Absorbing-State Phase Transitions:

Scaling and Universality

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Support: CNPq & Fapemig

OUTLINE

- Absorbing-State Phase Transitions
  - Basic Examples
  - Universality classes
  - Experimental Studies
- Contact process with diffusing vacancies
  - Relevance of mobile disorder
  - Weak dilution: apparent nonuniversality
  - Critical vacancy concentration: universality regained?
- Contact process with sublattice ordering

Absorbing state of a Markov process:

Consider a population of organisms, population size N(t)

N evolves via a stochastic dynamics with transitions from N to N+1 (reproduction), and to N-1 (death)

N=0 is an *absorbing state*: if N=0 at some time t, then N(t') = 0 for all times t' > t

Systems with spatial structure: **phase transitions** between active and absorbing states are possible in infinite-size limit

Of interest in population dynamics, epidemiology, self-organized criticality, condensed-matter physics, social system modelling...

General references on NEPT:
J Marro and R Dickman, *Nonequilibrium Phase Transitions in Lattice Models*, (Cambridge University Press, Cambridge, 1999).
H Hinrichsen, Adv. Phys. **49** 815 (2000).
G Odór, Rev. Mod. Phys. **76**, 663 (2004)

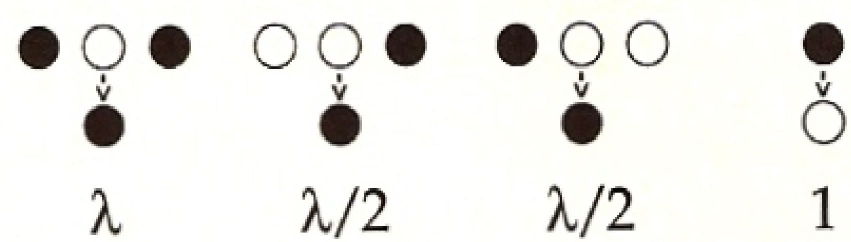
Contact Process (Harris 1972): a birth-and-death process with spatial structure

Lattice of L^d sites

Each site can be either active ($\sigma_i = 1$) or inactive ($\sigma_i = 0$)
An active site represents an organism

Active sites become inactive at a rate of unity, indep. of neighbors
An inactive site becomes active at a rate of λ times the fraction of active neighbors

The state with all sites inactive is ***absorbing***



Rates for the one-dimensional CP.

Contact Process: order parameter ρ is fraction of active sites

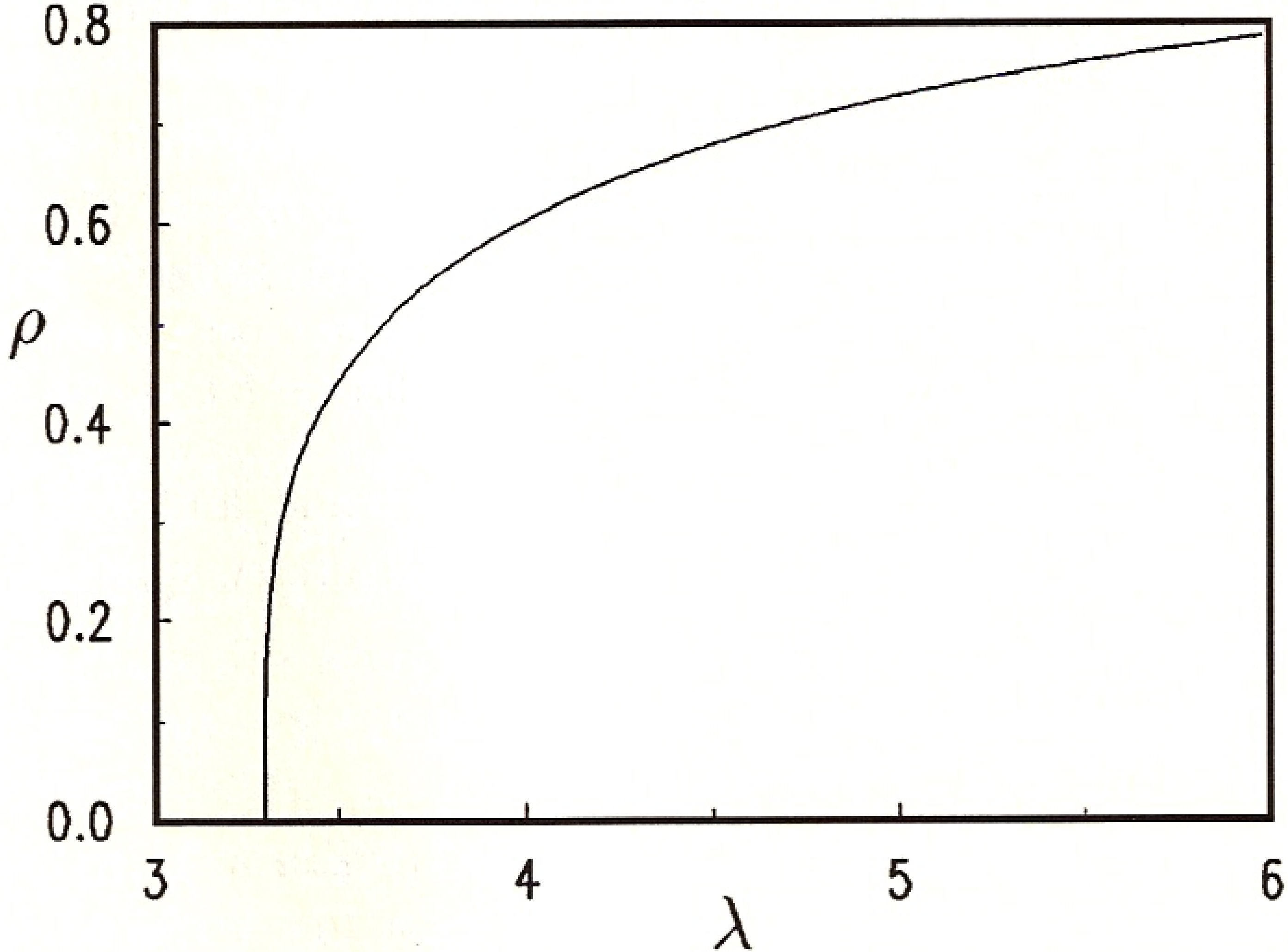
Rigorous results: continuous phase transition between active and absorbing state for d ≥ 1, at some λ_c (Harris, Grimmet...)

Order parameter: $\rho \sim (\lambda - \lambda_c)^{\beta}$

(Mean-field theory: $\lambda_c = 1$, $\beta = 1$)

Results for λ_c, critical exponents: series expansion, simulation, analysis of the master equation, ε-expansion

Types of critical behavior: static, dynamic, spread of activity



Order parameter in the one-dimensional contact process: series expansion analysis

Scaling behavior at an absorbing-state phase transition

Critical point: λ_c

Distance from critical point: $\Delta = \lambda - \lambda_c$

Order parameter: $\rho \sim \Delta^{\beta}$ for $\Delta > 0$

Correlation length: $\xi \sim |\Delta|^{-\nu_\perp}$

Lifetime: $\tau \sim |\Delta|^{-\nu_\parallel}$ for $\Delta < 0$

similarly for relaxation time

Weak external source h at $\lambda = \lambda_c$: $\rho \sim h^{1/\delta_h}$

Variance of order parameter: $L^d \, \mathrm{var}(\rho) \sim |\Delta|^{-\gamma}$

Spread of activity at critical point

Survival probability: $\mathsf{P(t)} \sim \mathsf{t}^{-\delta}$

Mean total activity: $\mathsf{n(t)} \sim \mathsf{t}^{\eta}$

Mean-square distance from seed: $\mathsf{R^2(t)} \sim \mathsf{t}^{z_{\mathsf{sp}}}$

For $\Delta > 0$: $\mathsf{P}_\infty \sim \Delta^{\beta'}$

scaling relation: $\delta = \beta/\nu_{||}$

hyperscaling: $2(1+\beta/\beta')\delta + 2\eta = dz_{\mathsf{sp}}$

For models in DP class, $\beta' = \beta$

subcritical critical supercritical

Spread of activity in contact process (avalanches)

Mean-field theory for the contact process

Let ρ denote particle density (probability that a given site is occupied)

$$\frac{d\rho}{dt} = -\rho + \lambda P(01)$$

where P(01) is joint probability for a given site to be empty, and its neighbor occupied

MF approximation: P(01) $\simeq \rho(1-\rho)$, giving

$$\frac{d\rho}{dt} = (\lambda - 1)\rho - \lambda\rho^2$$

i.e., the M-V equation

Bifurcation at $\lambda = 1$

$\rho_{st} = 1 - 1/\lambda$, for $\lambda > 1$.

