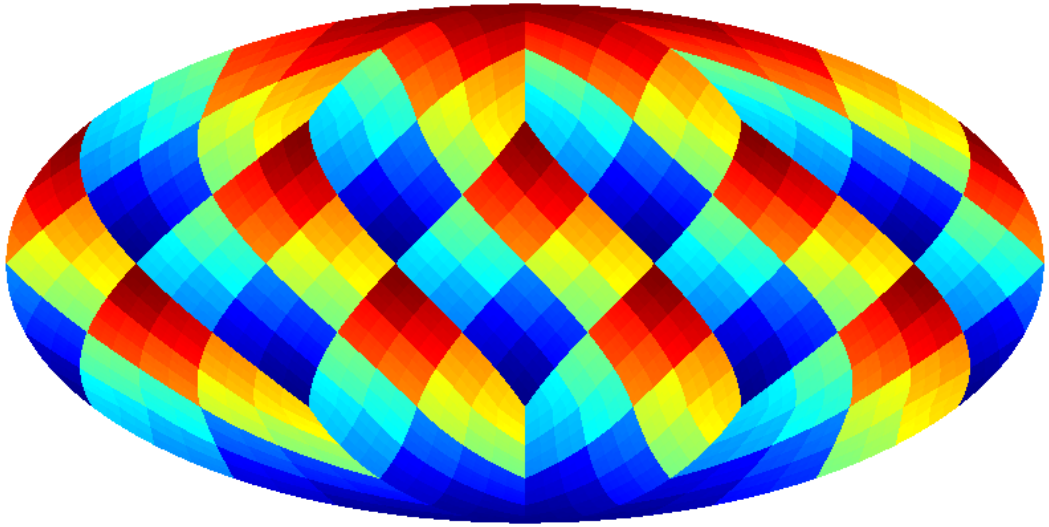


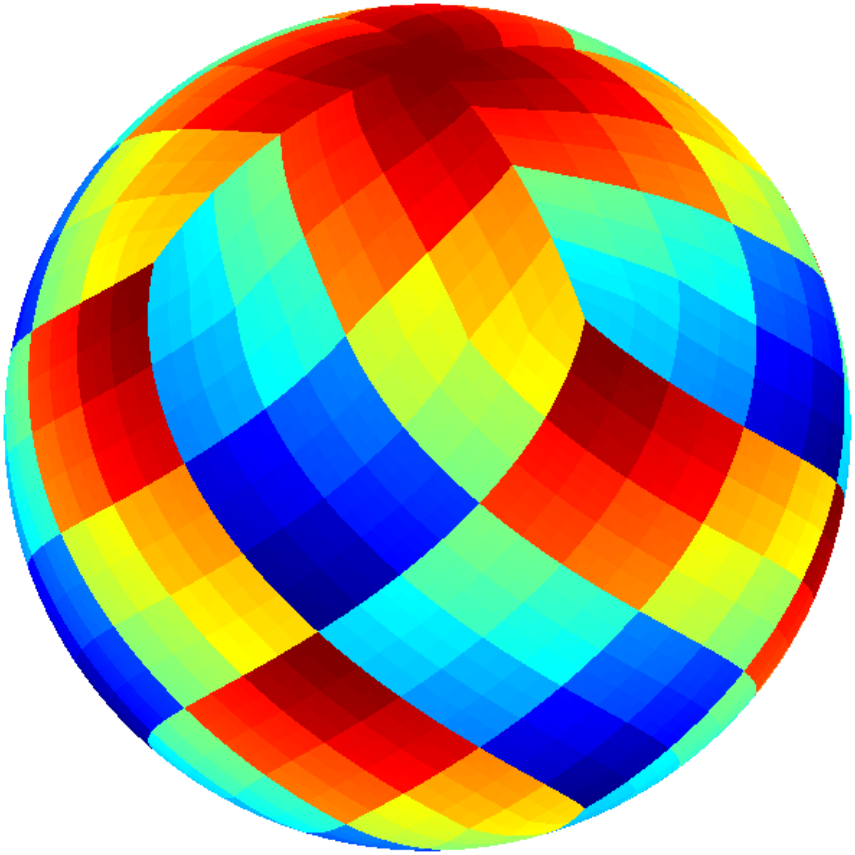


Data Structure of the HEALPix Sky Maps



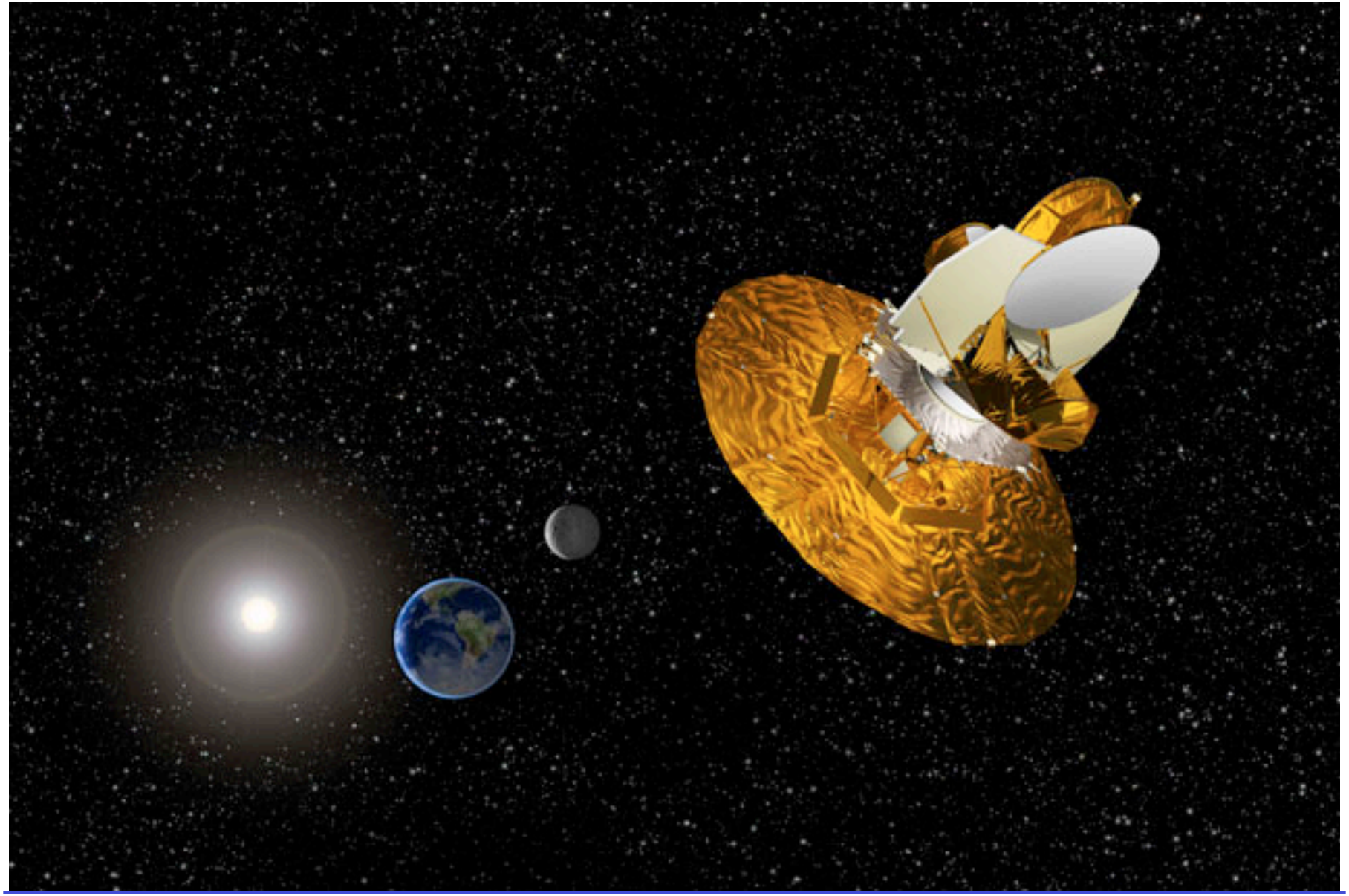
Novelty was required to make it possible to swallow "much more" …

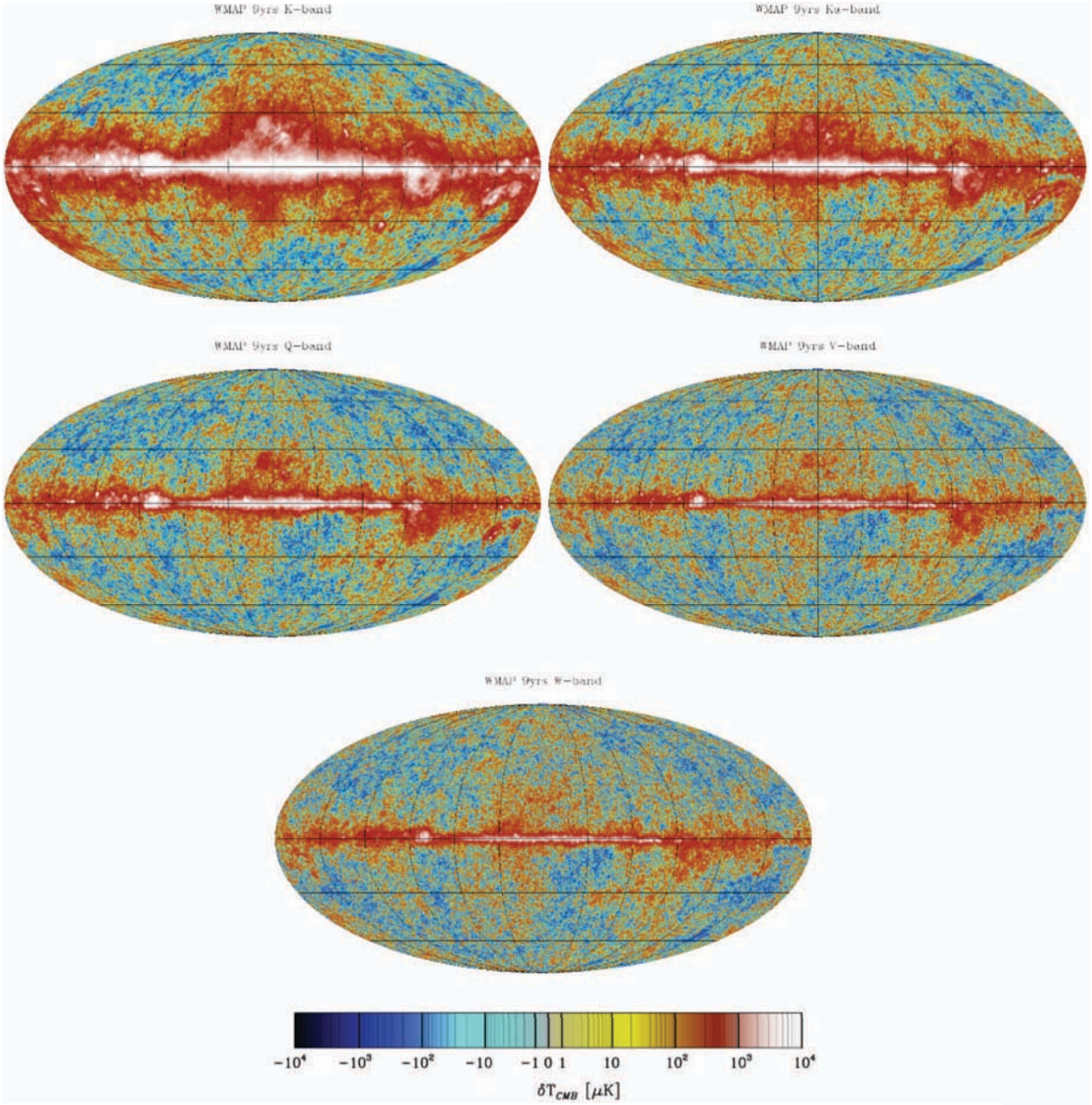




WMAP

WMAP in orbit at L-2





Pre-Planck microwave sky

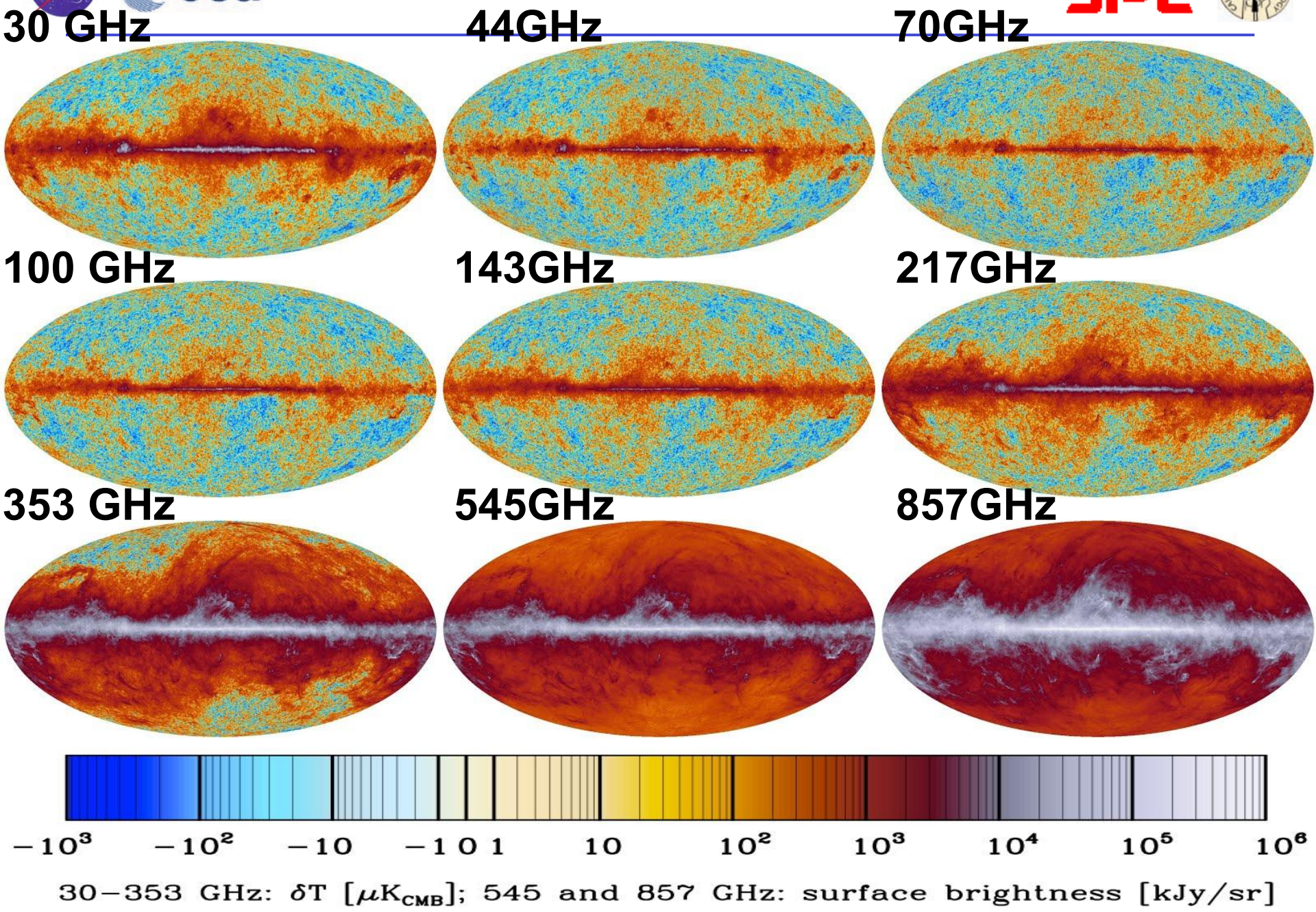
the great WMAP mission and its sky maps at 22-94 GHz

Planck

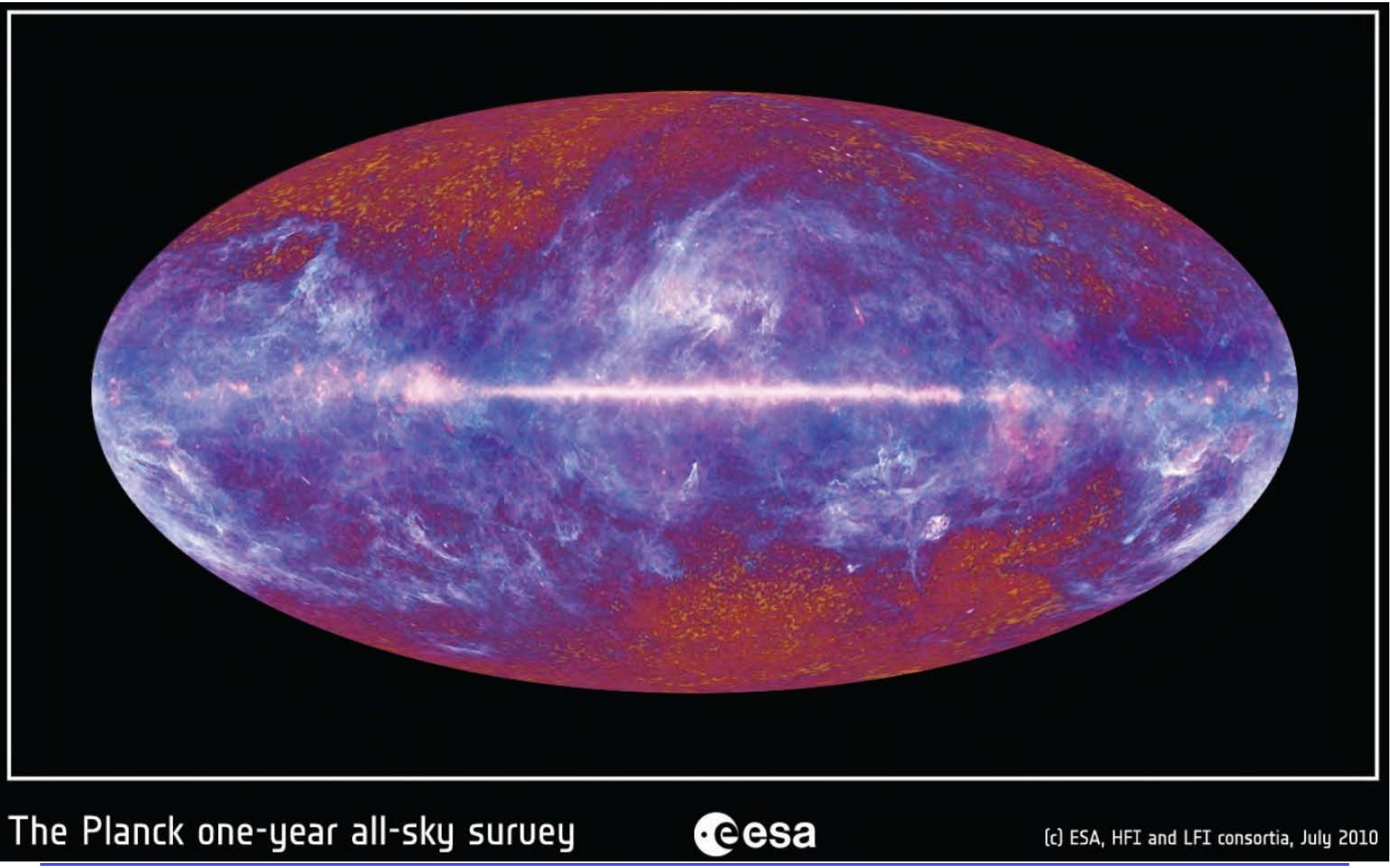
Planck - Ferrari of satellite CMB missions?

