# Estudo sobre o papel do ruído em sistemas estocásticos fora do equilíbrio

*W.A.M. Morgado (PUC-Rio), S.M. Duarte Queirós (ISI-CNR, Italy), D.O. Soares-Pinto (IFSC-USP)*

**Resumo:**
A partir das propriedades do ruído de Langevin podemos estudar o comportamento estacionário de vários modelos que representam sistemas fora do equilíbrio termodinâmico. Esta é uma técnica sistemática que pode ser utilizada para modelar sistemas econômicos ou biológicos.

**Modelo e Distribuição Estacionária:**

$$\dot{x}(t) = v(t),$$
$$M\,\dot{v}(t) = -k_0 x(t) - \gamma v(t) + \eta(t),$$

onde o ruído de Poisson é descrito por

$$\eta(t) = \sum_\ell \Phi(t)\,\delta(t - t_\ell),$$

variando com freqüência

$$\lambda(t) = \lambda_0\left[1 + A\cos(\omega\, t)\right],$$

Os cumulantes do ruído são dados por

$$\langle \eta(t_1)\dots\eta(t_n)\rangle_c = \lambda(t_1)\,\overline{\Phi^n}\,\delta(t_1 - t_2)\dots\delta(t_{n-1} - t_n).$$

Os cumulantes para os produtos das variáveis V e X são

$$\begin{aligned}\overline{\langle x^n v^m\rangle_c} &= \lim_{z\to 0} z\int_0^\infty dt\, e^{-zt}\,\langle x^n(t)\,v^m(t)\rangle_c\\ &= \lim_{z\to 0}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\prod_{h=1}^{n}\frac{dq_h}{2\pi}\int_{-\infty}^{+\infty}\prod_{j=1}^{m}\frac{dp_j}{2\pi}\,\frac{z}{z - \left[\sum_{h=1}^{n}(\mathrm{i}q_h+\epsilon) + \sum_{j=1}^{m}(\mathrm{i}p_j+\epsilon)\right]}\\ &\times\prod_{h=1}^{n}\frac{1}{[M\,R(\mathrm{i}q_h+\epsilon)]}\prod_{j=1}^{m}\frac{(\mathrm{i}p_j+\epsilon)}{[M\,R(\mathrm{i}p_j+\epsilon)]}\left\langle\prod_{h=1}^{n}\tilde\eta(\mathrm{i}q_h+\epsilon)\prod_{j=1}^{m}\tilde\eta(\mathrm{i}p_j+\epsilon)\right\rangle_c\end{aligned}$$

Podemos construir a distribuição de probabilidades estacionária:

$$p_{ss}(x,v) = \int_{-\infty}^{+\infty}\frac{dQ}{2\pi}\frac{dP}{2\pi}e^{iQx+iPv}\exp\left\{\sum_{n+m=0;(n+m>0)}^{\infty}\frac{(-\mathrm{i}Q)^n}{n!}\frac{(-\mathrm{i}P)^m}{m!}\mathcal{P}_{n,m}\right\}.$$

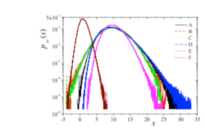

Figure 1. Numerically obtained probability density function $p_{ss}(x)$ vs position $x$ for various cases with $\lambda_0 = 10$, $\Phi = 1$ and the noise defined by Eq. (4) with $\omega = \pi$. Following the legend in the figure we have the respective cases, A: $M = 1, k_0 = 1, \gamma = 1, A = 0$, B: $M = 10, k_0 = 1, \gamma = 1, A = 1$, C: $M = 10, k_0 = 1, \gamma = 1, A = 1/2$, D: $M = 0.1, k_0 = 1, \gamma = 1, A = 0$, E: $M = 1, k_0 = 1, \gamma = 2, A = 0$ and F: $M = 1, k_0 = 10, \gamma = 1, A = 0$.

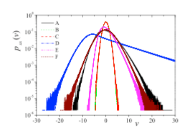

Figure 2. Numerically obtained probability density function $p_{ss}(v)$ vs scalar velocity $v$ for the same parameters sets of Fig. 1.