Finite-size scaling

order parameter: $\rho(\Delta, \mathsf{L}) = \mathsf{L}^{-\beta/\nu_\perp} \mathsf{f}(\Delta \mathsf{L}^{1/\nu_{||}})$
At critical point, $\rho \sim \mathsf{L}^{-\beta/\nu_\perp}$

Lifetime at critical point: $\tau \sim \mathsf{L}^{\mathsf{z}}$
$\mathsf{z} = \nu_{||}/\nu_\perp = 2/\mathsf{z}_{\mathsf{sp}}$

Moment ratios of form $\langle \rho^2 \rangle / \langle \rho \rangle^2$ etc.
also take universal values at critical point

CP: finite-size scaling

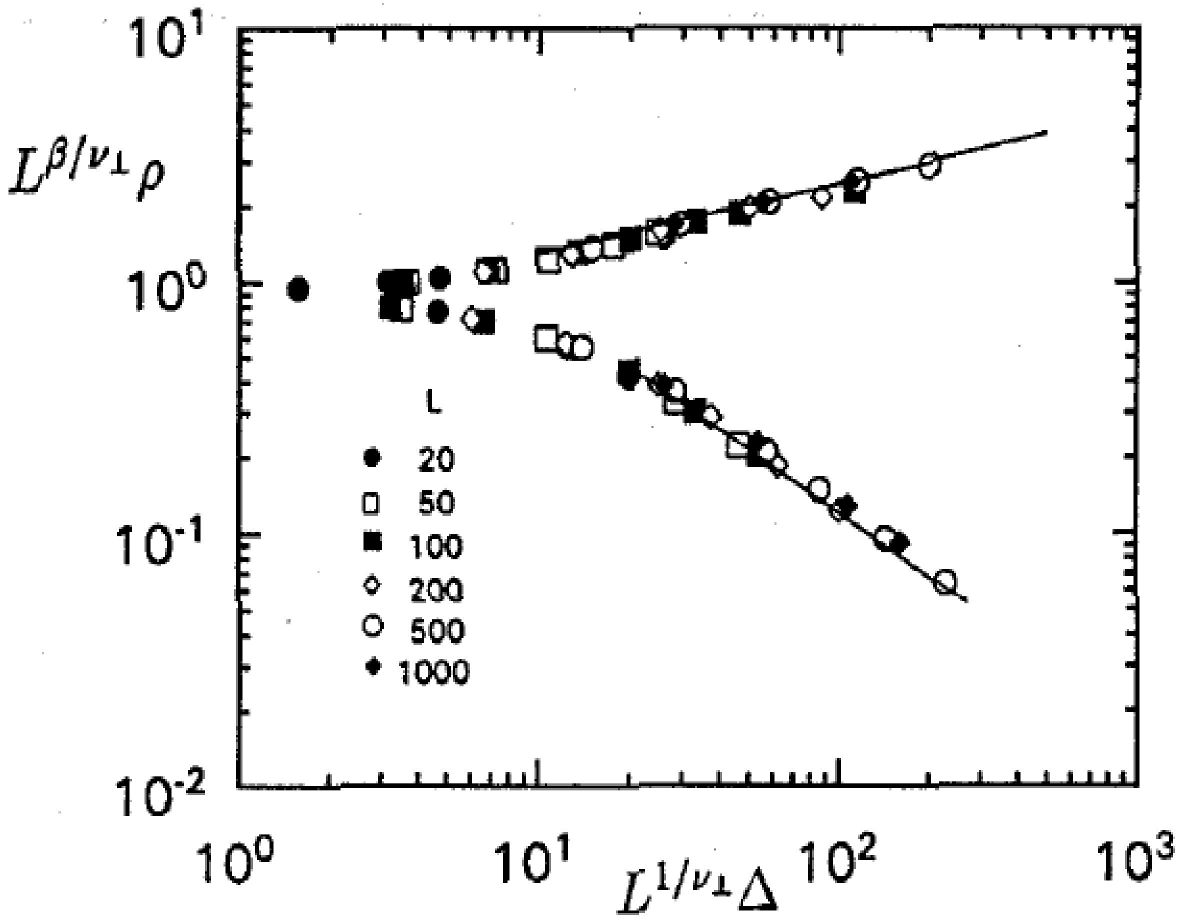


Fig. 6.6. Scaling plot of the quasistationary density in the one-dimensional CP. The slopes of the straight lines are 0.277 and −0.82.