Planck
ready to go
in French Guyana

Planck Maps of the microwave and sub-mm Sky



Planck's Universe

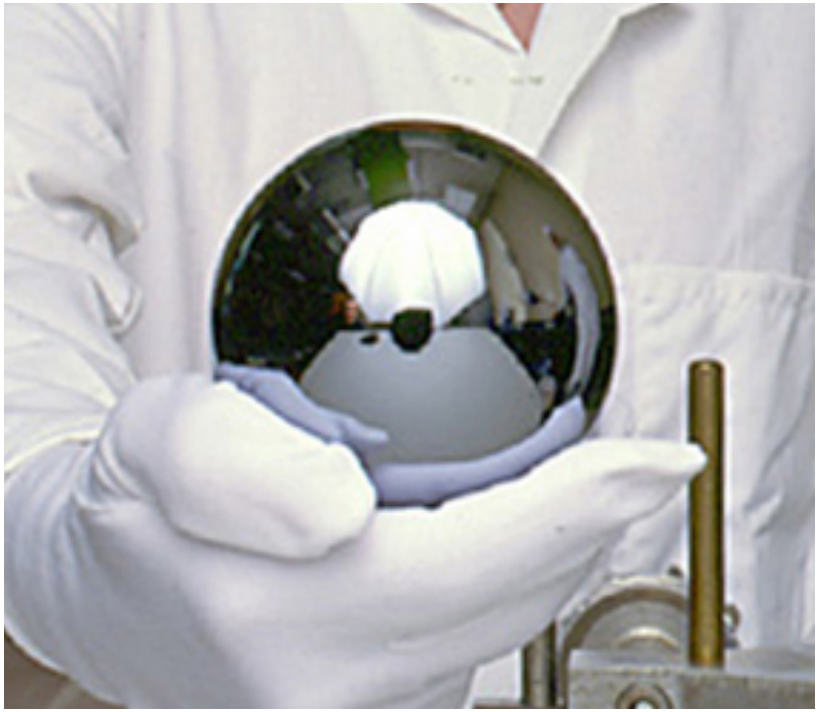


HEALPix

Some context

Before pushing ahead let's take a look at the #2 and #1 roundest man-madeob jects:

The Avogadro project's
1 kg of Si^{28} 93.6 mm ball



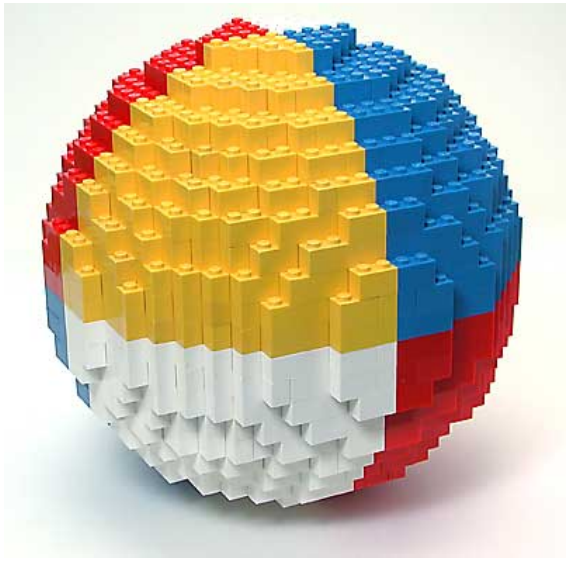
$\Delta r/r \sim 3.7 \times 10^{-7}$, $\Delta r \sim 35$ nm
(~2.8 m "hills" on the surface of the Earth)

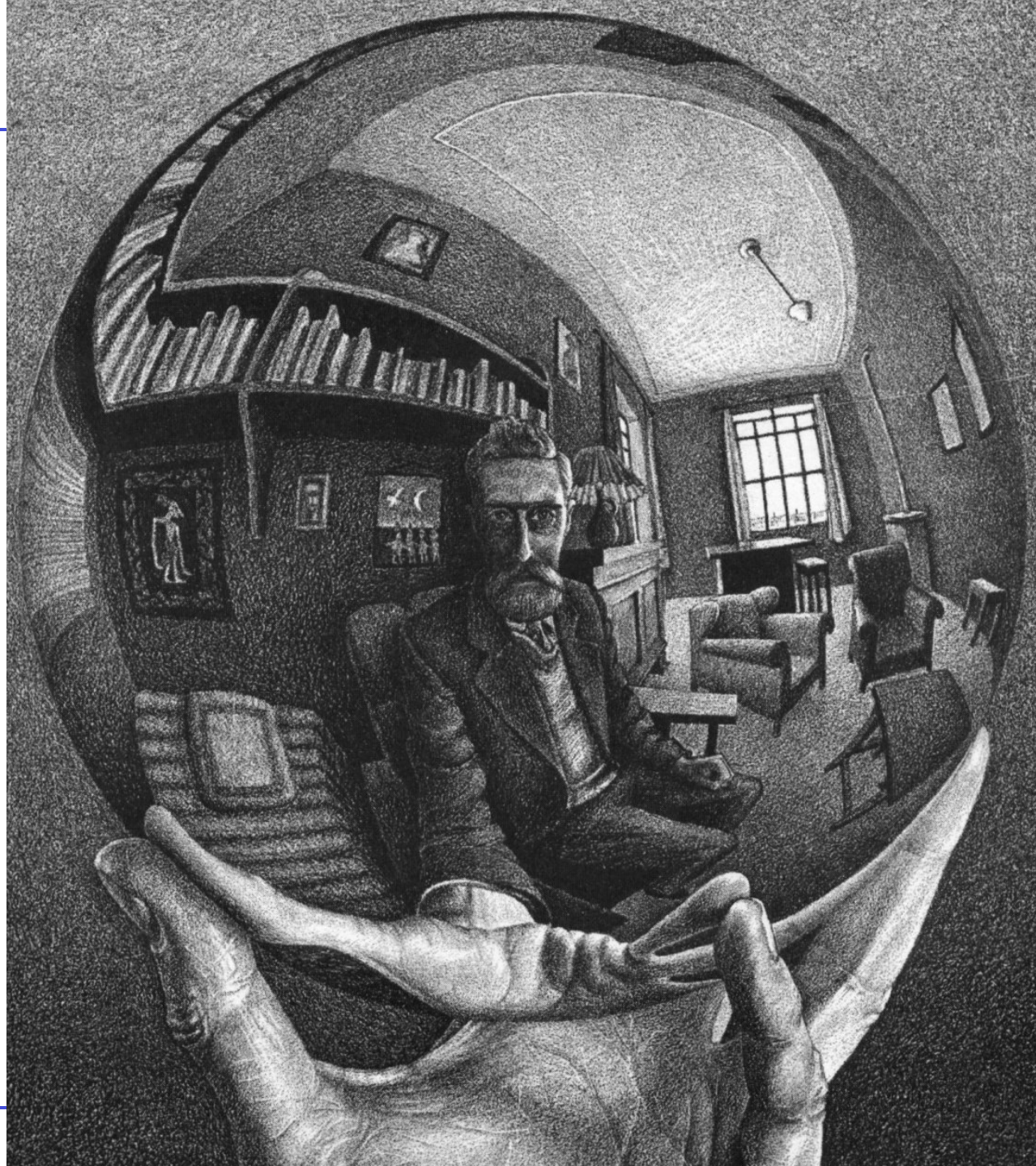
Gravity Probe B gyroscope
38.1 mm fused-quartz ball

$\Delta r/r \sim 1.8 \times 10^{-7}$, $\Delta r \sim 3.4$ nm
(~1.4 m "hills" on the surface of the Earth)

Studying Sphericity one can be

sophisticated,
like Escher's
(well, at least one can try),
or
simplified, as with LEGO,



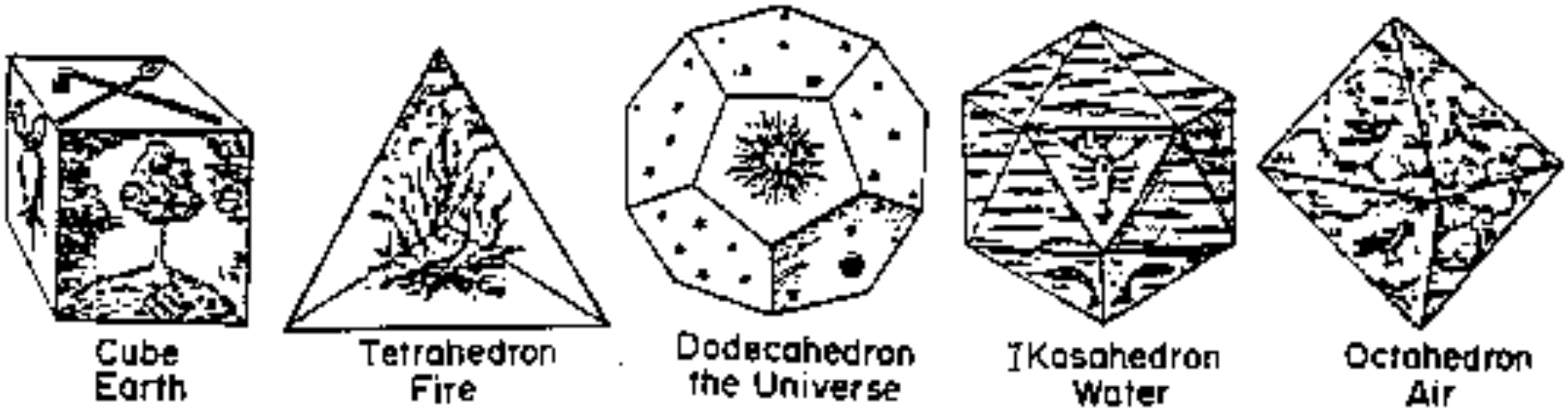


BUT IT IS ALWAYS FUN

When and where? 1997 in Copenhagen

5 Platonic Solids

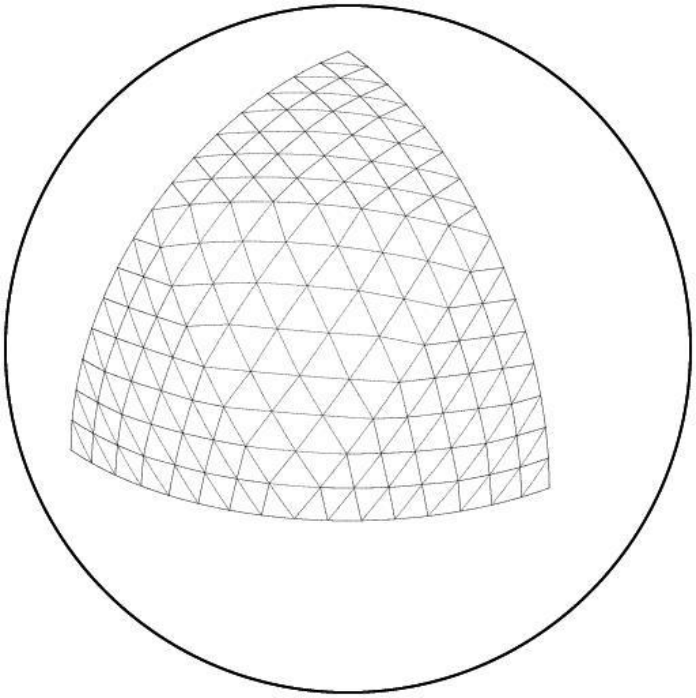
- Drawing from Kepler's Mysterium Cosmographicum

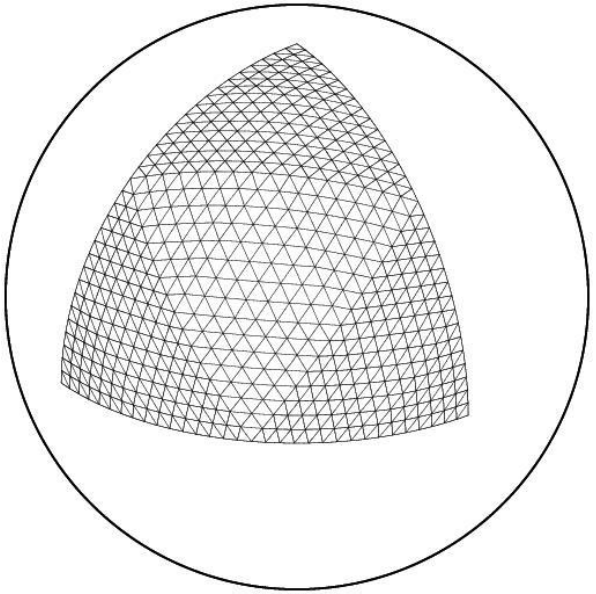


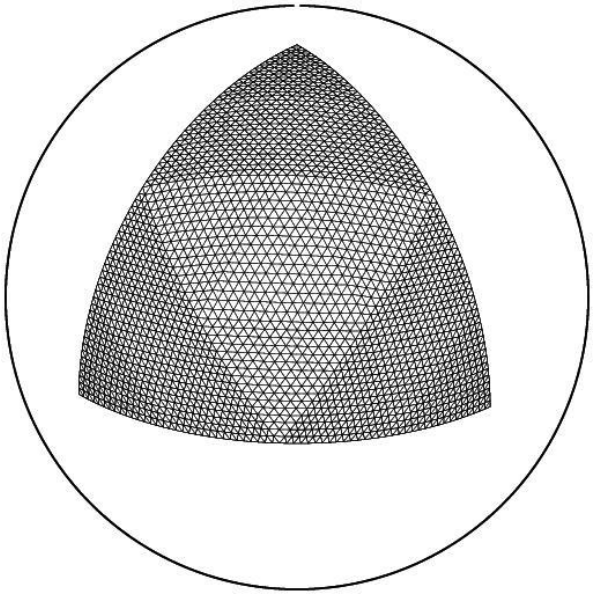
Icosahedral Dome, Epcot Center, Disneyworld, Florida