**Injeção de Energia e Balanço Energético**

$$\sum W_{F_{ext},i} = \Delta E_m$$
$$\int \eta\, dx - \gamma\int v\, dx = \frac{1}{2}M\,v(t)^2 + \frac{1}{2}k_0\,x(t)^2$$

Forma de v(t) e cumulantes via transformadas de Laplace do ruído

$$\begin{aligned}v(t)^2 &= \int\!\!\int v(t_1)\,v(t_2)\,\delta(t - t_1)\,\delta(t - t_2)\,dt_1\,dt_2\\ &= \lim_{\epsilon\to 0}\int\!\!\int e^{(\mathrm{i}q_1+\mathrm{i}q_2+2\epsilon)t}\,\tilde v(\mathrm{i}q_1+\epsilon)\,\tilde v(\mathrm{i}q_2+\epsilon)\,\frac{dq_1}{2\pi}\frac{dq_2}{2\pi},\end{aligned}$$

$$\langle v(t)^2\rangle - \langle v(t)\rangle^2 = \lim_{\epsilon\to 0}\frac{1}{M^2}\int\!\!\int e^{(\mathrm{i}q_1+\mathrm{i}q_2+2\epsilon)t}\,(\mathrm{i}q_1+\epsilon)(\mathrm{i}q_2+\epsilon)\times\frac{\langle\tilde\eta(\mathrm{i}q_1+\epsilon)\,\tilde\eta(\mathrm{i}q_2+\epsilon)\rangle_c\,dq_1\,dq_2}{R(\mathrm{i}q_1+\epsilon)\,R(\mathrm{i}q_2+\epsilon)\,2\pi\,2\pi},$$

Desenvolvimento assintótico de v(t) e x(t)

$$\begin{aligned}\langle v(t)^2\rangle_{asy} - \langle v(t)\rangle^2_{asy} &= \frac{\lambda_0\overline{\Phi^2}}{\gamma M} + 8\,\frac{A\left[4\left(\omega^2\theta^3 + \theta\,\omega_0{}^4\right)\cos(\omega t) + \omega^3\theta^2\sin(\omega t)\right]\lambda_0\overline{\Phi^2}}{M^2\left(4\theta^2+\omega^2\right)\left(\left(\omega_0^2-\omega^2\right)^2+4\theta^2\omega^2\right)} +\\ &4\,\frac{A\left[2\theta\left(\omega^4 - 3\,\omega_0{}^2\omega^2\right)\cos(\omega t) - 3\,\omega_0{}^2\omega^3\sin(\omega t)\right]\lambda_0\overline{\Phi^2}}{M^2\left(4\theta^2+\omega^2\right)\left(\left(\omega_0^2-\omega^2\right)^2+4\theta^2\omega^2\right)} +\\ &2\,\frac{A\left(\omega^5 + 8\,\omega_0{}^4\omega\right)\lambda_0\overline{\Phi^2}\sin(\omega t)}{M^2\left(4\theta^2+\omega^2\right)\left(\left(\omega_0^2-\omega^2\right)^2+4\theta^2\omega^2\right)},\end{aligned}$$

and

$$\begin{aligned}\langle x(t)^2\rangle_{asy} - \langle x(t)\rangle^2_{asy} &= \overline{\Phi^2}\frac{\lambda_0}{\gamma k_0} + \frac{4\omega\left(\omega^2 - 4\,\omega_0{}^2 - 8\,\theta^2\right)A\,\overline{\Phi^2}\lambda_0\sin(\omega t)}{\left(\omega^4 - 8\,\omega_0{}^2\omega^2 + 16\,\theta^2\omega^2 + 16\,\omega_0{}^4\right)\left(4\theta^2+\omega^2\right)M^2} +\\ &\frac{8\left(3\theta\,\omega^2 - 2\theta\,\omega_0{}^2\right)A\,\overline{\Phi^2}\lambda_0\cos(\omega t)}{\left(\omega^4 - 8\,\omega_0{}^2\omega^2 + 16\,\theta^2\omega^2 + 16\,\omega_0{}^4\right)\left(4\theta^2+\omega^2\right)M^2}.\end{aligned}$$

$$\langle v(t)\rangle_{asy} = \frac{A\left[(\omega^2 - \omega_0{}^2)\sin(\omega t) + 2\,\omega\,\theta\cos(\omega t)\right]\omega\,\lambda_0\bar\Phi}{M\left[\left(\omega_0^2-\omega^2\right)^2 + 4\,\theta^2\,\omega^2\right]},$$
$$\langle x(t)\rangle_{asy} = \frac{\bar\Phi\,\lambda_0}{\omega_0^2\,M} - \frac{A\left[(\omega^2 - \omega_0{}^2)\cos(\omega t) - 2\,\theta\,\omega\sin(\omega t)\right]\lambda_0\bar\Phi}{M\left[\left(\omega_0^2-\omega^2\right)^2 + 4\,\theta^2\,\omega^2\right]},$$

$$\sqrt{\langle x(t)\rangle^2_{asy}} = \frac{\bar\Phi\,\lambda_0}{\omega_0^2\,M}\sqrt{1 + A^2\frac{\omega_0^4}{2\left[4\theta^2\omega^2 + \left(\omega_0^2-\omega^2\right)^2\right]}},$$