Contact process on a ring: exact QS properties of small systems

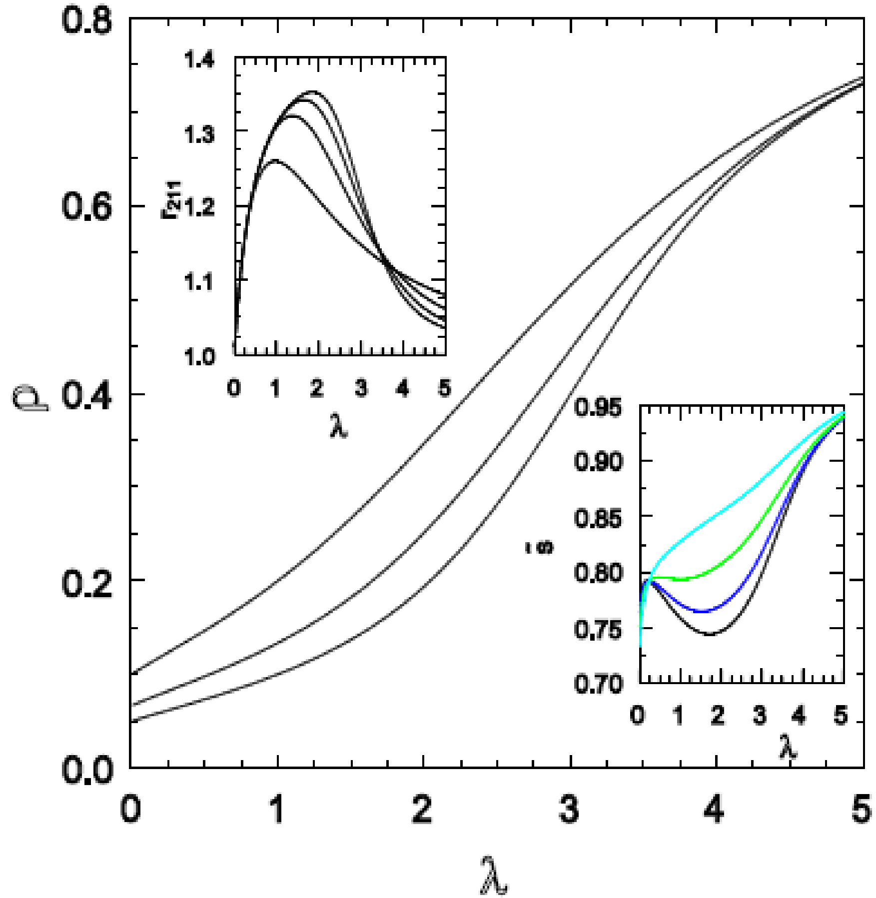


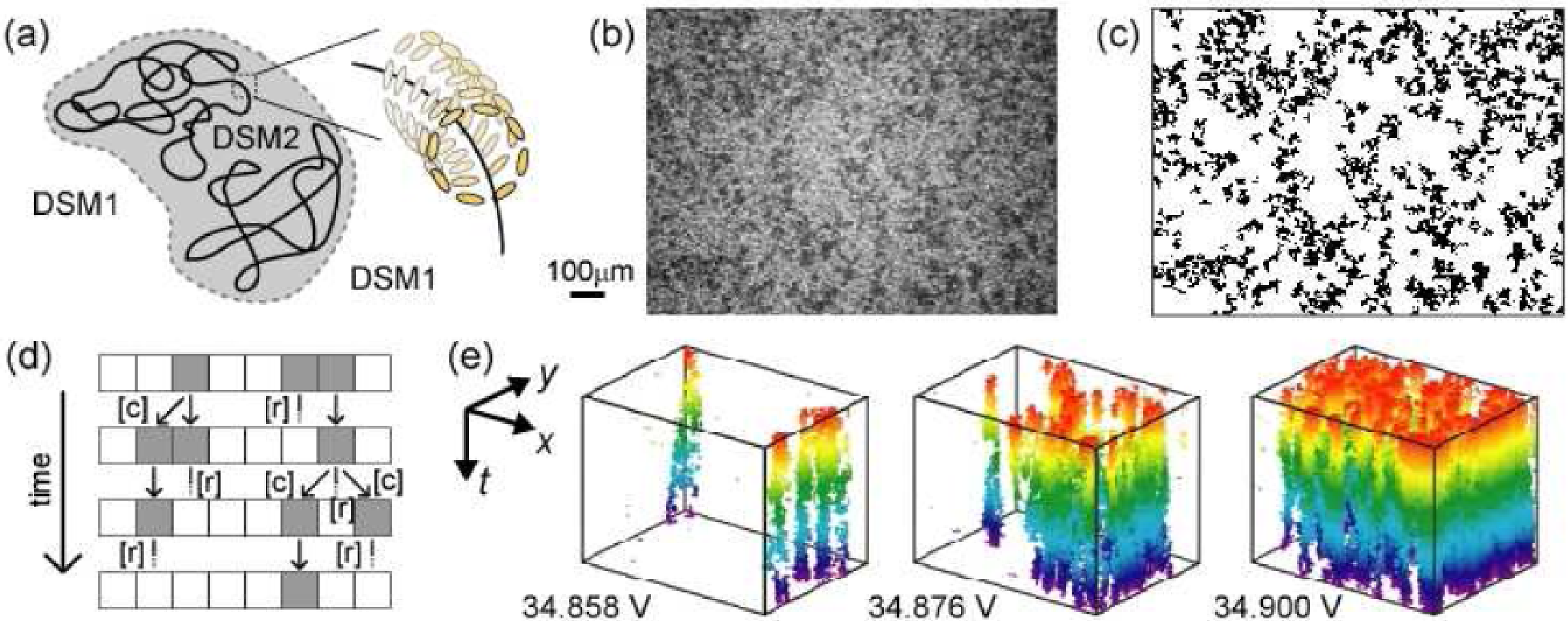
FIG. 1. (Color online) QS order parameter vs creation rate λ in the CP; system sizes $L=10$, 15, and 20 (upper to lower). Upper inset: moment ratio r_{211} for system sizes 5, 10, 15, and 20, in order of increasing maximum value. Lower inset: reduced entropy per site $\tilde{s}$ for sizes 12, 16, 20, and 23 (upper to lower).

Table 6.1. Critical creation rate and exponents for the contact process in d dimensions. Numbers in parentheses denote uncertainties.

	$d = 1$	$d = 2$	$d = 3$	$d = 4$
λ_c	3.29785(2)[a]	1.6488(1)[d]	1.3169(1)[h]	
β	0.27649(4)[b]	0.583(4)	0.805(10)[h]	1
δ_h^{-1}	0.111(3)[c]	0.285(35)[e]	0.45(2)[e]	$\frac{1}{2}$
γ	0.54386(7)[b]	0.35(1)	0.19(1)	0
$\nu_{\parallel}$	1.73383(3)[b]	1.295(6)[f]	1.105(5)[h]	1
$\nu_{\perp}$	1.09684(6)[b]	0.733(4)	0.581(5)	$\frac{1}{2}$
δ	0.15947(3)[b]	0.4505(10)[g]	0.730(4)[h]	1
η	0.31368(4)[b]	0.2295(10)[g]	0.114(4)[h]	0
z	1.26523(3)[b]	1.1325(10)[g]	1.052(3)[h]	1

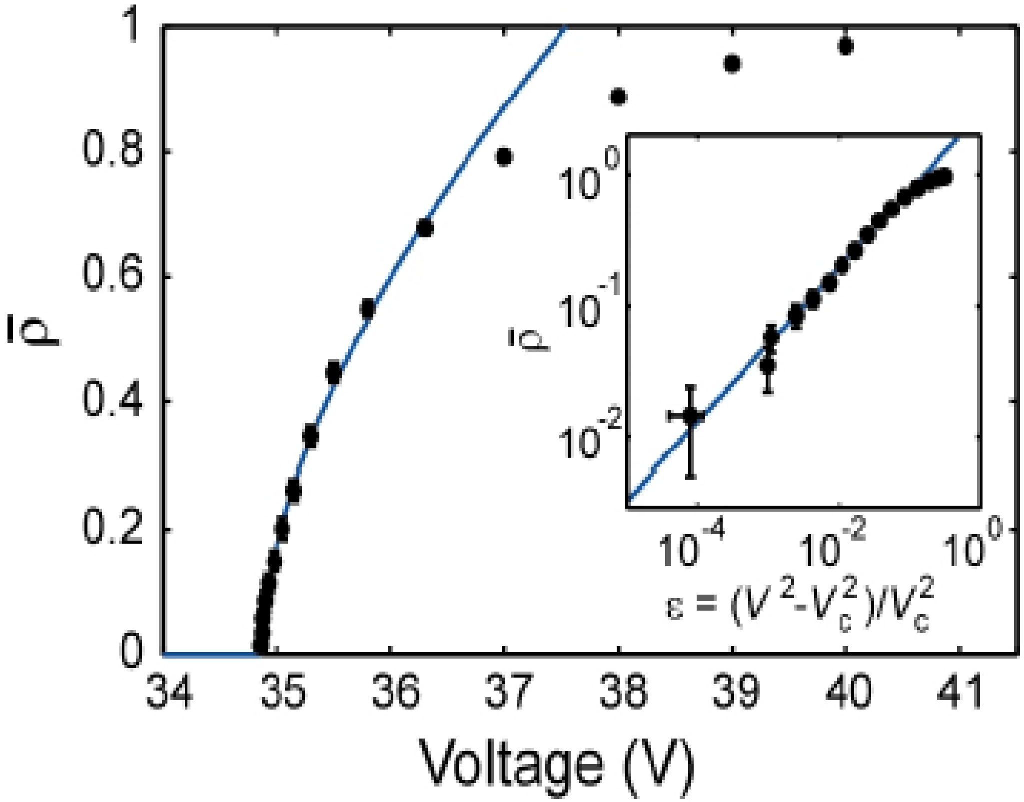
[a]Jensen & Dickman 1993b; [b]Jensen 1996; [c]Adler & Duarte 1987; [d]Grassberger 1989a; [e]Adler *et al.* 1988; [f]Grassberger & Zhang 1996; [g]Voigt & Ziff 1997; [h]Jensen 1992.