HTM Hierarchical Triangular Mesh

Used extensively in SDSS Archive







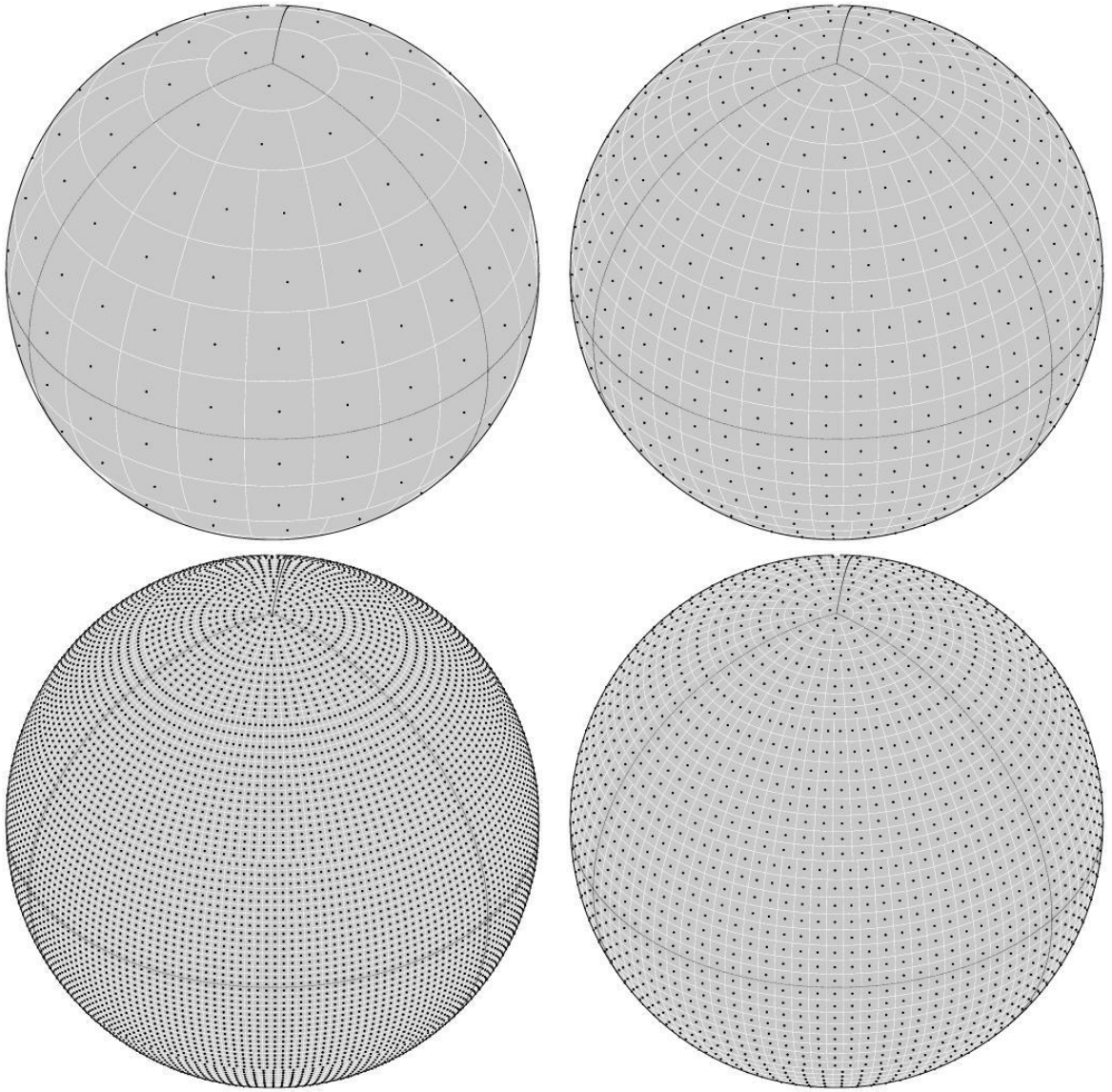
Discoball

- Used extensively in discos

IGLOO

Crittenden and Turok

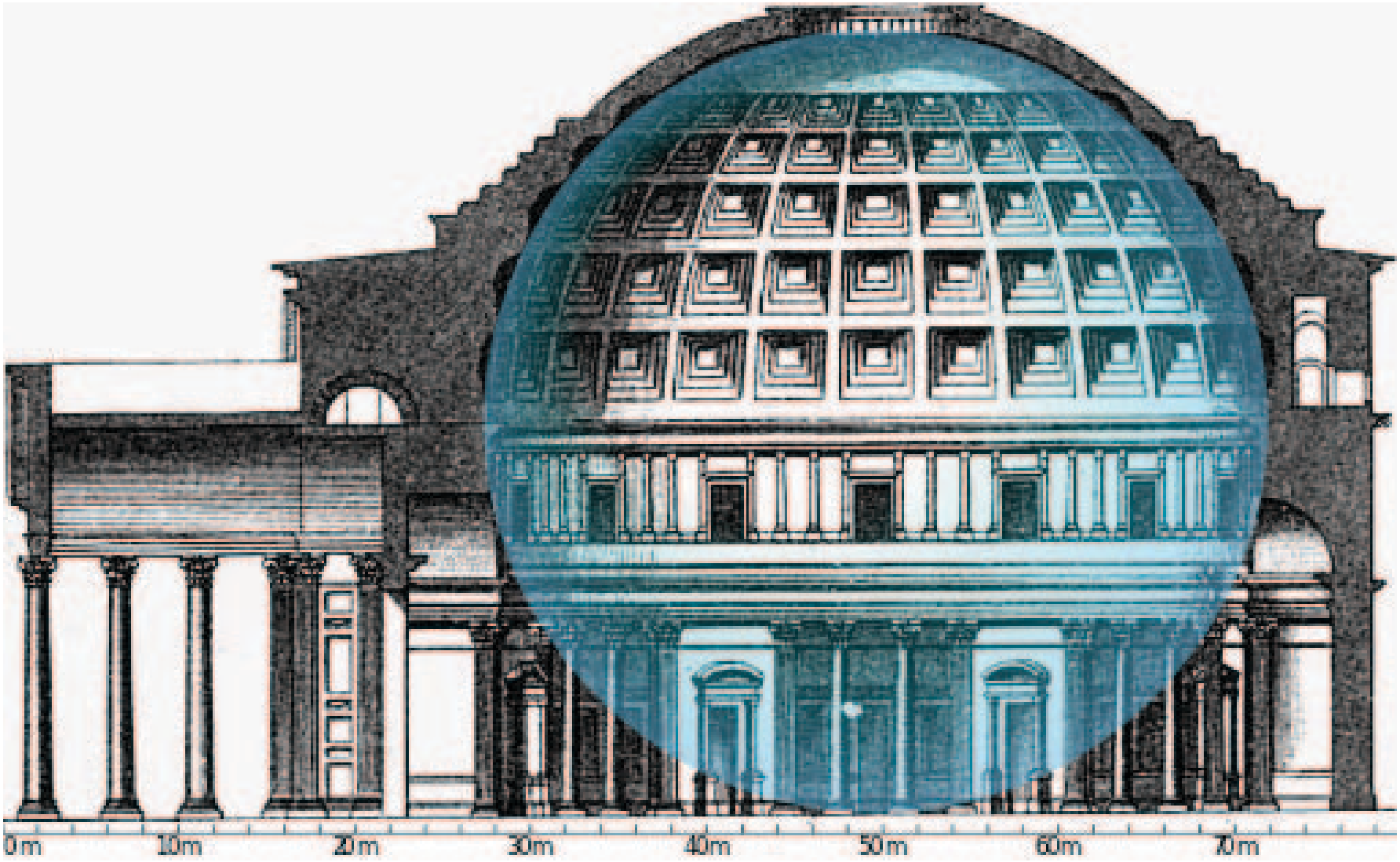
Used extensively in the Arctic …



Pantheon, Rome

- Built in 124 AD!
- 43m dome
- The largest in the world until the XXth century!!!

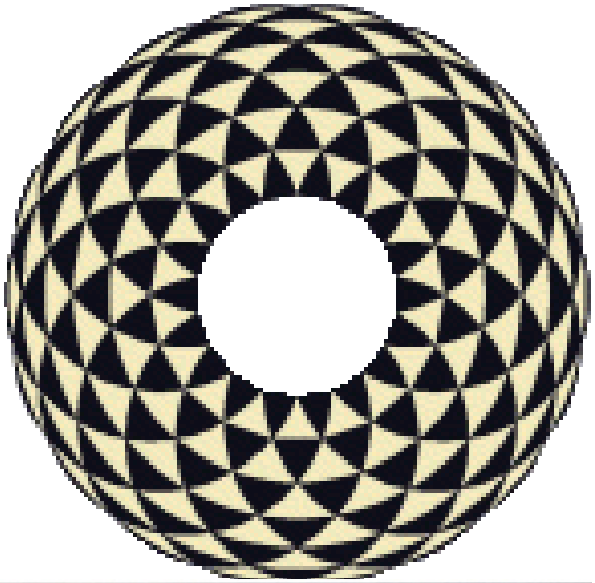
Pantheon – Spherical Magic



Temple of Venus and Roma

- On Forum Romanum, across the road from the Coloseum

Roman Floor Mosaics



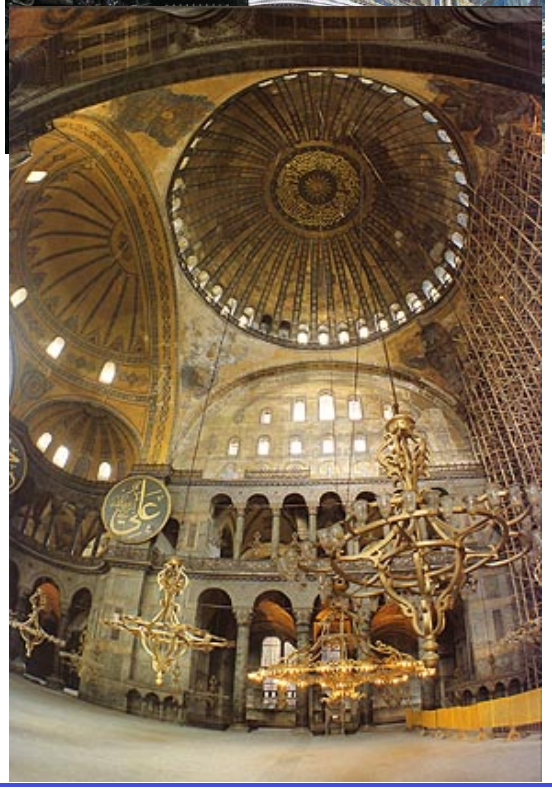
Chapelle d'Anet, P. De l'Orme, XVI c.

San Pietro Rome 42m 1593 AD

Santa Maria del Fiore Florence 42m 1420 AD

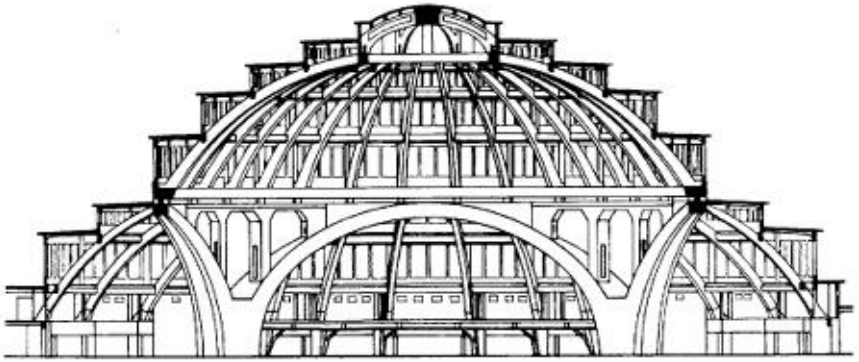
The Next Four

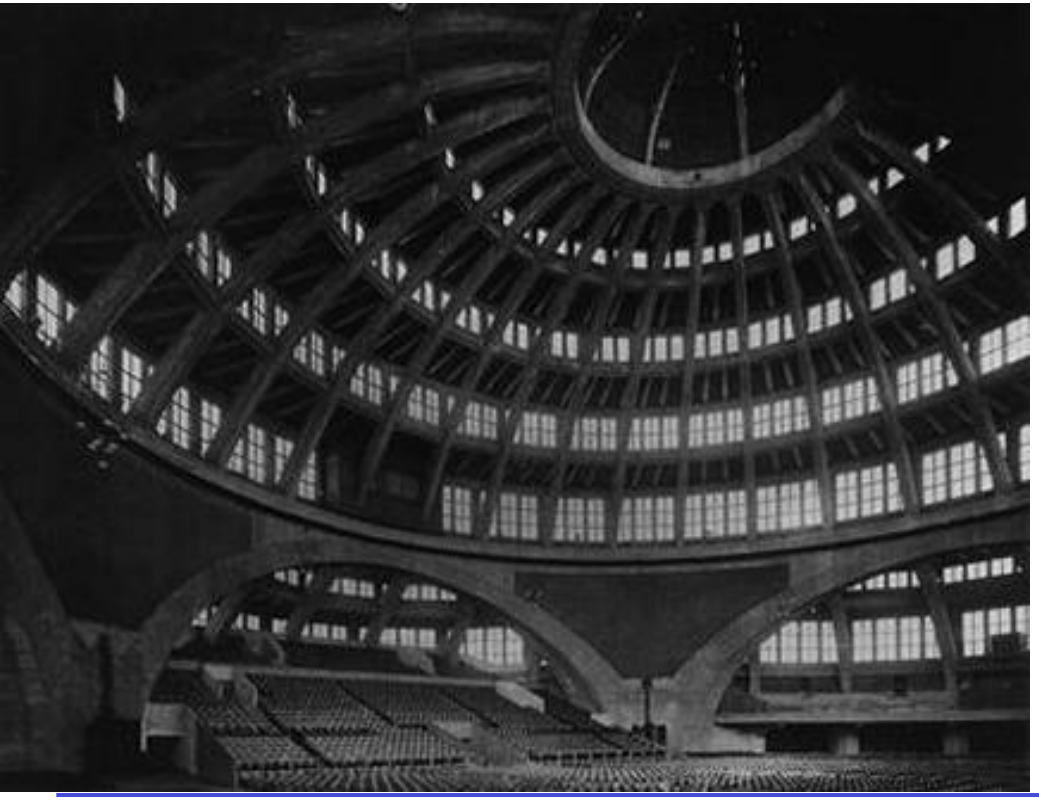
Hagia Sophia Istanbul 32m 537 AD

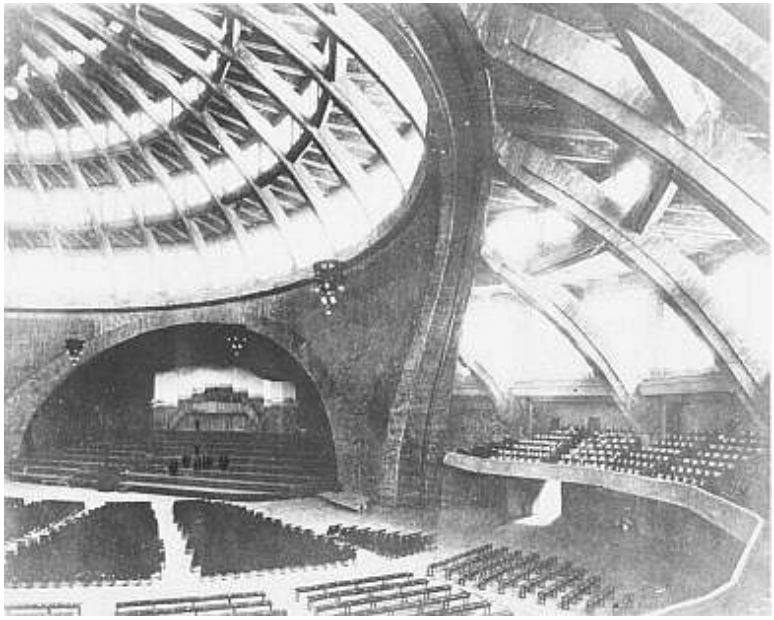


St. Paul's London 33m 1710 AD

Jahrhunderthalle by Max Berg, 1913. Breslau; 69m

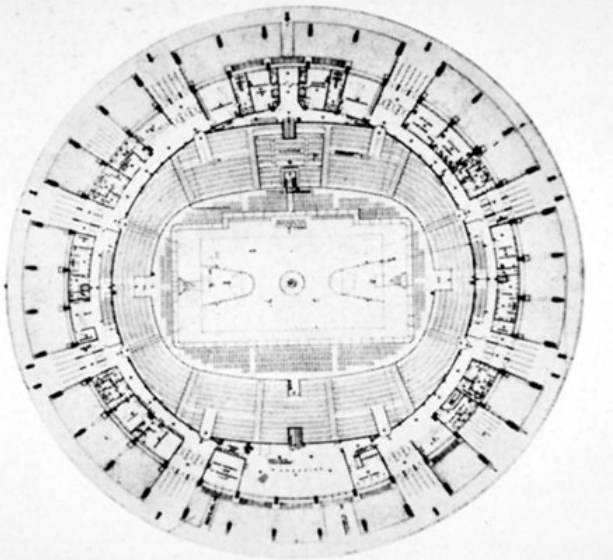






Italian God of Concrete – Pier Luigi Nervi

Pallazetto dello Sport
Rome, 1957
61 m
('assembled' in 40 days!)