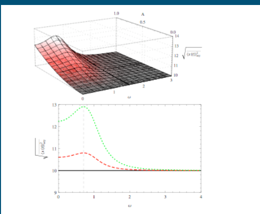

Figure 3. Upper panel: $\sqrt{\langle x(t)\rangle^2_{asy}}$ vs amplitude of the heterogeneous part, $A$, and frequency, $\omega$, with $M = 1$, $\gamma = 1$, $k_0 = 1$, $\lambda_0 = 10$, $\Phi = 1$. The maximum, characterising a stochastic resonance phenomenon, occurs at $\omega_{res} = \sqrt{\omega_0^2 - 2\theta^2}$ that defines the grey dashed line in the lower panel containing the plane cuts for $A = 0$ (black line), $A = \frac{1}{2}$ (dashed red line) and $A = 1$ (dotted green line). For this case the maximum occurs at $1/\sqrt{2}$.

Energia no regime assintótico (média temporal)

$$\begin{aligned}E^c_M &= \frac{1}{2}M\overline{\langle v(t)^2\rangle_{asy}} + \frac{1}{2}k_0\overline{\langle x(t)^2\rangle_{asy}}\\ &= \lambda_0\frac{\overline{\Phi^2}}{\gamma} + \frac{\lambda_0{}^2\bar\Phi^2}{2\,M\,\omega_0^2} + \frac{\lambda_0{}^2\bar\Phi^2A^2\left(\omega^2 + \omega_0{}^2\right)}{4\,M\left[\left(\omega_0^2-\omega^2\right)^2 + 4\,\theta^2\,\omega^2\right]}.\end{aligned}$$

**Injeção e Dissipação de Energia:**

$$J_{IT} = \int_0^\tau dt\, v(t)\,\eta(t),$$ e $$J_{DT} = -\gamma\int_0^\tau dt\, v^2(t).$$

$$\begin{aligned}J_{DT} &= -\gamma\int_0^\tau dt\int_0^\infty dt_1\,\delta(t - t_1)\int_0^\infty dt_2\,\delta(t - t_2)\,v(t_1)\,\eta(t_2),\\ &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\int_{-\infty}^{\infty}\frac{dq_2}{2\pi}\frac{e^{(\mathrm{i}q_1+\mathrm{i}q_2+2\epsilon)\tau} - 1}{(\mathrm{i}q_1 + \mathrm{i}q_2 + 2\epsilon)}\tilde v(\mathrm{i}q_1+\epsilon)\,\tilde v(\mathrm{i}q_2+\epsilon),\\ &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\int_{-\infty}^{\infty}\frac{dq_2}{2\pi}\frac{e^{(\mathrm{i}q_1+\mathrm{i}q_2+2\epsilon)\tau} - 1}{(\mathrm{i}q_1 + \mathrm{i}q_2 + 2\epsilon)}\times\\ &\left[\frac{(\mathrm{i}q_1+\epsilon)}{M\,R(\mathrm{i}q_1+\epsilon)}\frac{(\mathrm{i}q_2+\epsilon)}{M\,R(\mathrm{i}q_2+\epsilon)}\tilde\eta(\mathrm{i}q_1+\epsilon)\tilde\eta(\mathrm{i}q_2+\epsilon)\right],\end{aligned}$$

onde

$$R(s) \equiv s^2 + \frac{\gamma}{M}s + \frac{k_0}{M} = (s - \kappa_+)(s - \kappa_-),$$

to $\tau$,

$$J_{DT0}(\tau) = -\left(\frac{\bar\Phi^2\lambda_0}{M} + \frac{A^2\omega^2\bar\Phi^2\lambda_0^2\,\theta}{\left[\omega^2\left(\omega^2 - 2\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]M}\right)\tau,$$

an oscillating term,

$$\begin{aligned}J_{D,osc}(\tau) &= 8\,\frac{A\left(3\,\theta\,\omega^2\omega_0{}^2 - 4\,\theta\,\omega_0^4 - 2\,\theta\,\omega^4 - 4\,\theta^3\omega^2\right)\bar\Phi^2\lambda_0^2\gamma\sin(\omega\tau)}{M^2\omega\left(4\theta^2+\omega^2\right)\left(\omega^4 + 16\,\theta^2\omega^2 + 16\,\omega_0^4 - 8\,\omega^2\omega_0^2\right)} +\\ &2\,\frac{A\left(8\,\omega\,\omega_0{}^4 + 4\,\theta^2\omega^3 + \omega^5 - 6\,\omega^3\omega_0{}^2\right)\bar\Phi^2\lambda_0^2\gamma\cos(\omega\tau)}{M^2\omega\left(\omega^2 + 4\,\theta^2\right)\left(\omega^4 + 16\,\omega^2\theta^2 + 16\,\omega_0^4 - 8\,\omega^2\omega_0^2\right)} -\\ &2\,\frac{A^2\left[\omega\,\theta^2\sin(2\,\omega\tau) + \left(\omega_0^2 - \omega^2\right)\cos(2\,\omega\tau)\right]\bar\Phi^2\lambda_0^2\theta^2\omega^2}{M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]^2} +\\ &\frac{A^2\left[\omega^2\left(\omega^4 + -2\,\omega_0{}^2\right) + \omega_0{}^4\right]\bar\Phi^2\lambda_0{}^2\theta\,\omega\sin(2\,\omega\tau)}{2\,M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]^2},\end{aligned}$$