After 30 years!
Experimental realization of the contact process/directed percolation
(Takeuchi et al, PRL **99** 234503 (2007))



Absorbing-state phase transiton between two turbulent regimes in electrohydrodynamic convection of liquid crystals in a thin layer

Takeuchi et al: order parameter vs control parameter



Experiments confirm critical exponents of DP in 2 space dimensions, for example: $\beta = 0.59(4)$ (expt), $\beta = 0.583(3)$ (sim)

Physics

Physics **2**, 96 (2009)

Viewpoint

Observation of directed percolation—a class of nonequilibrium phase transitions

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Published November 16, 2009

Directed percolation, a class of nonequilibrium phase transitions as prominent as the Ising model in equilibrium statistical mechanics, is realized experimentally for the first time, after more than fifty years of research.

Subject Areas: **Statistical Mechanics, Soft Matter**

A Viewpoint on:
Experimental realization of directed percolation criticality in turbulent liquid crystals
Kazumasa A. Takeuchi, Masafumi Kuroda, Hugues Chaté and Masaki Sano
Phys. Rev. E **80**, 051116 (2009) – Published November 16, 2009

Principal universality classes of absorbing-state phase transitions:

Directed percolation (DP) (contact process)

Parity-conserving (branching-annihilating random walks)

Conserved DP* (conserved stochastic sandpile)

Pair contact process with diffusion (PDPC)

*Experiment: L Corté, P M Chaikin, J P Gollub and D J Pine, Nature Phys 2008
Transition between reversible and irreversible deformation in sheared colloidal suspension

Stochastic Sandpile Model (Manna model)

Conserved Version

A Markov process defined on a lattice of $\mathbf{L^d}$ sites with ***periodic boundaries***

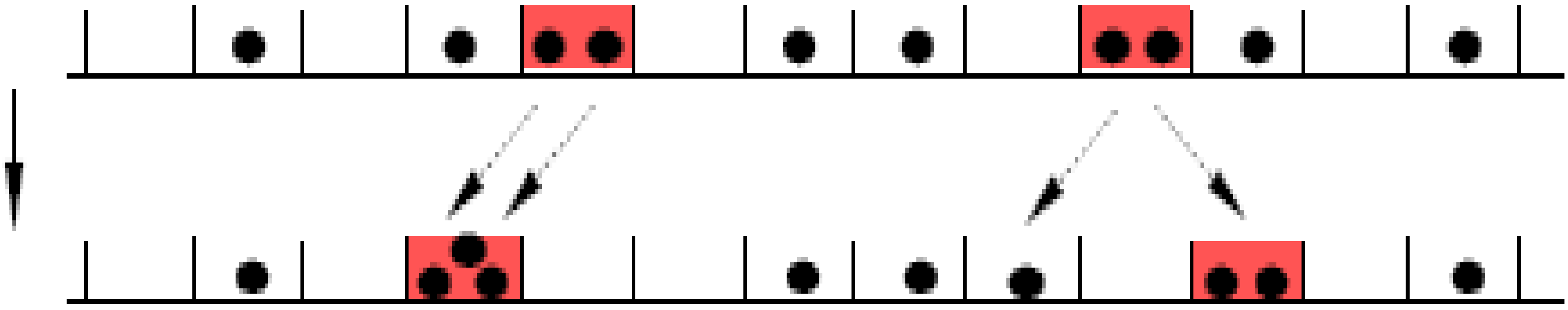
Particles perform random walks on the lattice
Let n_i denote the number of particles at site i (n_i = 0, 1, 2,...)

Initially **N** particles are distributed randomly over the lattice

Dynamics: any site with $n_i \geq 2$ is ***active***
Active sites ***topple*** at a rate of unity, sending two particles to nearest-neighbor sites, chosen at random

Number of particles remains **constant** throughout the evolution
Particle density $\zeta = N/L^d$ is a ***control parameter***