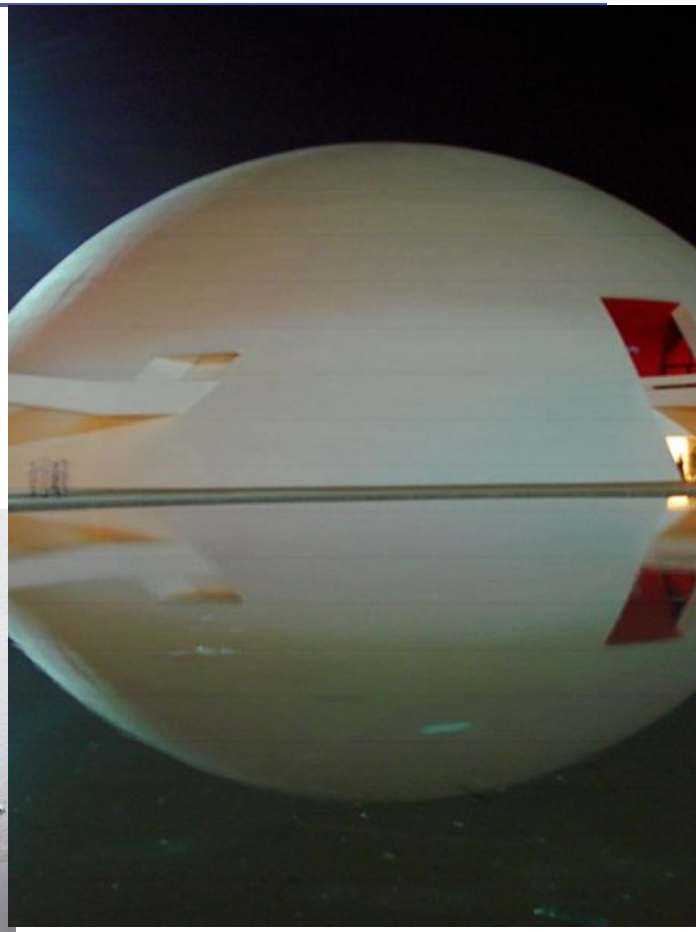
Nervi, PalaLottomatica, Rome 1960, 100m

Nervi, Norfolk Scope, Virginia, 1971, 136m

Norfolk Scope

Oscar Niemeyer, Brasilia, 2006

Museu Nacional da República

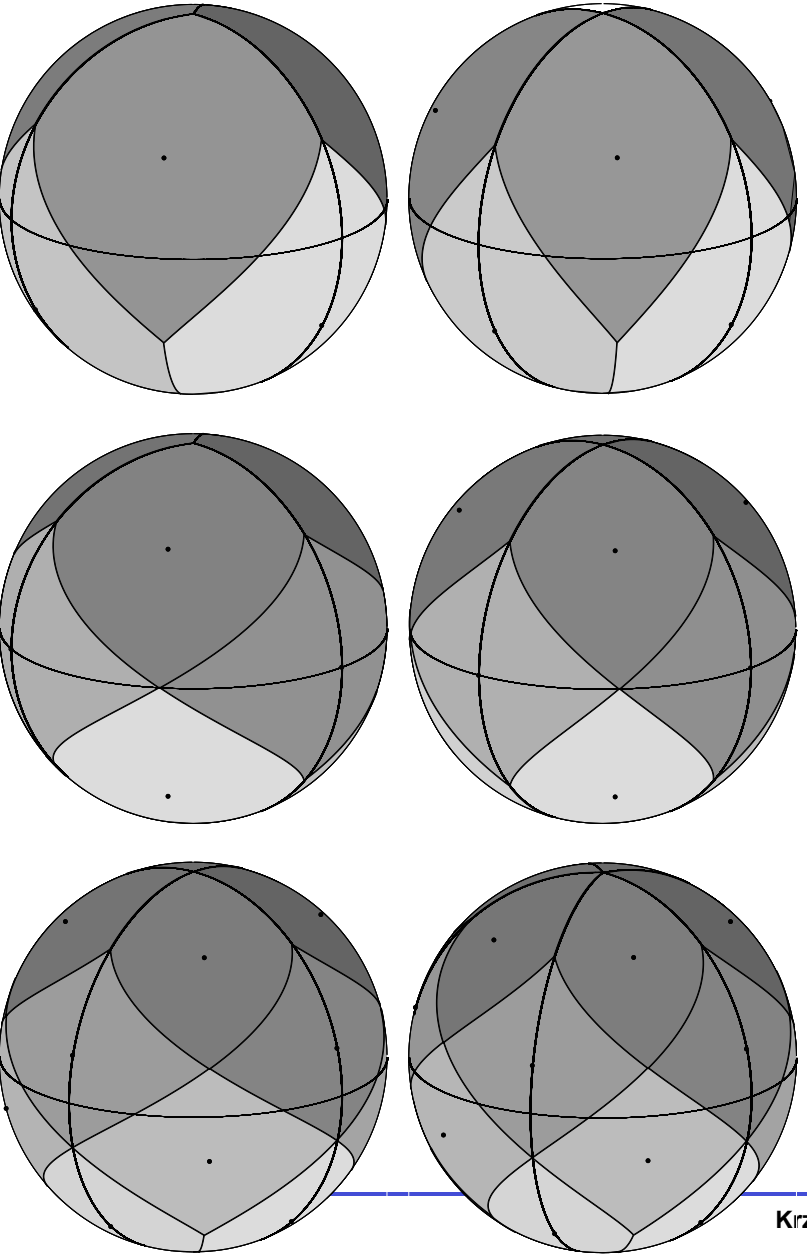


HEALPix

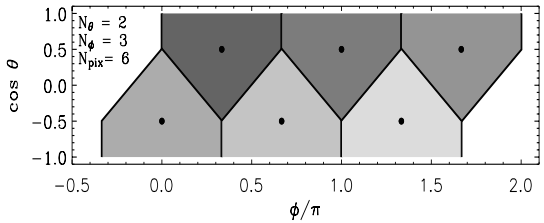
Desiderata

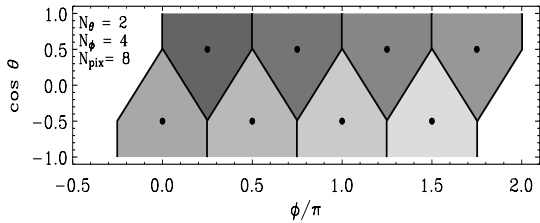
- **What do we want from an acceptable scheme for discretization of the sphere?**
  - **Isolatitude pointset (to support effective harmonic analysis), "a la Pantheon", but …**
  - **Hierarchical subdivisibility (to support efficient data handling – e.g. region definitions, data selection, map resolution changes, etc.), inherited from QuadCube, and**
  - **Uniformity of pixel size (to preserve photometric integrity of the maps), like QuadCube (or indeed better)**

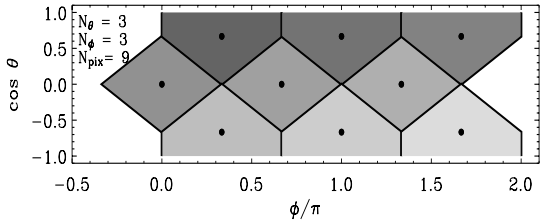
Alternative possibilities for base-level tessellation

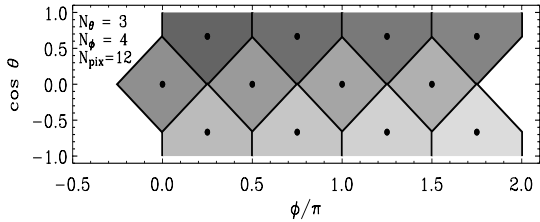


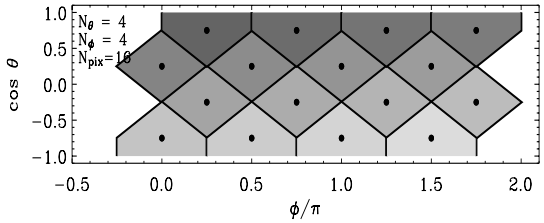
Figuring this out was essential for the eventual emergence of HEALPix

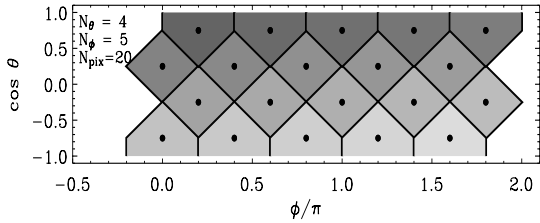


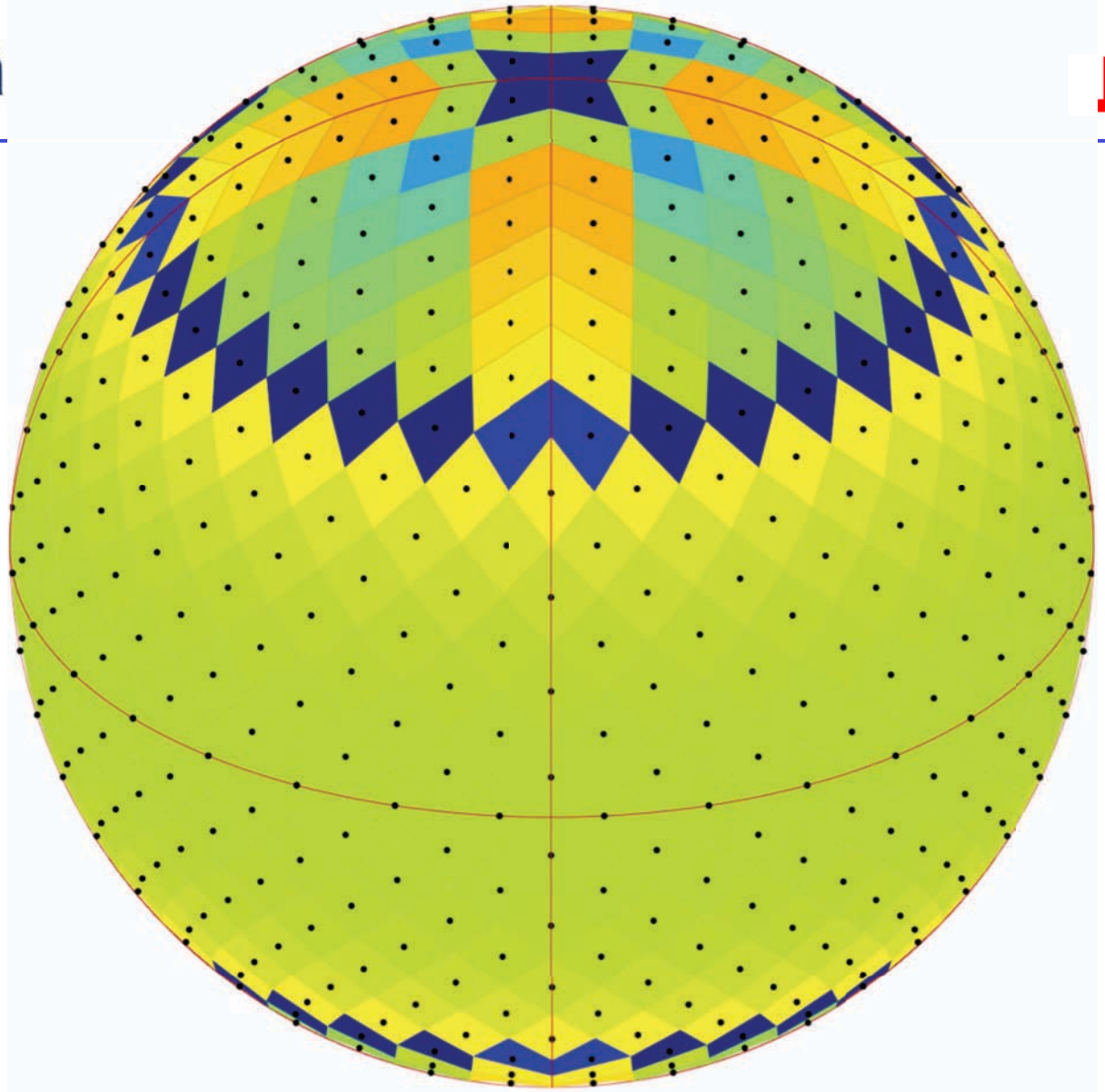












Early stage plot, before we figured out the analytical description of pixel boundaries

HEALPix

- **H**ierarchical
- **E**qual **A**rea
- Iso-**L**atitude
- **Pix**elization

http://healpix.jpl.nasa.gov

Freely available, GNU-licensed software library (F90, C++, C, IDL) for discretization, synthesis, analysis, etc. of functions on the sphere;

Developed and supported since 1997;

Used by WMAP and Planck; also Boomerang, Archeops, and other suborbital experiments;

Gorski, Hivon, Banday, Reinecke, Wandelt, Hansen, …

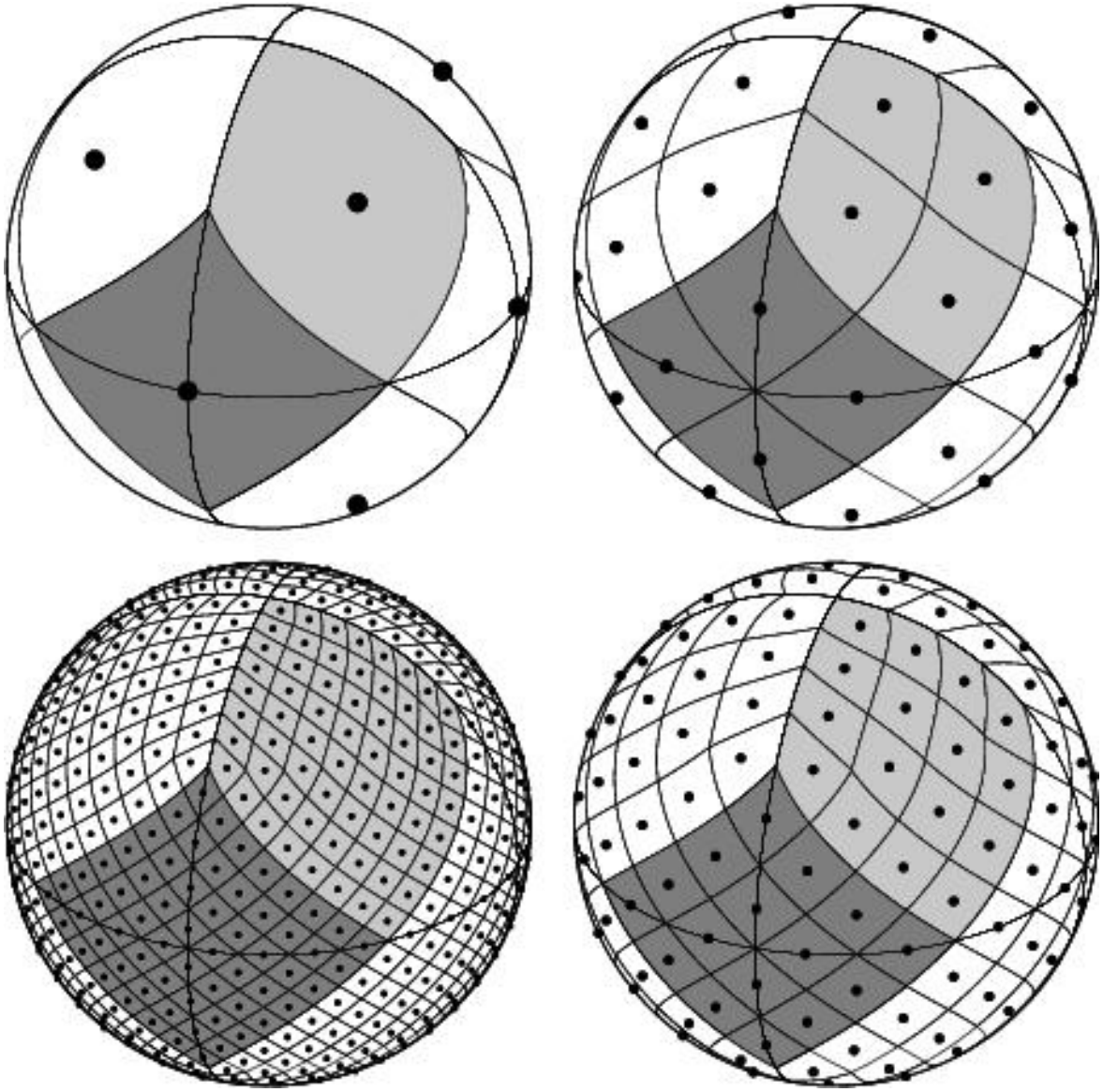


TABLE 1

SUMMARY OF NUMBER OF PIXELS AND PIXEL SIZES ACCESSIBLE TO HEALPIX

k	$N_{\rm side} = 2^k$	$N_{\rm pix} = 12N_{\rm side}^2$	$\theta_{\rm pix} = \Omega_{\rm pix}^{1/2}$
0	1	12	58.°6
1	2	48	29.°3
2	4	192	14.°7
3	8	768	7.°33
4	16	3072	3.°66
5	32	12,288	1.°83
6	64	49,152	55.′0
7	128	196,608	27.′5
8	256	786,432	13.′7
9	512	3,145,728	6.′87
10	1024	12,582,912	3.′44
11	2048	50,331,648	1.′72
12	4096	201,326,592	51.″5
13	8192	805,306,368	25.″8
14	2^{14}	3.22×10^9	12.″9
15	2^{15}	1.29×10^{10}	6.″44
16	2^{16}	5.15×10^{10}	3.″22
17	2^{17}	2.06×10^{11}	1.″61
⋮	⋮	⋮	⋮
29	2^{29}	3.46×10^{18}	$3.″93 \times 10^{-4}$

NOTES.—Currently, the use of 32 bit signed integers for the pixel indexing restricts the resolution accessible to $N_{\rm side} \leq 8192$. The use of 64 bit signed integers will allow a value $N_{\rm side} = 2^{29}$ to be achieved. Note that k corresponds to the "resolution parameter" referred to by the *WMAP* team.

What range of angular scales can be handled by the HEALPix software?