and a constant term,

$$\begin{aligned}J_{DTc} &= \frac{\bar\Phi^2\lambda_0}{\gamma} - \frac{\bar\Phi^2\lambda_0^2}{2\,M\,\omega_0^2} + \frac{A\bar\Phi^2\lambda_0^2\left(\omega^2 - \omega_0^2\right)}{M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]} +\\ &\frac{A^2\left[\left[2\,\omega^2\left(\omega_0^2 + 2\,\theta^2\right) - \omega_0^4\right]\omega_0^2 + 8\,\omega^4\left(\theta^2 - \omega_0{}^2\right)\right]\bar\Phi^2\lambda_0^2}{2\,M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]^2},\end{aligned}$$

where $J_{DT} = J_{DT0}(\tau) + J_{D,osc}(\tau) + J_{DTc}$.

Then finally, we write the contributions for the injection of energy as

$$J_{IT0}(\tau) = \left(\frac{\lambda_0\bar\Phi^2}{M} + \frac{A^2\omega^2\,\theta\,\lambda_0^2\,\bar\Phi^2}{M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]}\right)\tau,$$

and

$$\begin{aligned}J_{IT,osc}(\tau) &= \frac{\lambda_0\bar\Phi^2A\sin(\omega\tau)}{M\omega} + \frac{A\left[A\left(\omega_0{}^2 - \omega^2\right)\cos(2\,\omega\tau) + 4\,\omega_0{}^2\cos(\omega\tau)\right]\lambda_0^2\,\bar\Phi^2}{4\,M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]}\\ &+ \frac{A\left[2\omega\theta\left(A\sin(2\,\omega\tau) + 4\sin(\omega\tau)\right) - 4\,\omega^2\cos(\omega\tau)\right]\lambda_0^2\,\bar\Phi^2}{4\,M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]},\end{aligned}\qquad(47)$$

and

$$\begin{aligned}J_{ITc} &= \frac{\lambda_0^2\bar\Phi^2}{M\omega_0^2} + \frac{A\lambda_0^2\bar\Phi^2\left(\omega_0{}^2 - \omega^2\right)}{M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]} +\\ &\frac{\left[\omega^4\left(\omega^2 - 12\,\theta^2\right) + \omega_0^2\left(3\,\omega_0^4 - 5\,\omega_0^2\omega^2 + \omega^4 - 4\,\theta^2\right)\right]A^2\lambda_0^2\,\bar\Phi^2}{4\,M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]^2}.\end{aligned}\qquad(48)$$

Somando tudo:

$$J_E^{no} = \frac{\bar\Phi^2\lambda_0\left(2\,M\omega_0{}^2 + \lambda_0\gamma\right)}{2\gamma\,M\omega_0^2} + \frac{\left(\omega^2 + \omega_0{}^2\right)A^2\bar\Phi^2\lambda_0{}^2}{4\,M\left[\omega^2\left(\omega^2 - 2\,\omega_0^2 + 4\,\theta^2\right) + \omega_0^4\right]},$$

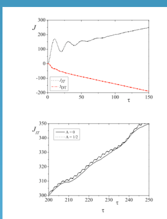

Figure 5. Upper panel: Total injected, $J_{IT}$, and dissipated power, $J_{DT}$, vs time, $\tau$, according to the definitions in the legend for the heterogeneous case $M = 10$, $k_0 = 1$, $\gamma = 1$, $\theta = 1$, $\lambda_0 = 10$ and $A = 1/2$. Lower panel: Comparison of $J_{IT}$ for the heterogeneous process with a homogeneous process ($A = 0$) under the same remaining parameters. The impact of the oscillations in the noise are clearly visible.

**Conclusões:**
Temos uma metodologia que permite uma análise detalhada e exata da ação do ruído sobre sistemas pequenos. Este tipo de abordagem pode ser estendido para tratarmos do comportamento térmico de proteínas e moléculas idealizadas.

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