Examples of topplings in one dimension

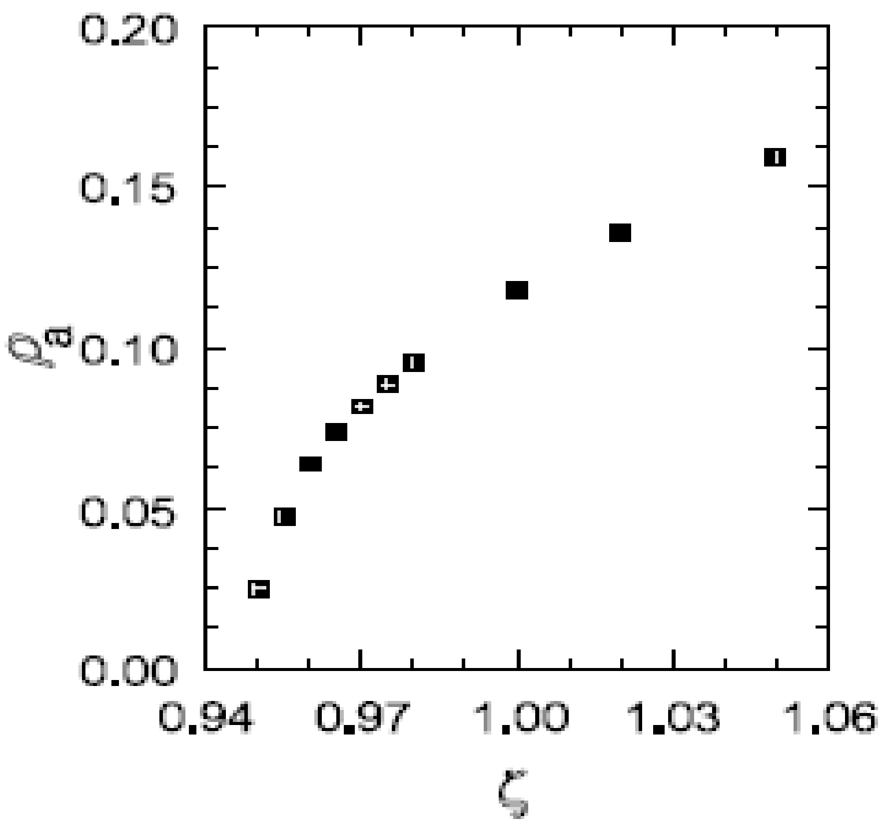


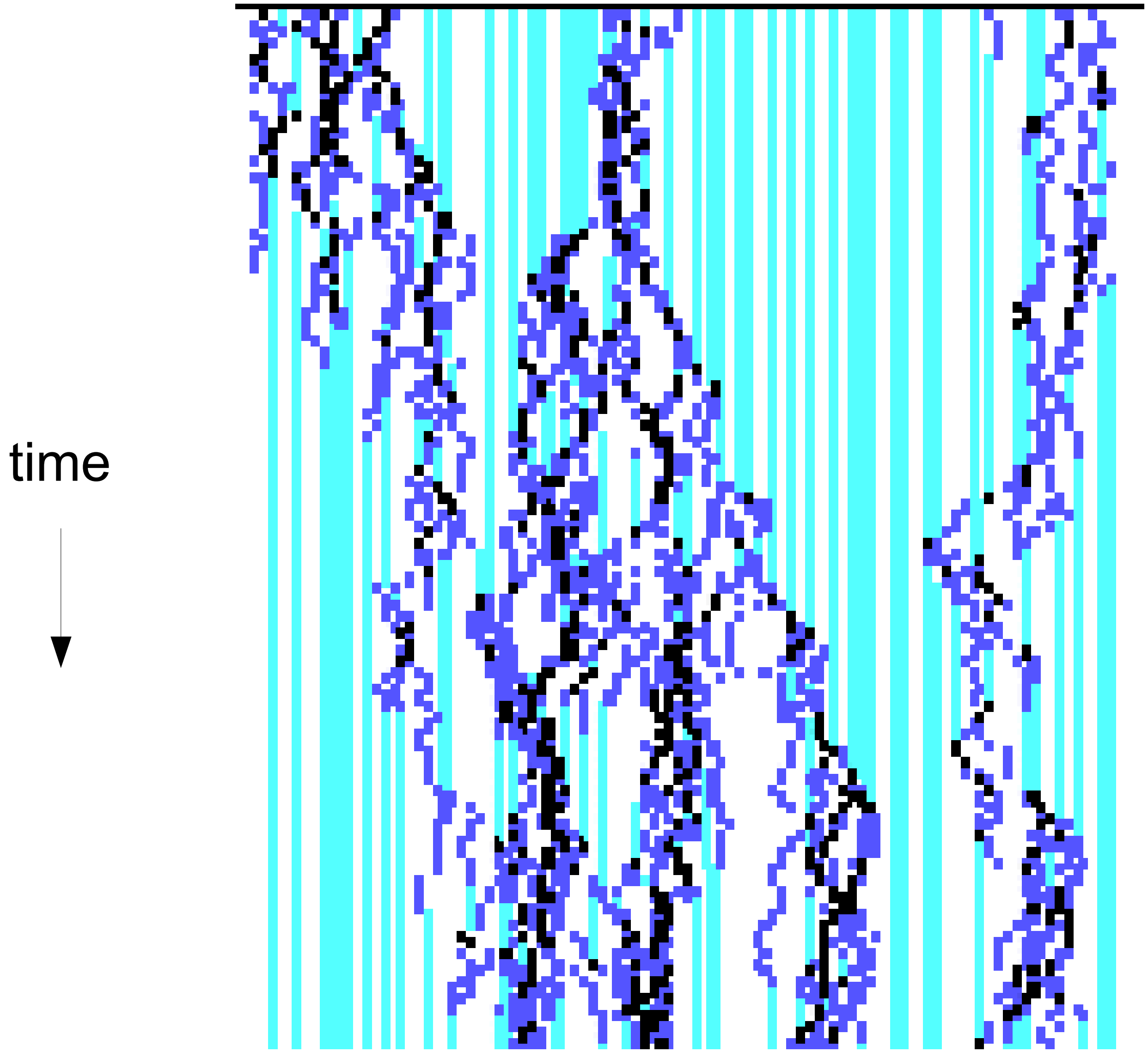
Stochastic sandpile: any configuration without active sites is absorbing

Such configurations exist for $\zeta < 1$

There is an continuous absorbing-state phase transition at $\zeta = \zeta_c$ (= 0.94885 in one dimension)

Order parameter: $\boldsymbol{\rho}$, the fraction of active sites





Typical evolution of stochastic sandpile

Like the contact process, the conserved stochastic sandpile exhibits scale-invariance at the critical point

To reach this point we must tune ζ to its critical value

Now we make two simple changes in the model:

1. Replace the periodic boundary condition with open boundaries
   When a site at the edge topples, particles may be lost

2. Eventually the system reaches an absorbing configuration
   When this happens a new particle is added at a randomly chosen site

This converts the model into the ***nonconserved*** Manna sandpile

These changes ***force*** the model to its critical point:

If $\zeta > \zeta_c$ there is activity and ζ can only decrease

If $\zeta < \zeta_c$ activity will stop and then ζ will increase

Nonconserved Manna sandpile: avalanche distribution

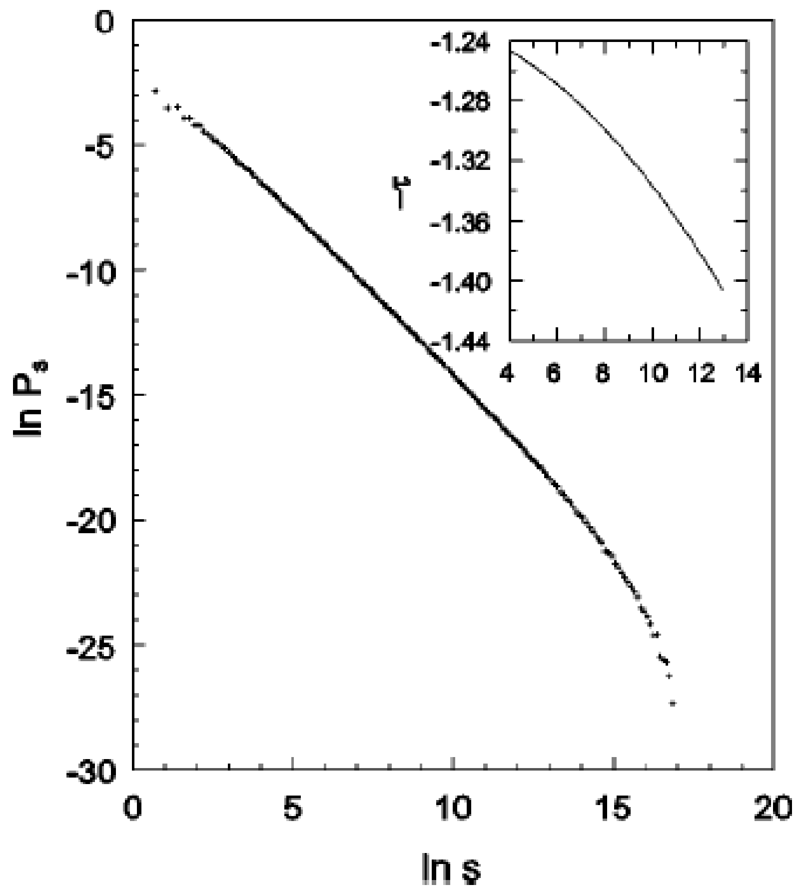


FIG. 1. Main graph: avalanche size distribution for the two-dimensional model, $L=1280$; the data points are simulation results and the superposed smooth curve a cubic fit to the data on the interval $4<\ln s<13$. Inset: estimate for the slope $-\tau$ of the cubic fit, versus $\ln s$.

Absorbing-state mechanism for SOC:
(A Vespignani, RD, S Zapperi & M A Muñoz, PRE 2000)

self-organized criticality in a slowly driven system corresponds to an absorbing-state phase transition in the model with the same local dynamics, but with strict conservation

Simulations confirm that the critical exponents in SOC and in the absorbing phase transition are related

As the system size increases, the fluctuations of ζ in the ***driven*** sandpile are restricted to an ever smaller region centered on the critical density of the ***conserved*** model

The SOC and absorbing "ensembles" are however distinct
(Pruessner and Peters, Phys. Rev. E, 2006, arXiv:0912.2305)

In deterministic sandpiles, the critical density in the conserved version is a ***tiny*** bit higher than in the SOC version! (Fey et al., Phys Rev Lett, 2010)
But the two densities are ***the same*** if system starts with a larger negative height! (Poghosyan et al, arXiv:1104.3458)

Critical exponents for 1d conserved stochastic sandpile/CDP

TABLE I. Summary of exponent values for one-dimensional models in the CDP universality class. L_{max} denotes the largest system size studied. Abbreviations: CAM, coherent anomaly method; FT, field theory.