1. Pixel Position

For a resolution parameter N_{side} $4N_{side}$-1 isolatitude rings. Pixel centers are located on (z=cosθ, ϕ)

North Polar Spur, ring index, $i < N_{side}$ and pixel-in-ring index, $1 \le j \le 4i$

$$\phi = \frac{\pi}{2i}\left(j - \frac{s}{2}\right), \quad s = 1$$

$$z = 1 - \frac{i^2}{3N_{side}^2}$$

$$j = p + 1 - 2i(i-1), \quad i = I\left(\sqrt{p_h - \sqrt{I(p_h)}}\right) + 1, \quad p_h = (p+1)/2$$

North Equatorial Belt, ring index, $2N_{side} \ge i \ge N_{side}$ and pixel-in-ring index, $1 \le j \le N_{side}$

$$\phi = \frac{\pi}{2N_{side}}\left(j - \frac{s}{2}\right), \quad s = (i - N_{side} + 1) \bmod (2)$$

$$z = \frac{4}{3} - \frac{2i}{3N_{side}}$$

$$j = p' \bmod (4N_{side}) + 1, \quad i = I\left(p'/4N_{side}\right) + N_{side}, \quad p' = p - 2N_{side}(N_{side} - 1)$$

Pixel centers in southern hemisphere are obtained by the mirror-symmetry of the grid with respect to the equator.

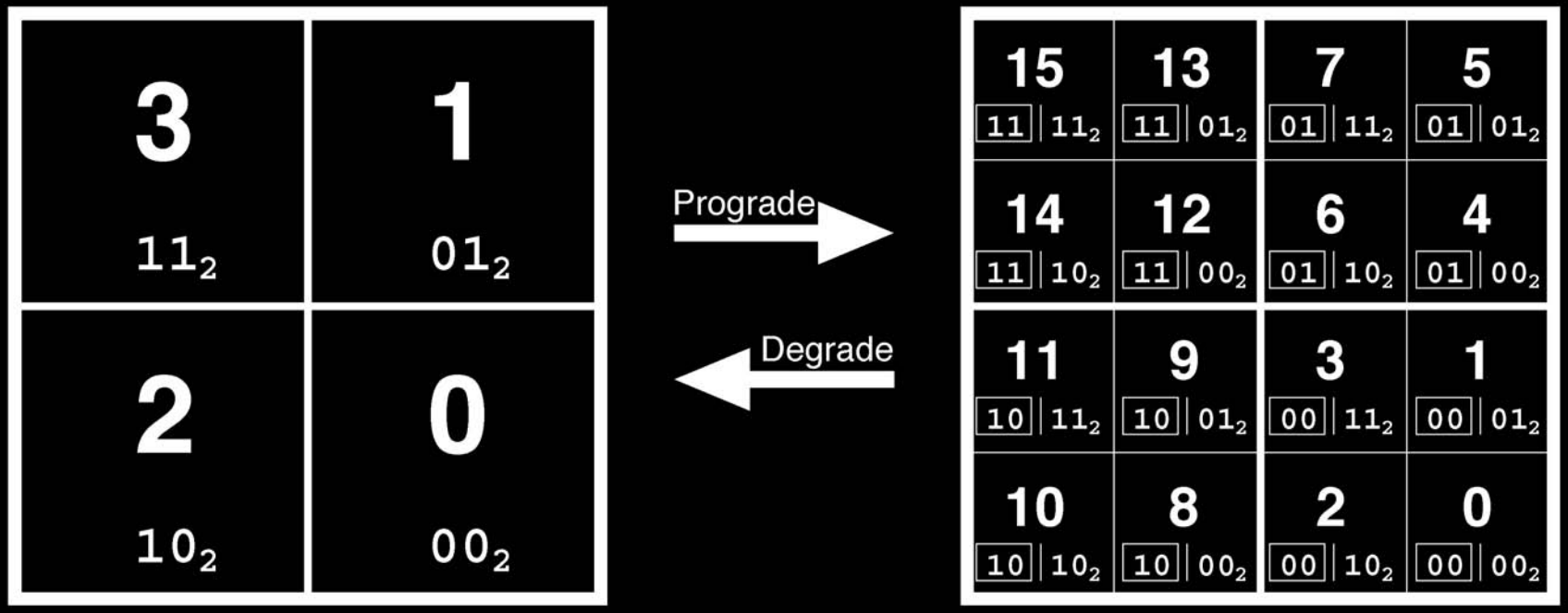
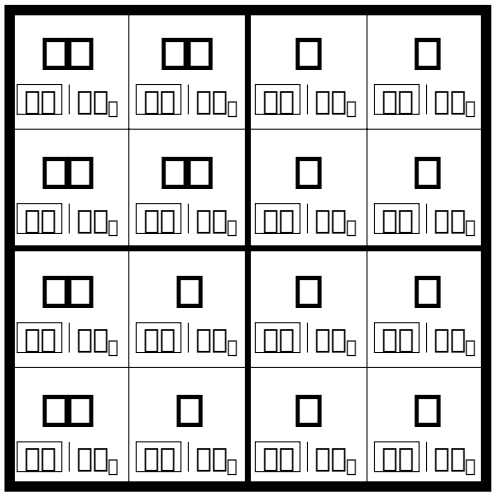


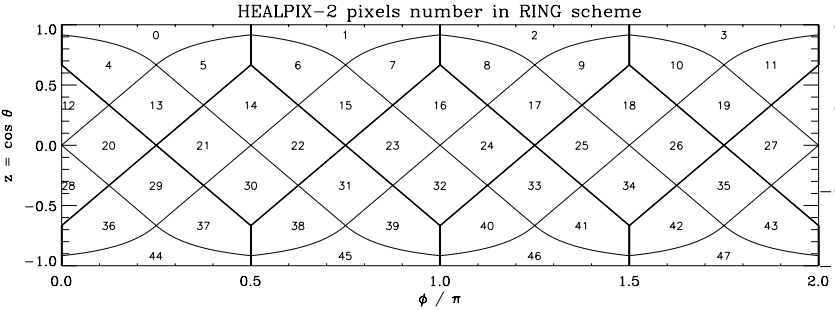
Figure 1: Quadrilateral tree pixel numbering. The coarsely pixelised coordinate patch on the left consists of four pixels. Two bits suffice to label the pixels. To increase the resolution, every pixel splits into 4 daughter pixels shown on the right. These daughters inherit the pixel index of their parent (boxed) and acquire two new bits to give the new pixel index. Several such curvilinearly mapped coordinate patches (12 in the case of **HEALPix**, and 6 in the case of the *COBE* quad-sphere) are joined at the boundaries to cover the sphere. All pixels indices carry a prefix (here omitted for clarity) which identifies which base-resolution pixel they belong to.

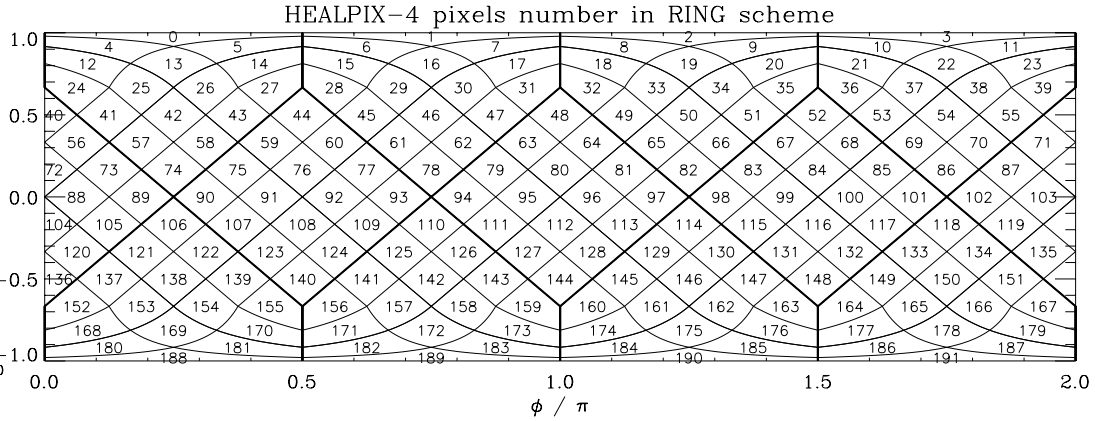
Z-order, or Morton order

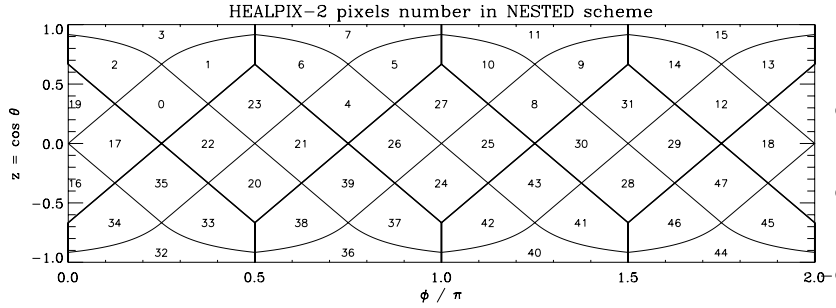


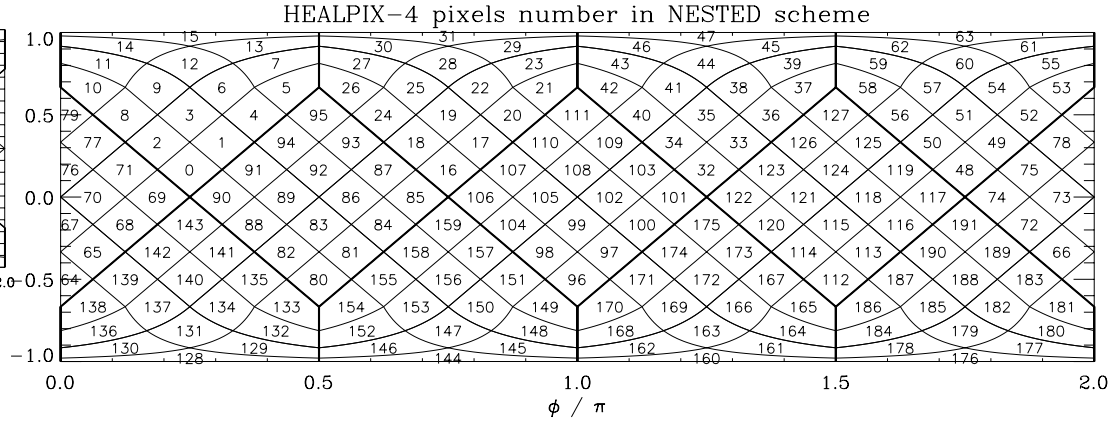
Z-order, **Morton order**, or **Morton code is a function which maps multidimensional data to one dimension while preserving locality of the data points.**

Two indexing schemes – comparative merits









Again, 1-d data representation of the 2-d configuration space that preserves proximity in both spaces

The HEALPixP rojection

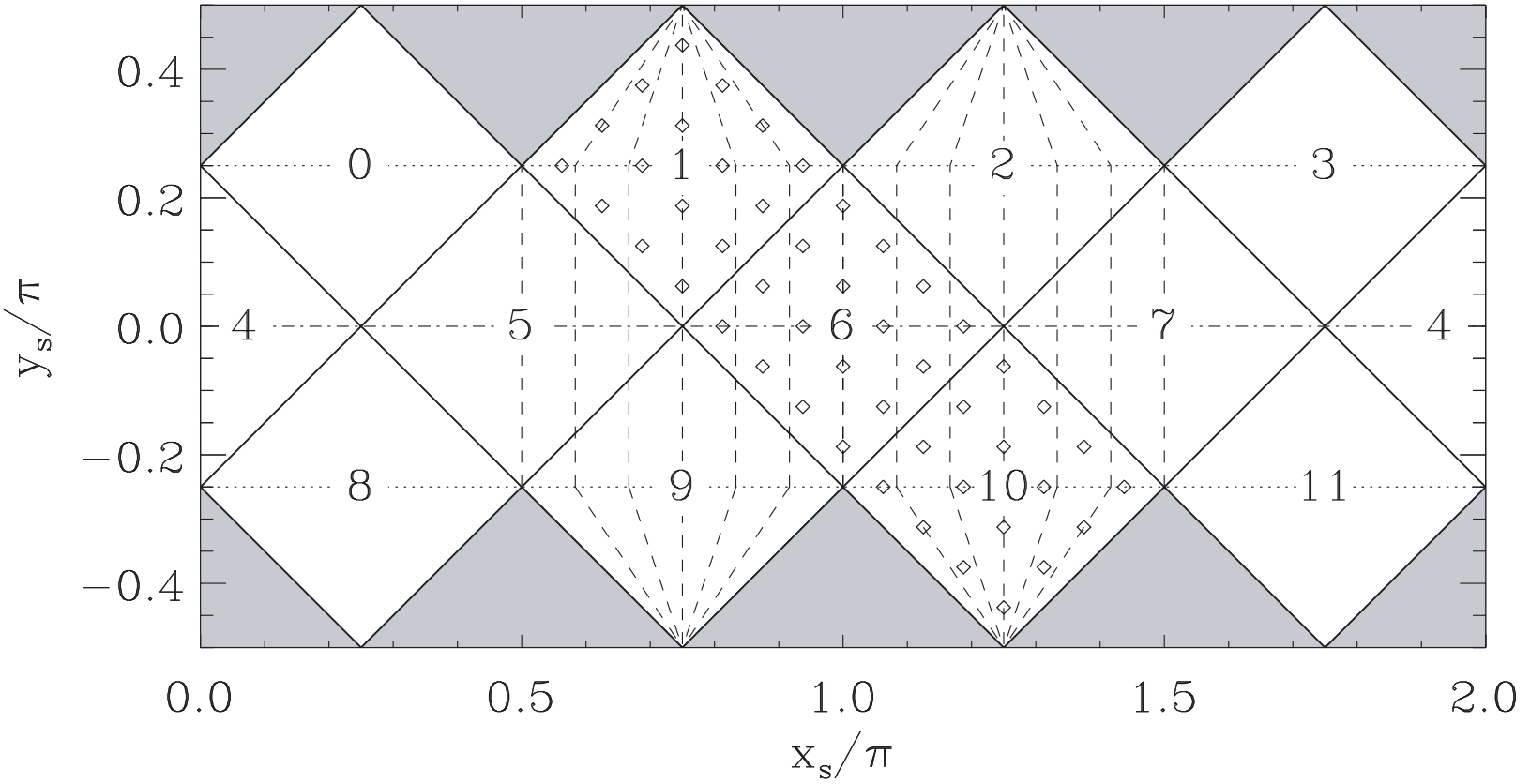
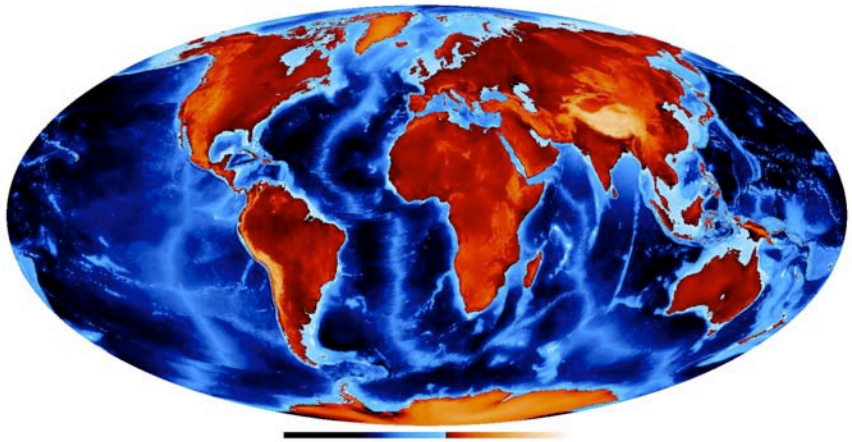
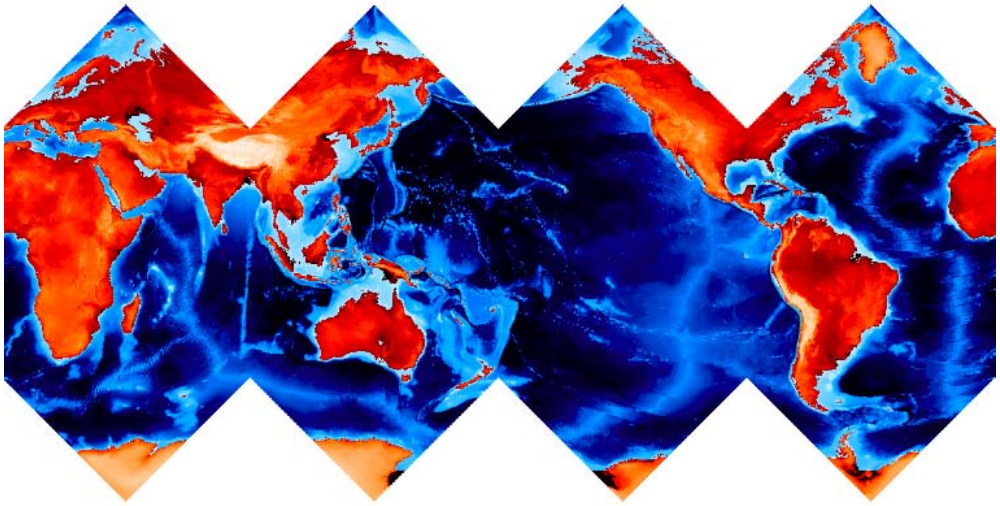


FIG. 5.—Spherical HEALPix projection onto the plane. Base-resolution HEALPix pixels (indexed here from 0 to 11, as in the HEALPix software) project into 12 identical square pixels in the plane. Hierarchical subdivision of the HEALPix grid generates identical square pixel images over the entire planar image of the HEALPix tessellation ($N_{\rm side} = 4$ pixel centers are shown within the base pixels 1, 6, and 10). Constant-latitude lines map into horizontal lines on the plane (*dashed lines*), and the HEALPix spherical projection mapping of meridians on one hemisphere is shown by the dashed lines.

HEALPix Cartography





- Old subject in cartography - Flattening the Earth
- Górski, K.M., Hivon, E., Banday, A.J., Wandelt, B.D., Hansen, F.K., Reinecke, M., Bartelmann, M., 2005, Ap.J., 622, pp. 759-771, "HEALPix: A Framework for High-Resolution Discretization and Fast Analysis of Data Distributed on the Sphere"
  - Description of "HEALPix projection" from the sphere to the plane
- Calabretta, M.R., and Roukema, B.F., 2007, MNRAS, 381, pp. 865-872, "Mapping on the HEALPix grid"
  - HEALPix - a hybrid of the cylindrical equal area (equatorial region), and interrupted Collignon (polar regions) cartographic projections

From C.A. Furuti's Cartography Webpage

The HEALPix Projection

HEALPix, **H**ierarchical **E**qual **A**rea and iso**L**atitude **P**ixelisation (Górski and others, 1999), is a collection of standards and resources for efficient storage and processing of large sets of data for astronomical and cosmological research. At discrete points ("pixels") covering a conceptual celestial sphere surrounding the Earth, satellite probes detect incoming radiation, like gamma rays and the cosmic microwave background; the measured values are saved on a raster grid for further analysis.

HEALPix defines a family of hybrid interrupted pseudocylindrical projections mapping from the celestial sphere to the plane. A HEALPix map comprises *H* lobes; in each lobe in the normal aspect, the equatorial band is mapped to a square using Lambert's equal-area cylindrical projection; the polar areas are mapped to two right isosceles triangles using a rescaled, interrupted form of Collignon's projection. The boundary parallels, approximately 41°48'37" N and S, are chosen in order to make the triangular regions cover 1/3 of the total area.

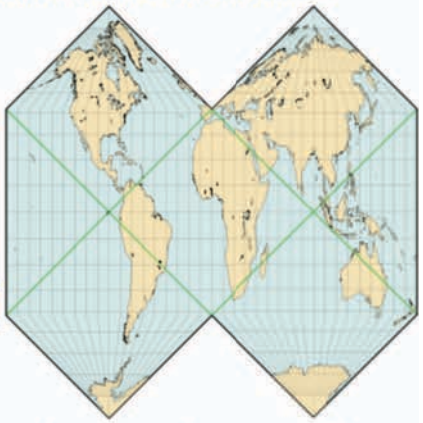
For a special case, with *H* = 2 and dispensing with the equatorial band, the result is an interrupted Collignon map in two squares. In general, changing *H* affects the unscaled aspect ratio (much like in variations of Lambert's projection) and the parallel of least shape distortion; the most common case, *H* = 4, can be trivially folded into a cube, like polyhedral projections.

The whole map may be divided into 3*H* identical facets, each a square with vertical and horizontal diagonals; a single facet is split along the boundary opposite the central meridian, but this can be fixed moving one half to the opposite side of the map, leaving all facets whole in a herringbone lay-out. Further, each facet may be recursively divided into 4 smaller squares. This hierarchical organization allows data processing in different levels of detail. In the last level, pixels are also squares with horizontal and vertical diagonals; this unorthodox orientation may be fixed by the artifice of rotating the entire map by 45 degrees. Since the overall result is equal-area, raster computation yields consistent results. The pseudocylindrical property provides uniform pixel distribution along parallels. Another favorable feature for large data sets is the easily predictable number and location of neighbor pixels.

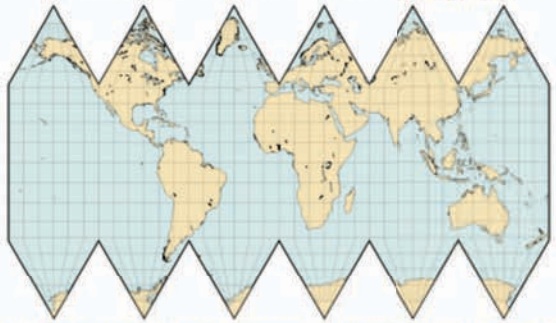
World maps in the HEALPix projection

Although the main purpose of HEALPix is not cartographical, using its projection for world maps shows some interesting properties.

This HEALPix map with *H* = 2 has facet edges marked in green. Each facet can be further hierarchically subdivided into smaller squares.



HEALPix map with *H* = 4. Compared with the map above, facets are still square, but the overall aspect ratio differs.



HEALPix map with *H* = 6, rescaled in order to show facets as equilateral triangles. Each facet may be subdivided into four triangles.

The angular power spectrum of the CMB anisotropy

- **Current measurements are quite consistent with the primary fluctuations in the CMB being a Gaussian random field.**
- **The primary fluctuations in the CMB are thus well described by their angular power spectrum.**

$$\frac{\Delta T}{T} \equiv \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

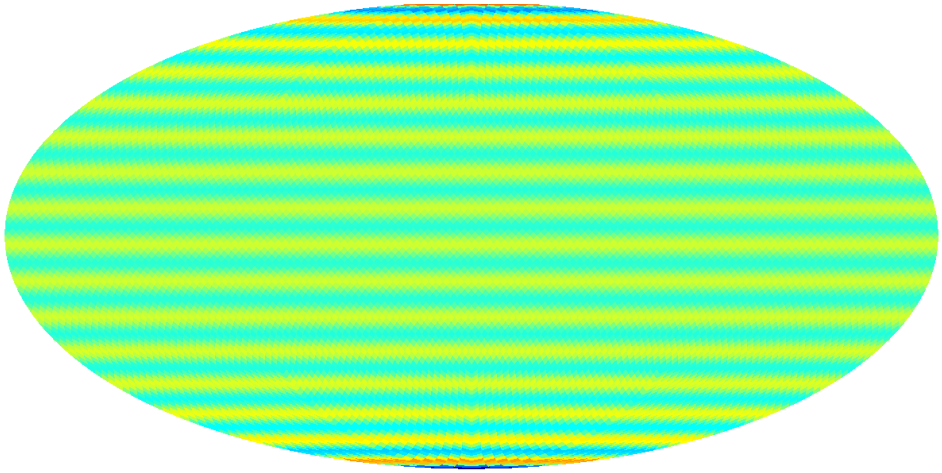
Curved sky analogue of FT

$$\langle a_{\ell m} a_{\ell' m'} \rangle = C_\ell \, \delta_{\ell \ell'} \delta_{m m'}$$