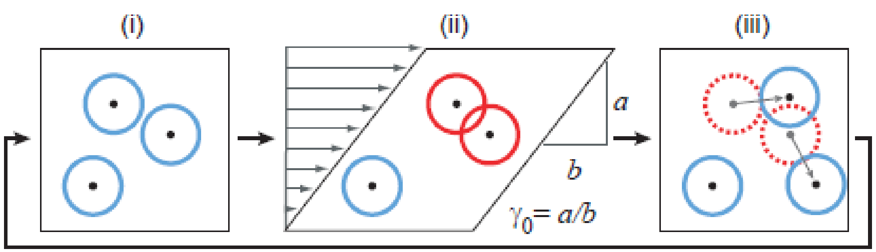
Model	L_{max}	β	$\beta/\nu_\perp$	z
Manna [15]	10000	0.42(2)	0.24(1)	1.66(7)
Manna [26]	8192		0.28(3)	1.39(11)
CTTP [18]	131072	0.38(2)	0.24(1)	1.66(7)
Rest. Manna [16]	5000	0.416(4)	0.246(5)	1.50(9)
Rest. Manna CAM [17]		0.41(1)		
CDP [20]	4.2×10^6	0.29(2)		1.55(3)
CDP FT [13]	4000	0.28(2)	0.214(8)	1.47(4)
Rest. Manna (present work)	50000	0.289(12)	0.213(6)	1.50(4)
DP [28]		0.2765	0.2521	1.5807

CDP class: still no precise value for z

Experimental realization of the CDP transition

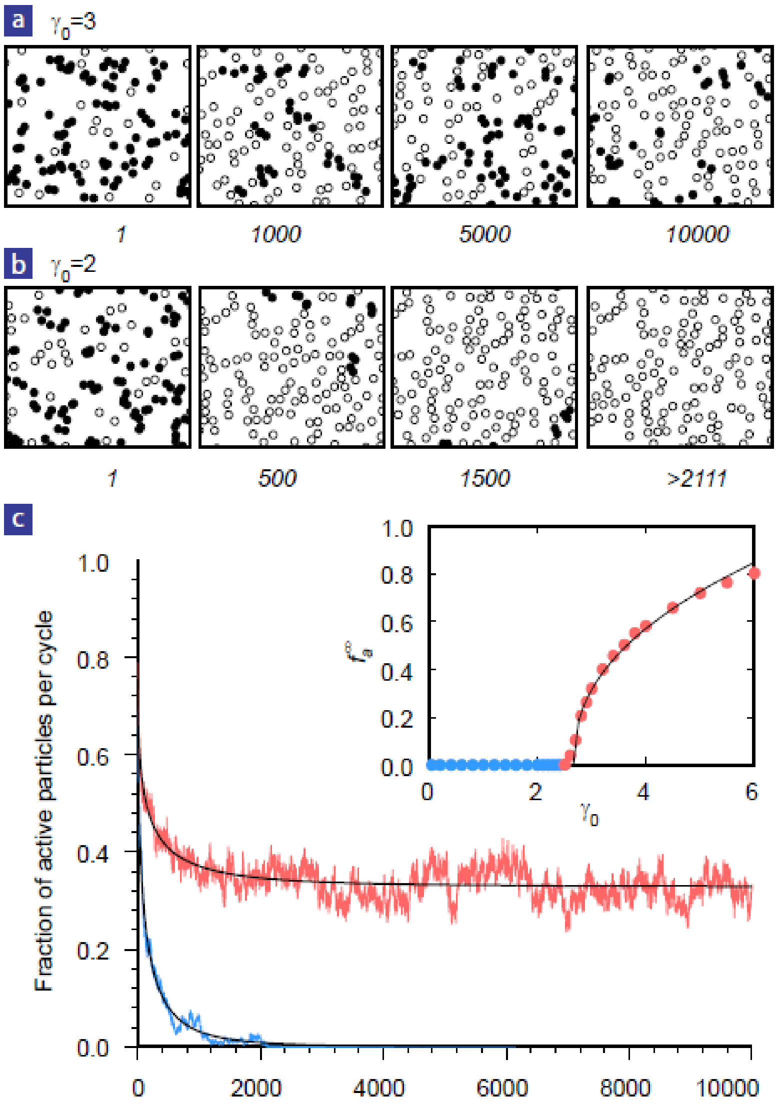
L Corté, P M Chaikin, J P Gollub and D J Pine, Nature Phys 2008

Transition between reversible and irreversible deformation in sheared colloidal suspension

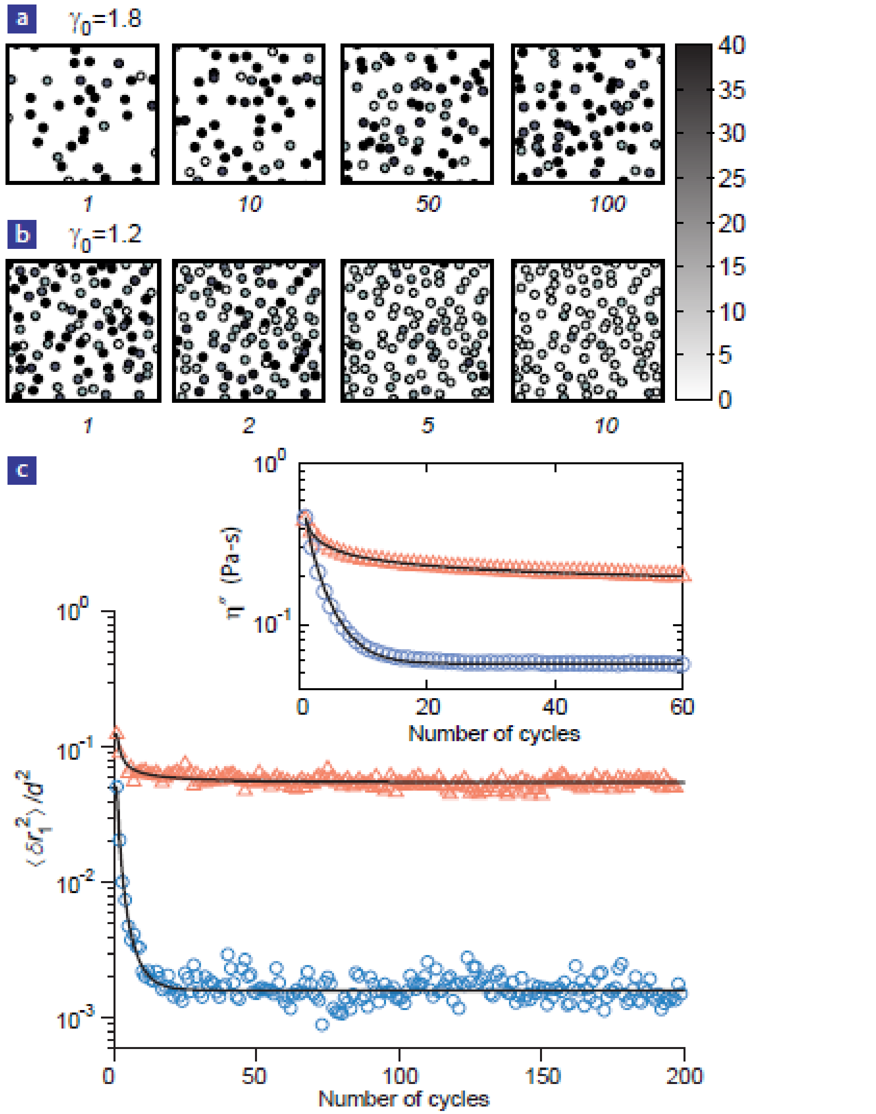


Non-Brownian colloidal suspension in Couette geometry: deformation is reversible at low shear rates, for which particles arrange to avoid collisions
Above a critical shear rate the deformation is irreversible
Corté et al interpret irreversible particle motion as activity, observe a phase transition between absorbing and active phases

Corté et al 2008



model



experiment

Interpretation of experiment of Corté et al:
(Menon & Ramaswamy, PRE 2009)

The system suffers an absorbing-state phase transition

Activity is carried by the particles, which are conserved

This suggests that the transition belongs to the conserved DP universality class – no precise results for critical exponents yet