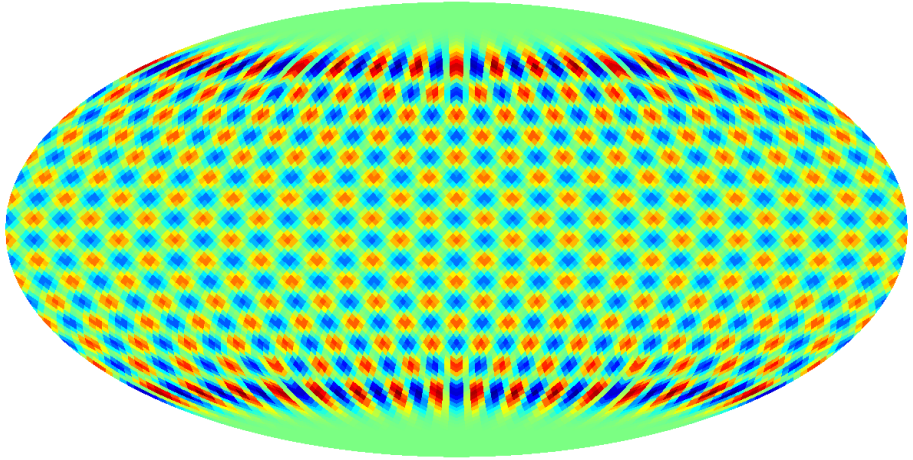
Statistical isotropy

Rotational invariance

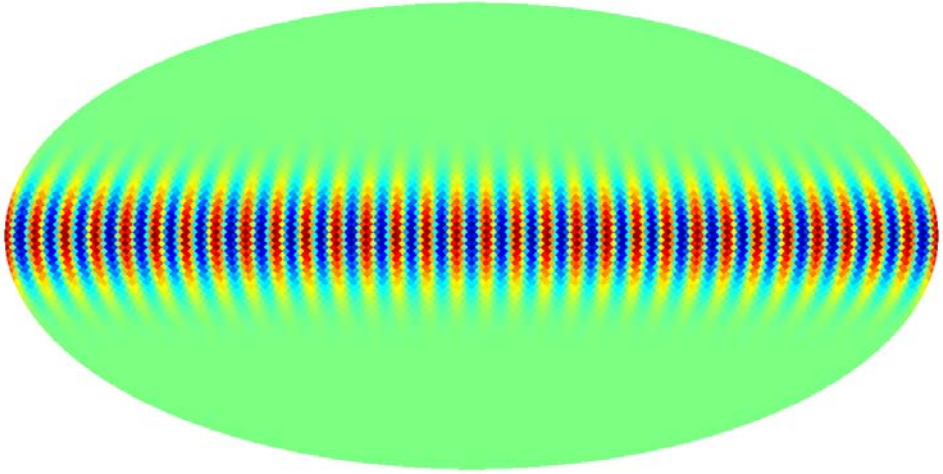
l=31 m=0



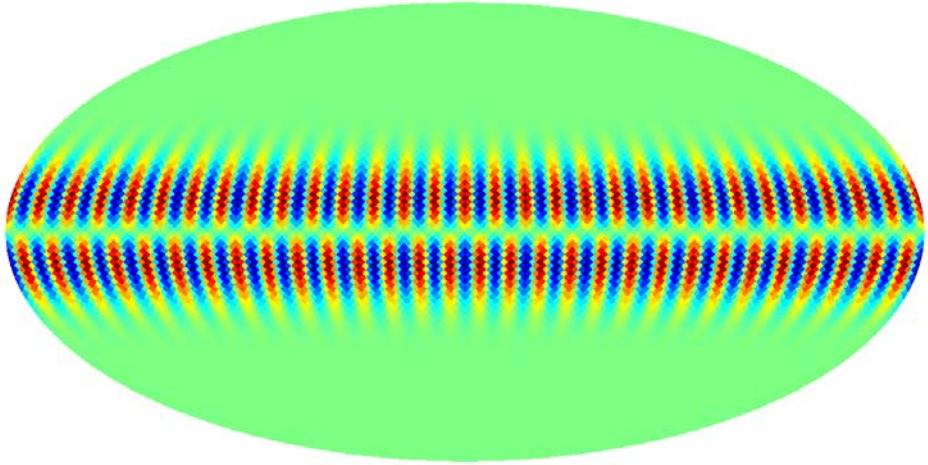
L=31 m=16



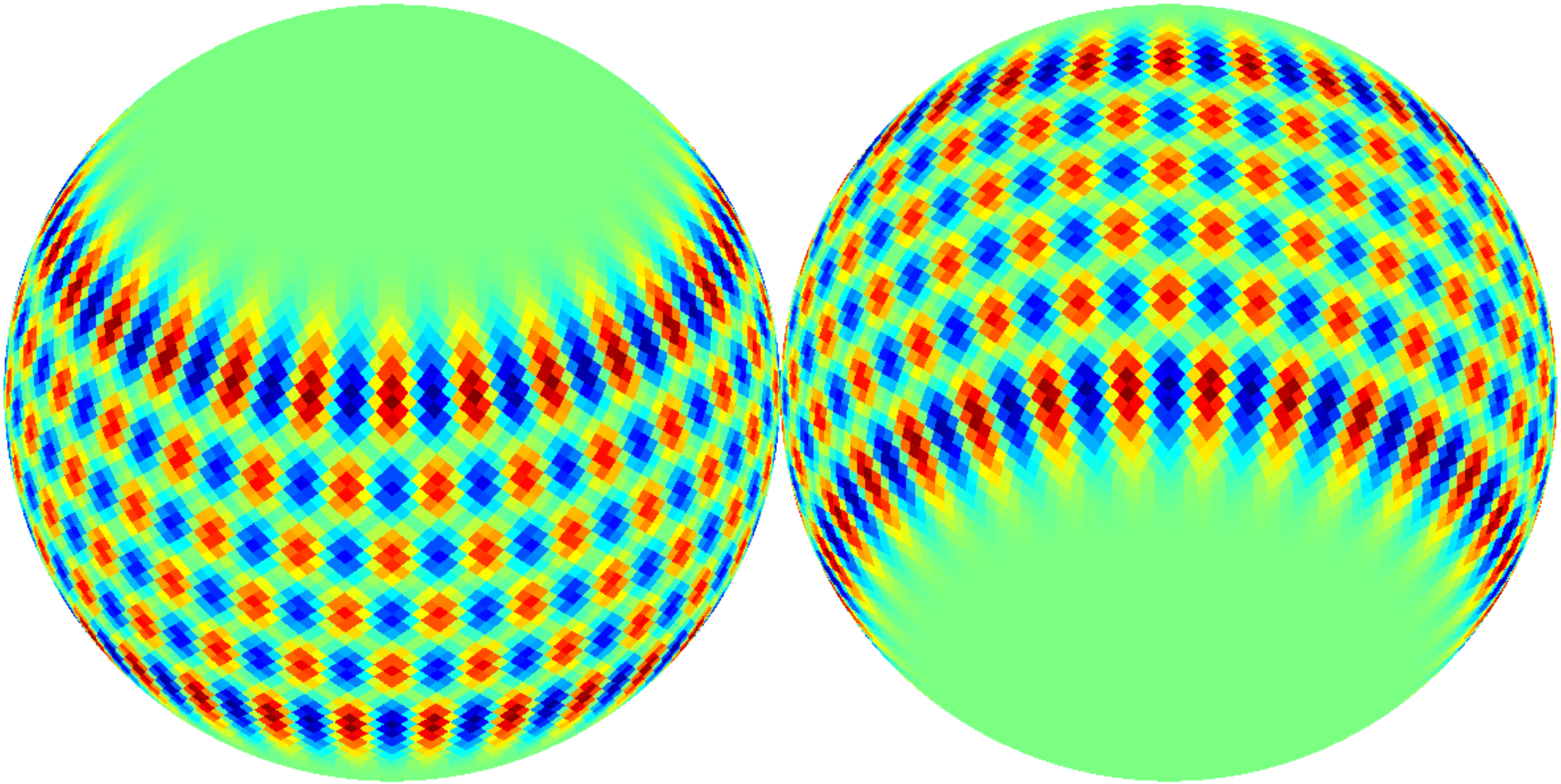
l=31 m=31



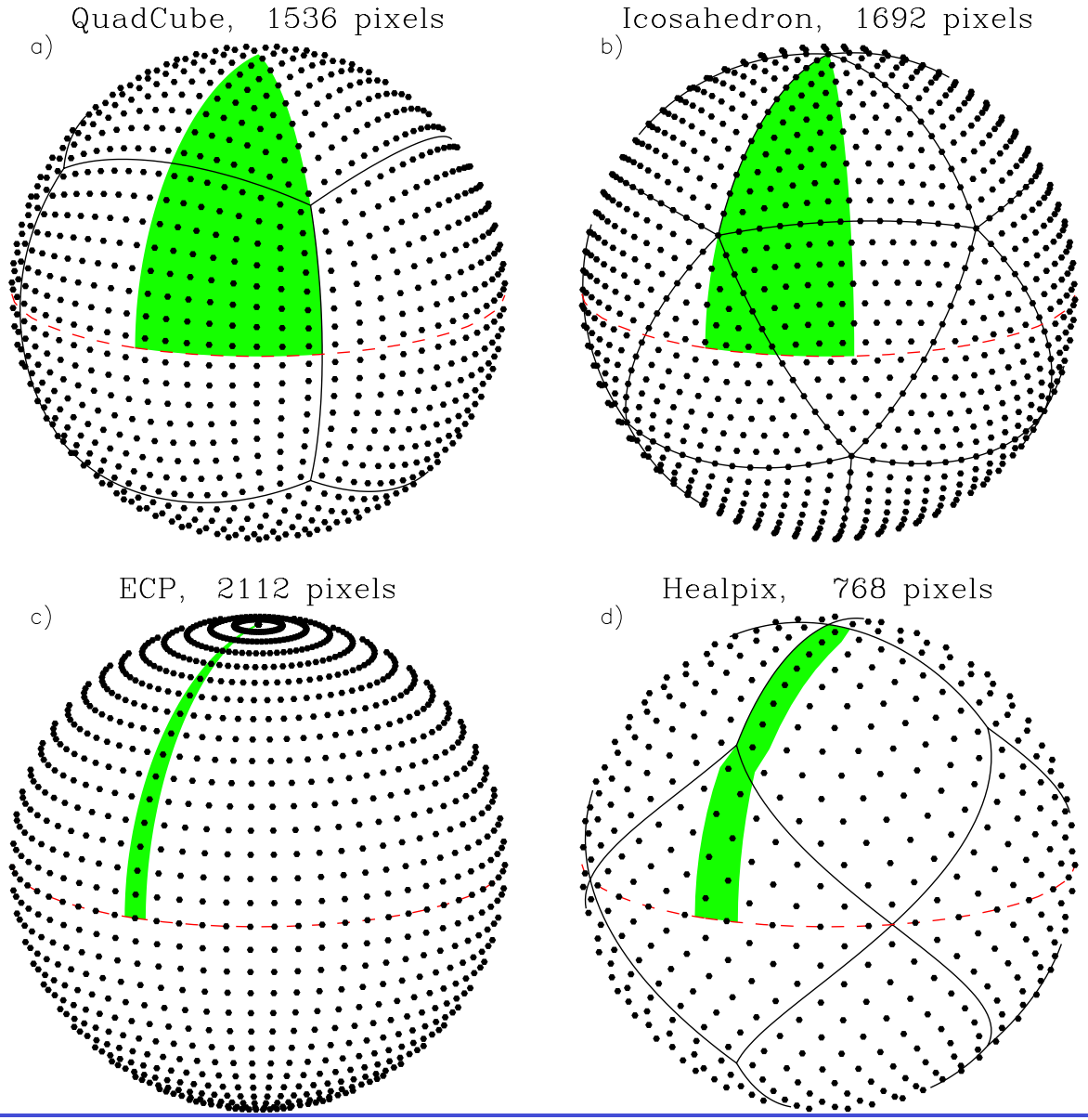
L=31 m=30



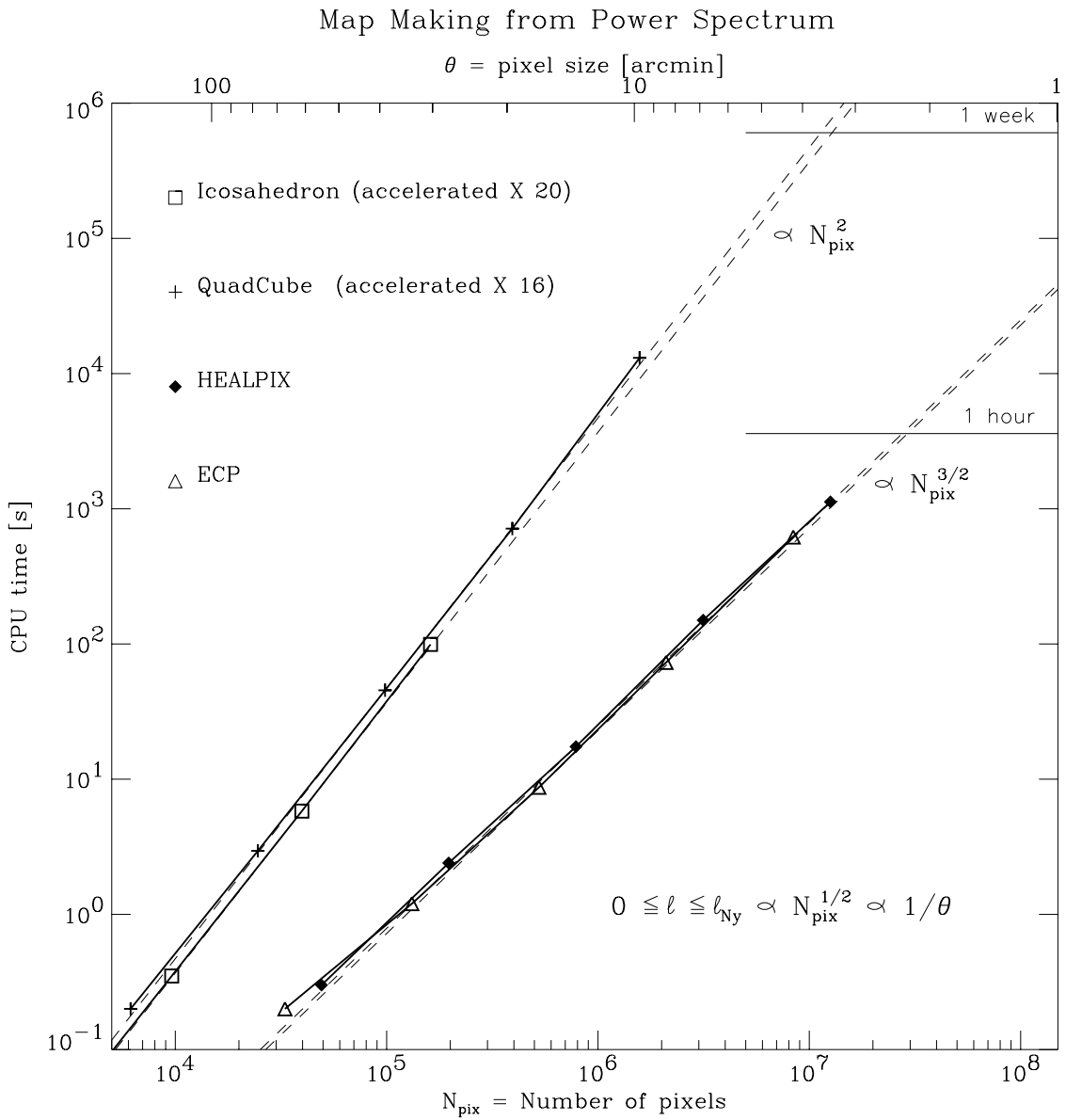
l=31 m=16



How should one do this? And - Why?

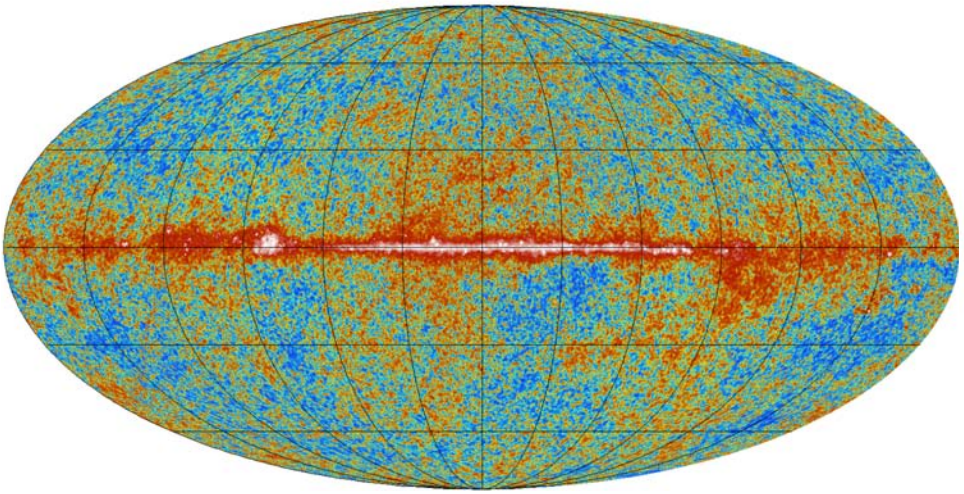


Computing speed in SH-analysis

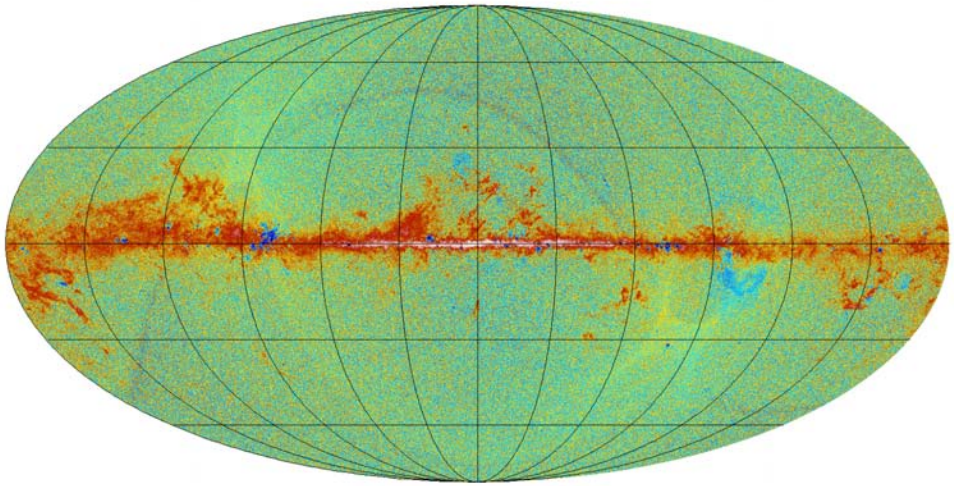


Quick look at large and intermediate angular scales: 70 vs. 100 GHz

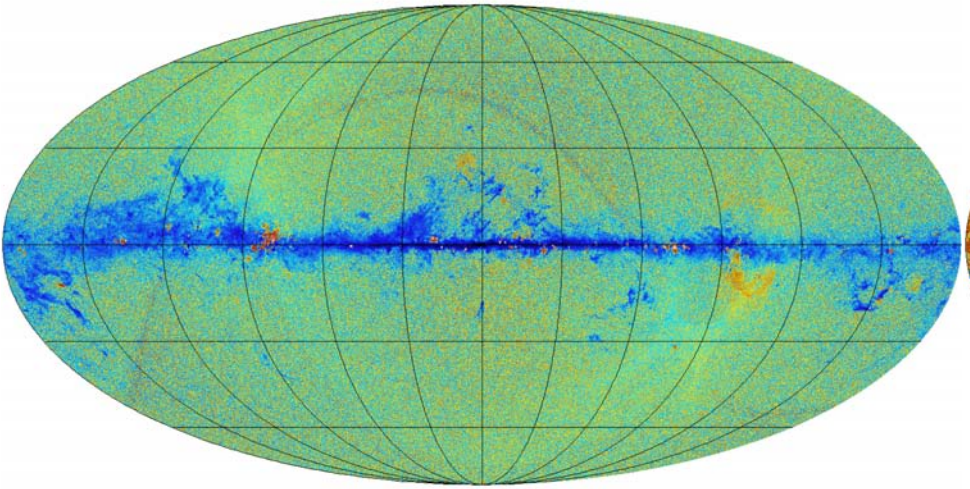
Planck Nominal Mission LFI 70 GHz



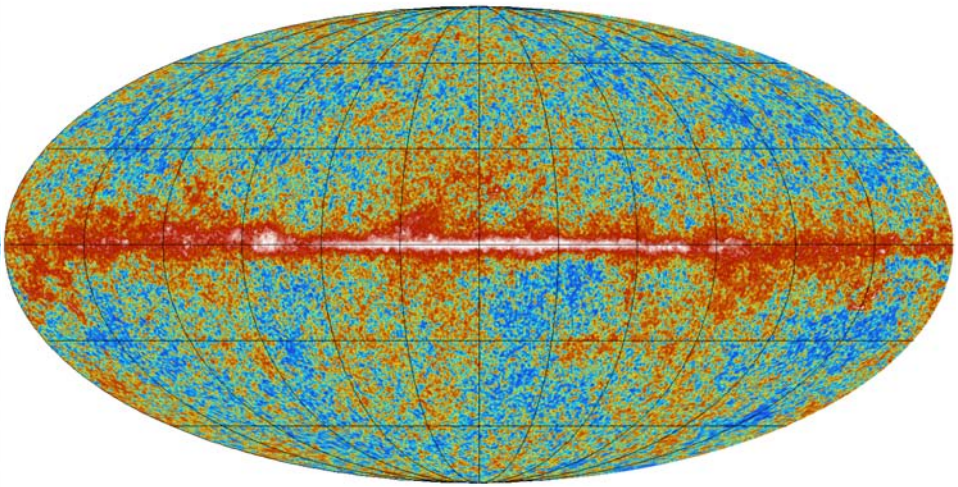
Planck Nominal Mission "100−70"

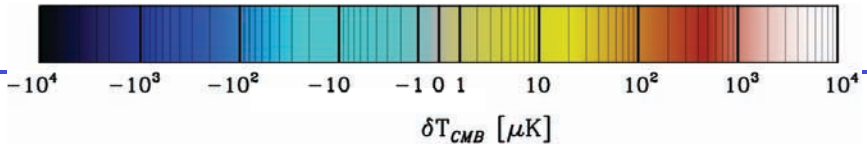


Planck Nominal Mission "70−100"

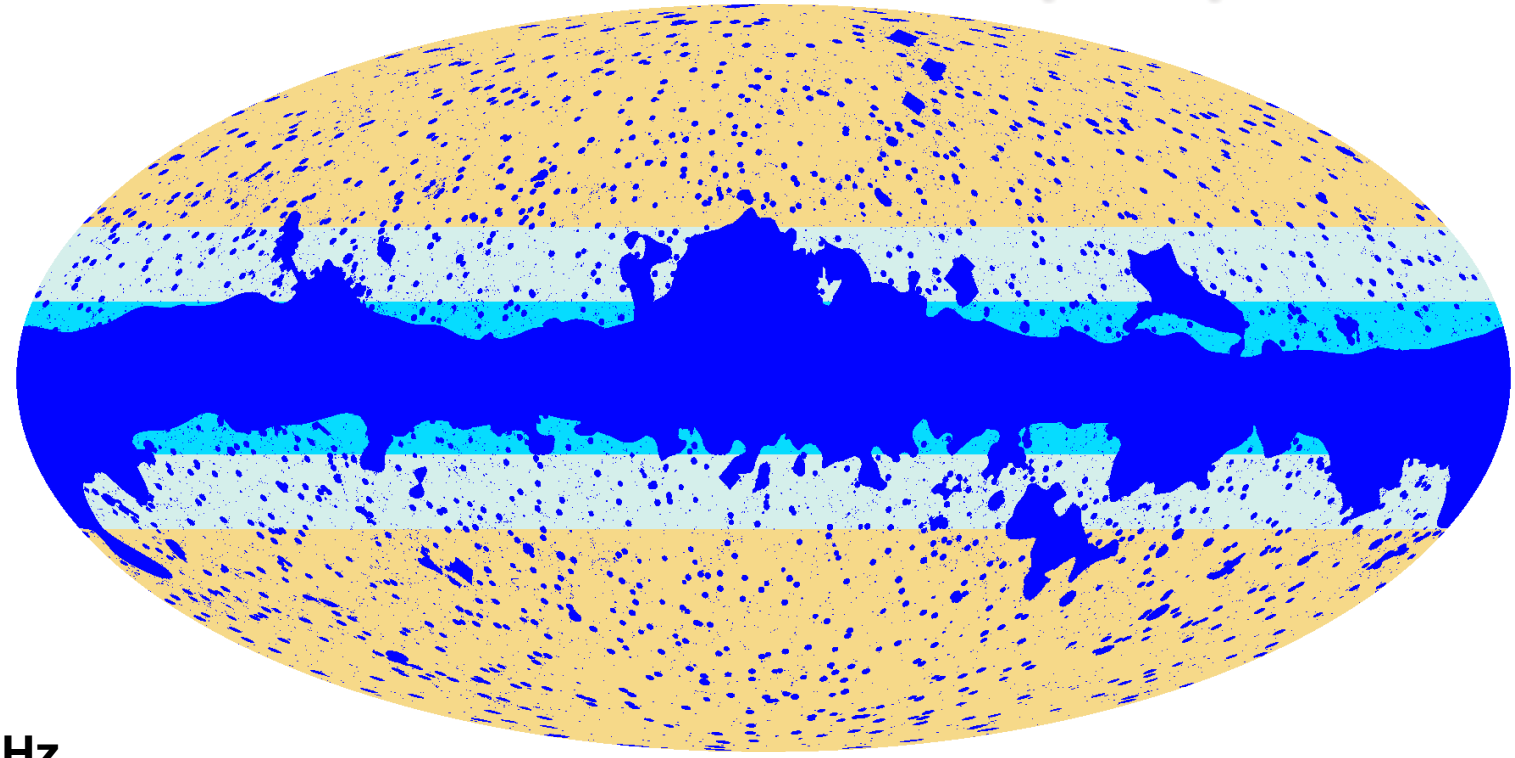


Planck Nominal Mission HFI 100 GHz



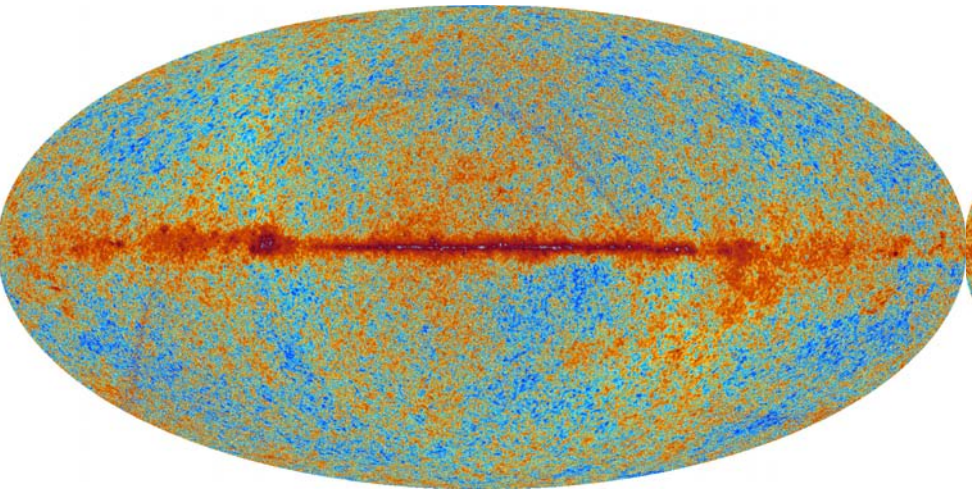


Some masks that we use for *Planck* vs. WMAP consistency analysis

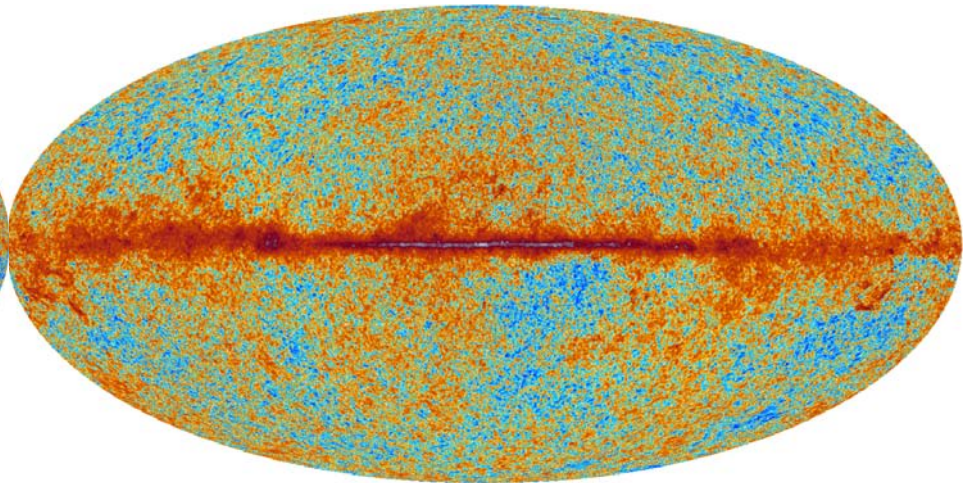


Union of WMAP kq85 and Planck comp-sep mask

70 GHz

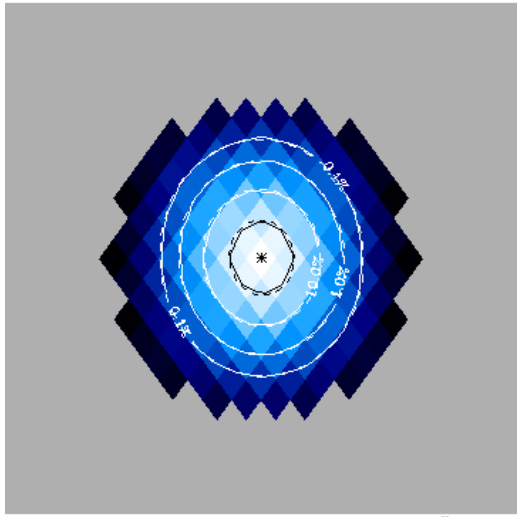


100 GHz



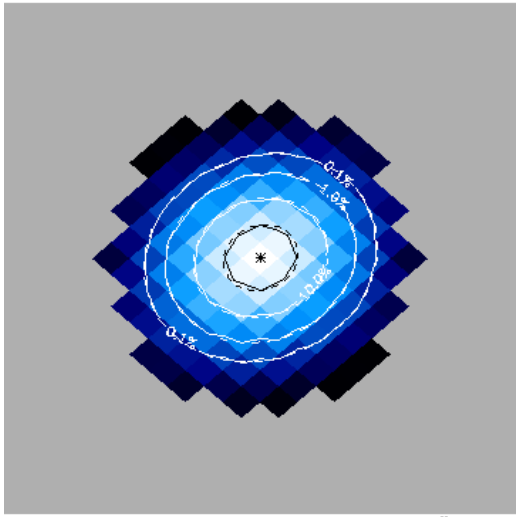
Effective beam variation on the sky

CoP_dx5 EG 100 survey1 PSF : pix #1017
Fit:θ_{FWHM}=9.75′,e=1.168,ψ_{eff}=4.7°,Δ_{MMSE}=0.72%



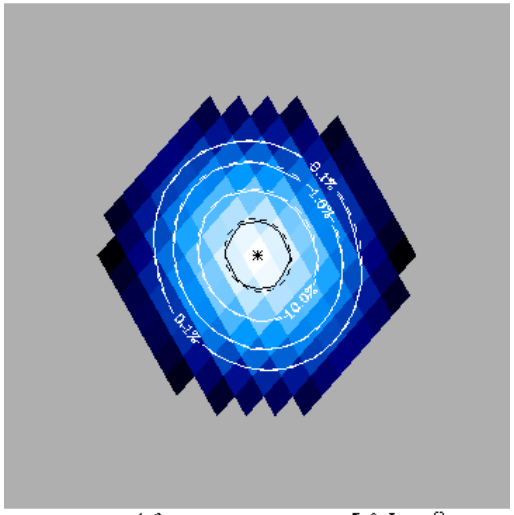
−1.0 5.0 Log ()
(86.1, −38.2) Galactic

CoP_dx5 EG 100 survey1 PSF : pix #692
Fit:θ_{FWHM}=9.77′,e=1.173,ψ_{eff}=−68.4°,Δ_{MMSE}=1.71%



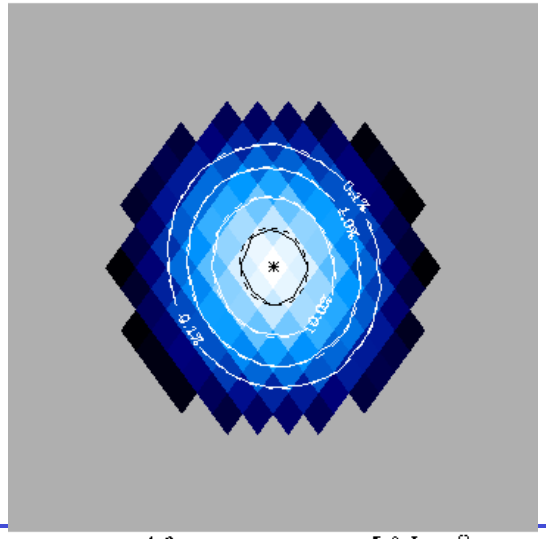
−1.0 5.0 Log ()
(164.5, −5.6) Galactic

CoP_dx5 EG 100 survey1 PSF : pix #101
Fit:θ_{FWHM}=9.80′,e=1.152,ψ_{eff}=23.5°,Δ_{MMSE}=1.07%



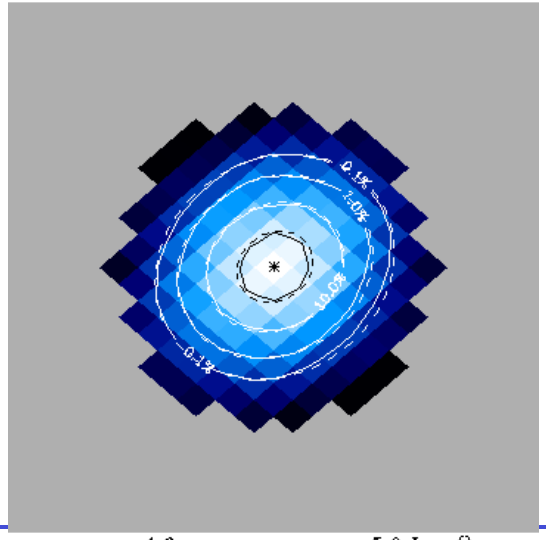
−1.0 5.0 Log ()
(305.1, 57.1) Galactic

CoP_dx5 EG 100 survey2 PSF : pix #1017
Fit:θ_{FWHM}=9.79′,e=1.183,ψ_{eff}=14.9°,Δ_{MMSE}=2.57%



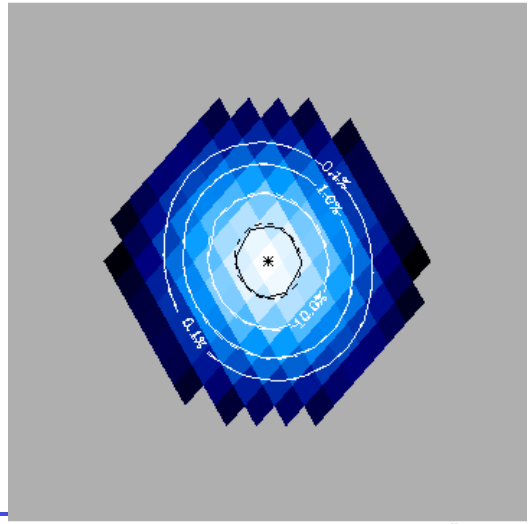
−1.0 5.0 Log ()
(86.1, −38.2) Galactic

CoP_dx5 EG 100 survey2 PSF : pix #692
Fit:θ_{FWHM}=9.89′,e=1.160,ψ_{eff}=−57.3°,Δ_{MMSE}=3.38%



−1.0 5.0 Log ()
(164.5, −5.6) Galactic

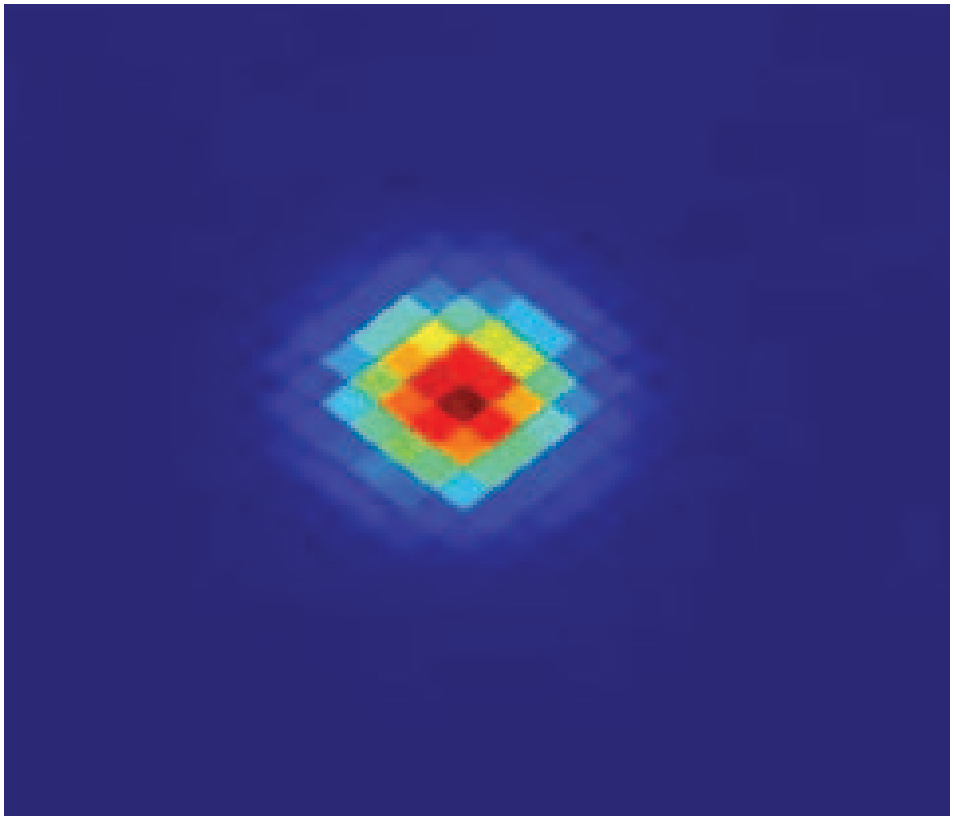
CoP_dx5 EG 100 survey2 PSF : pix #101
Fit:θ_{FWHM}=9.77′,e=1.170,ψ_{eff}=16.3°,Δ_{MMSE}=1.39%



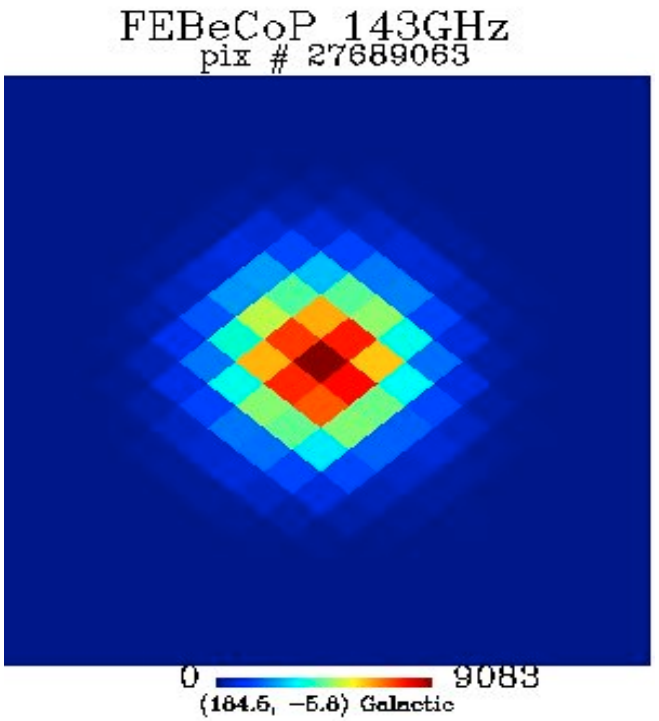
−1.0 5.0 Log ()
(305.1, 57.1) Galactic

PSF and flux estimates: FEBeCoP

- Modeling Planck response to the sky signal, both diffuse and form discrete sources

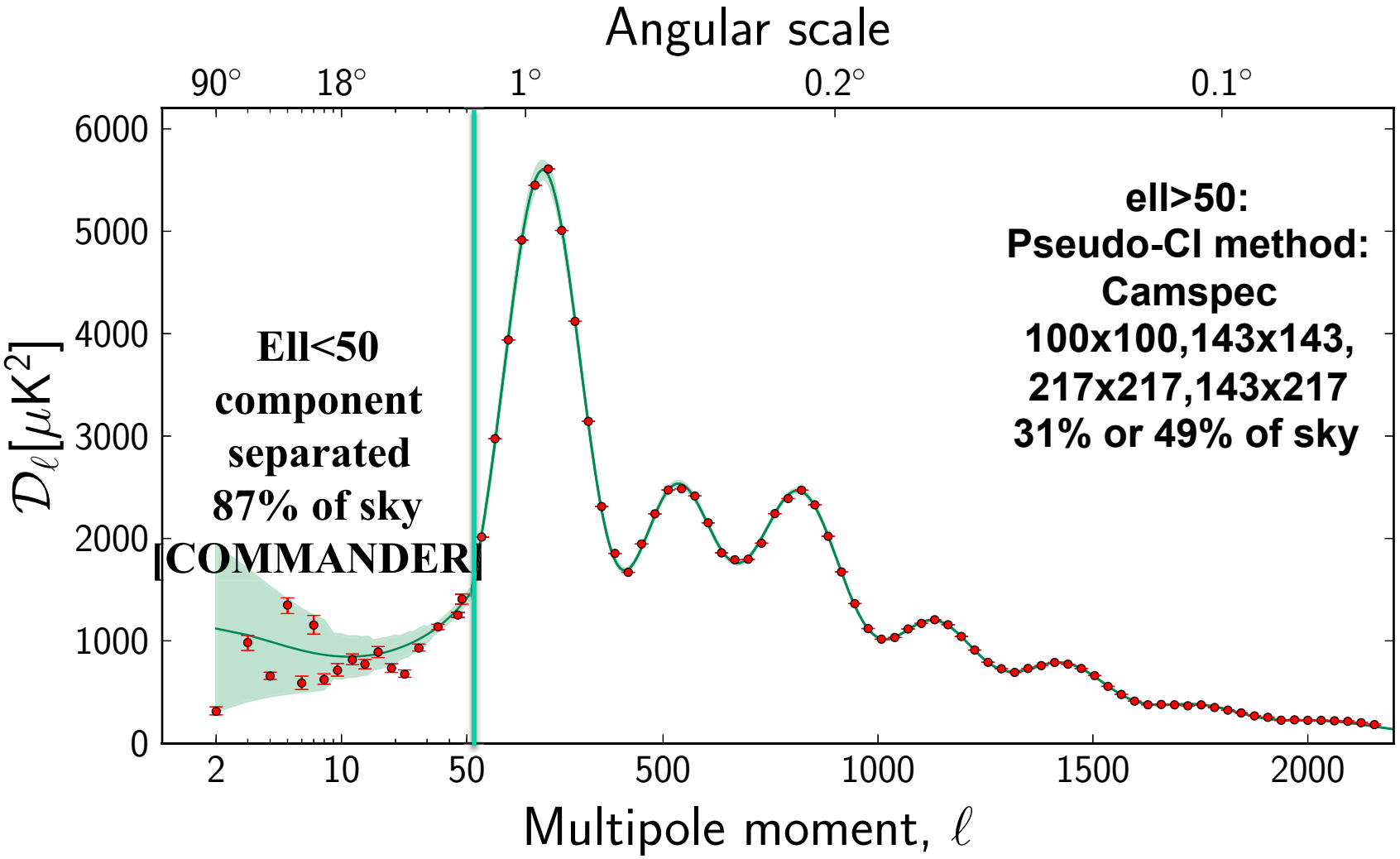


Crab @ 143 GHz, I



FEBeCoP @ 143 GHz, I

Planck's CMB anisotropy angular power spectrum



… but that is an altogether different story

TBC

… but all of that (sky maps -> science) works with the methodology, data structure, and computing/ analytical and data handling tools conceptualized and assembled in

HEALPix