Corté et al, PRL 2009: Study the same system but with slow sedimentation
Now the bulk particle density is subcritical
Under shear the density profile exhibits a plateau at the critical density, verifying absorbing-state transition/SOC connection

Corté et al 2009

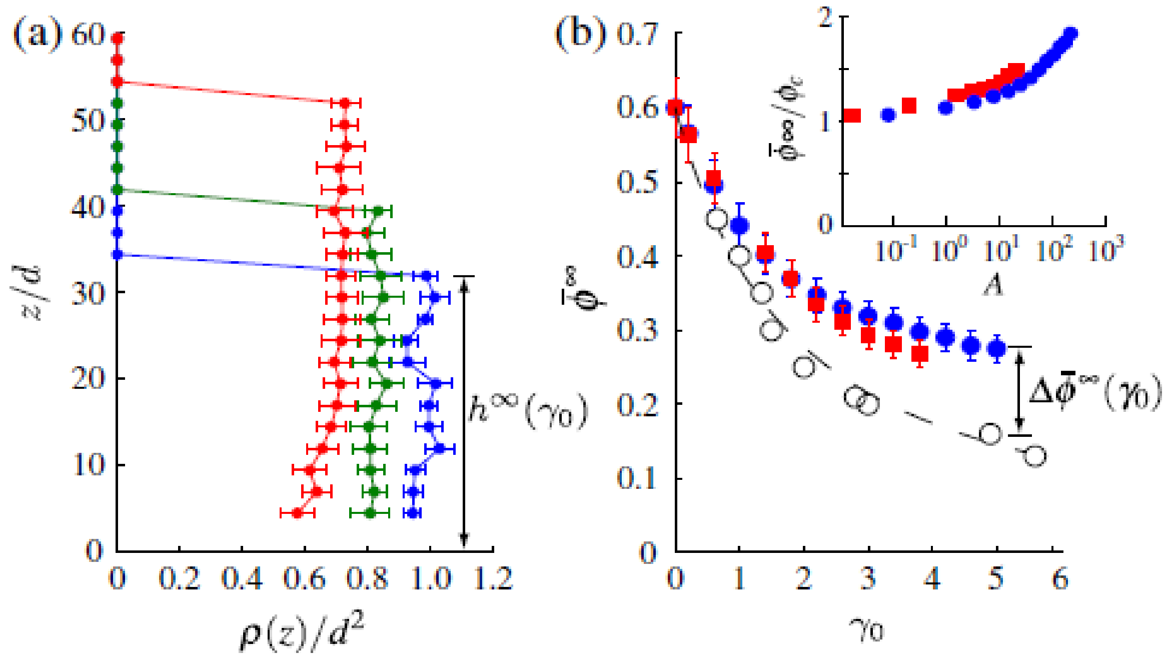


FIG. 4 (color online). Experiment: (a) Particle density profiles in steady state for $\gamma_0 = 2.8$ (left), 1.0 (middle), and 0.2 (right). Profiles are averaged over 50 shear cycles. Error bars correspond to standard deviations. (b) Steady-state volume fraction as a function of γ_0. Solid symbols show mean volume fraction $\bar{\phi}^{\infty}(\gamma_0)$ measured in the sedimenting case for two different oscillation periods (●: 25 s, ■: 5 s). Open symbols (○) show the critical line $\phi_c(\gamma_0)$ as measured with neutrally buoyant suspensions. The line is a power-law fit scaling as $(1/0.6 + \gamma_0)^{-1}$.

Effect of disorder on the contact process

Harris criterion ($d\nu < 2$): quenched disorder relevant for contact process (CP) and directed percolation (DP)
(For recent perspective: T Vojta and M Dickison, PRE **72**)

Harris criterion for CP:

Local fluctuation in λ (temperature-like variable) is ~v
In a block of length b, summed fluctuation is $\sim b^{d/2}$ v, by central limit theorem

Treat this as equivalent to a uniform variation over block, $\sim v b^{-d/2}$

Under a block transformation (Kadanoff) $v \to v' = v b^{y} b^{-d/2}$

Note: $y = 1/\nu_\perp$
Then disorder is relevant if $d\nu_\perp < 2$

Effect of disorder on the contact process

By the Harris criterion, quenched disorder is expected to be relevant for contact process/directed percolation
(For recent perspective: T Vojta and M Dickison, PRE **72**)

What about **diffusing disorder**?

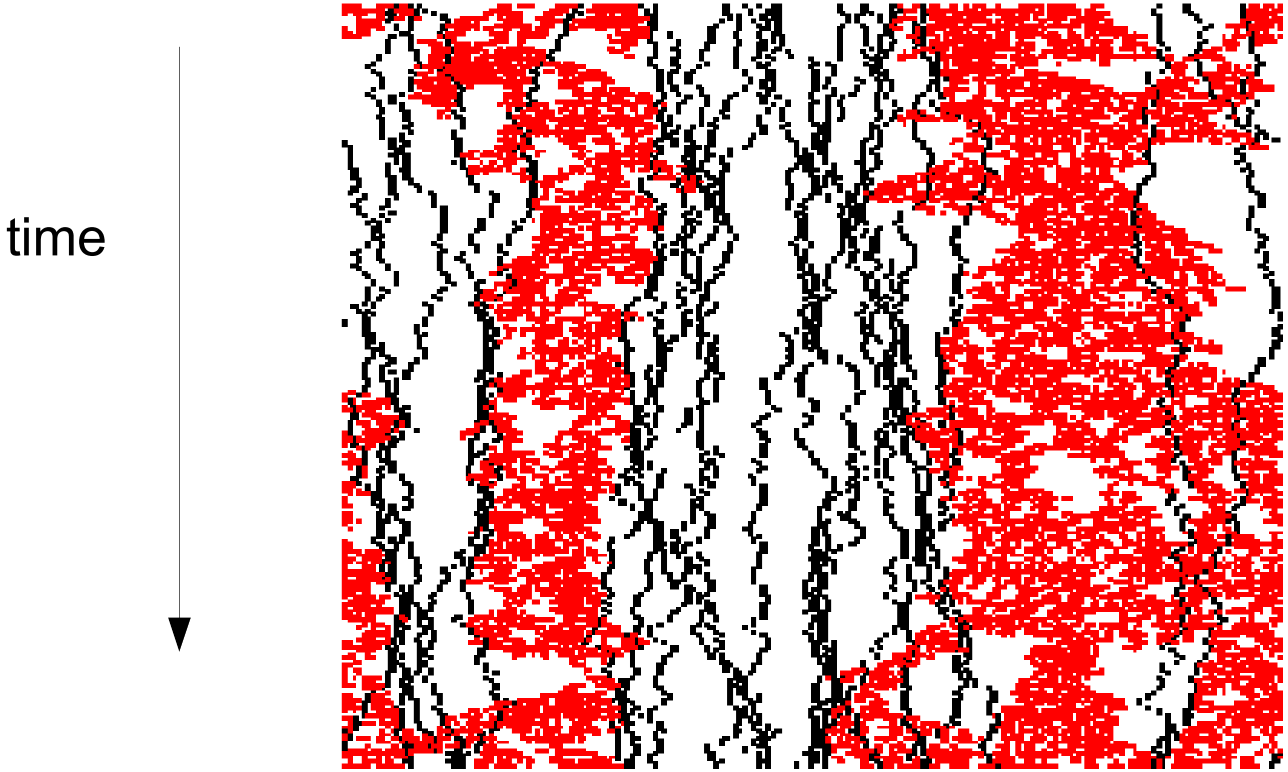
Model: Contact process with mobile vacancies (CPMV)

Vacancies are permanently inactive but diffuse at rate D, exchanging positions with the other sites, which host a basic contact process (Individuals with permanent immunity)

A fraction **v** of sites are vacancies

Nondiluted sites may be active or inactive

CP with mobile vacancies: simulation in one dimension



Typical evolution near critical point. Red: active; black: vacancies
v=0.1, D=1, λ = 4.1

Related model: CP with diffusive background (Evron et al., arXiv:0808-0592)
"good" (large λ) and "bad" (small λ) sites instead of vacancies

In principle both models should have the same continuum description:

$$\partial_t \boldsymbol{\rho} = D_a \boldsymbol{\nabla}^{\mathbf{2}} \rho + (a+\gamma\boldsymbol{\phi})\boldsymbol{\rho} - b\boldsymbol{\rho}^{\mathbf{2}} + \boldsymbol{\eta}(x,t)$$

$$\partial_t \boldsymbol{\phi} = \boldsymbol{\nabla}^{\mathbf{2}}\boldsymbol{\phi} + \boldsymbol{\nabla}\bullet\boldsymbol{\xi}(x,t)$$

$\boldsymbol{\rho}$: order parameter density; $\boldsymbol{\phi}$: density of nondiluted (or "good") sites

η and ξ are suitable noise terms.

Mobile disorder is relevant for finite D

Consider a correlated region in the CP, with characteristic size ξ and duration τ

If fluctuations in the vacancy density on this spatial scale relax on a time scale $\tau_\phi \ll \tau$, then the CP will be subject, effectively, to a disorder that is uncorrelated in time, which is ***irrelevant***

But fluctuations in ϕ relax via diffusion, so $\tau_\phi \sim \xi^2$

In the neighborhood of the critical point, $\xi \sim |\lambda - \lambda_c|^{-\nu_\perp}$

and $\tau \sim \xi^z$, so that $\tau_\phi \sim \tau^{2/z}$

This suggests that diffusing disorder is **relevant** for z < 2, provided that quenched disorder is relevant

In directed percolation these conditions are satisfied in d < 4 space dimensions

CP with mobile vacancies: limiting situations

D = 0: In ***one-dimension,*** this corresponds to a CP on finite strips, which must always fall in the absorbing state.
Thus for any $v > 0$, $\lambda_c \rightarrow \infty$ as $D \rightarrow 0$.

In ***two or more dimensions,*** the CP with fixed vacancies is active (for suff. large λ) if nondiluted sites percolate ($v < 1\text{-}pc$).
Thus $\lambda_c \rightarrow \infty$ as $D \rightarrow 0$ for $v > 1\text{-}pc$

D$\rightarrow \infty$: In ***one dimension***, diffusing vacancies do not change order of active and inactive (nondiluted) sites
Thus $D \rightarrow \infty$ is not a mean-field limit
Instead it represents a regular CP with $\lambda_{eff} = (1-v)\lambda$, so one expects
$\lambda_c \rightarrow \lambda_{c,pure} / (1-v)$, with DP scaling, in this limit

In two or more dimensions $D \rightarrow \infty$ should correspond to a mean-field limit

Studies of CPMV in one dimension

(RD, J Stat Mech (2009) P08016)

Determine λ_c and scaling properties as functions of vacancy fraction v and diffusion rate D

Three kinds of simulation:

conventional (stationary regime)

quasistationary

spreading

A “first look”: moderate dilution (v=0.1), vary D

Monte Carlo simulations

Rings of $L = 100, 200,...,1600$ sites - all nondiluted sites initially active

Determine (1) fraction $\boldsymbol{\rho}(t)$ of active sites

(2) moment ratio $m(t) = \boldsymbol{< \rho^2 > / \rho^2}$ in averages over surviving realizations

(3) mean lifetime τ from the decay of the survival probability, $P_s(t) \sim \exp[-t/\boldsymbol{\tau}]$

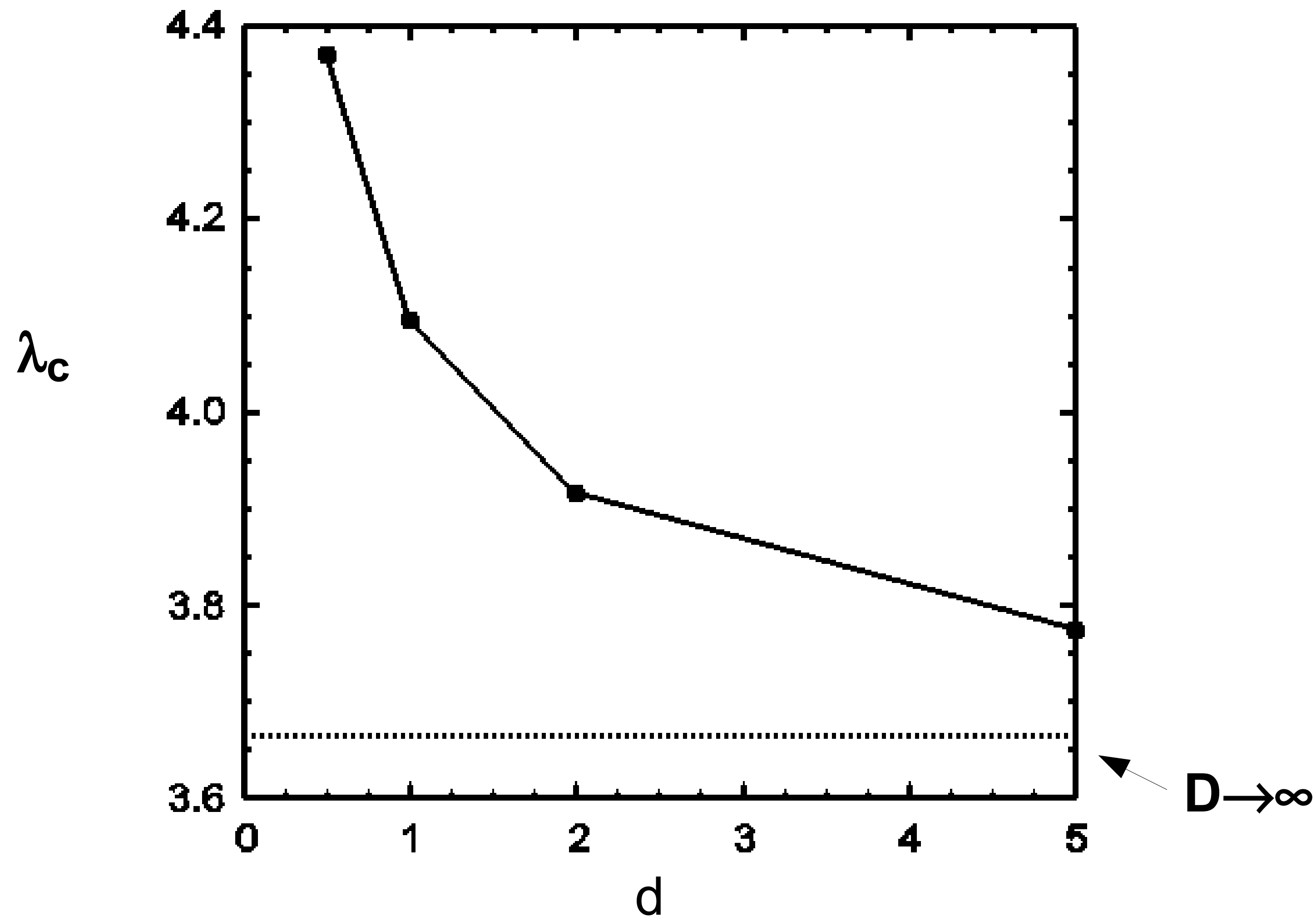
In large (pure) systems at critical point, ρ and m approach their quasistationary (QS) values via

$$\rho(t) \sim t^{-\delta} \quad \text{and} \quad m(t) - 1 \sim t^{1/z}$$

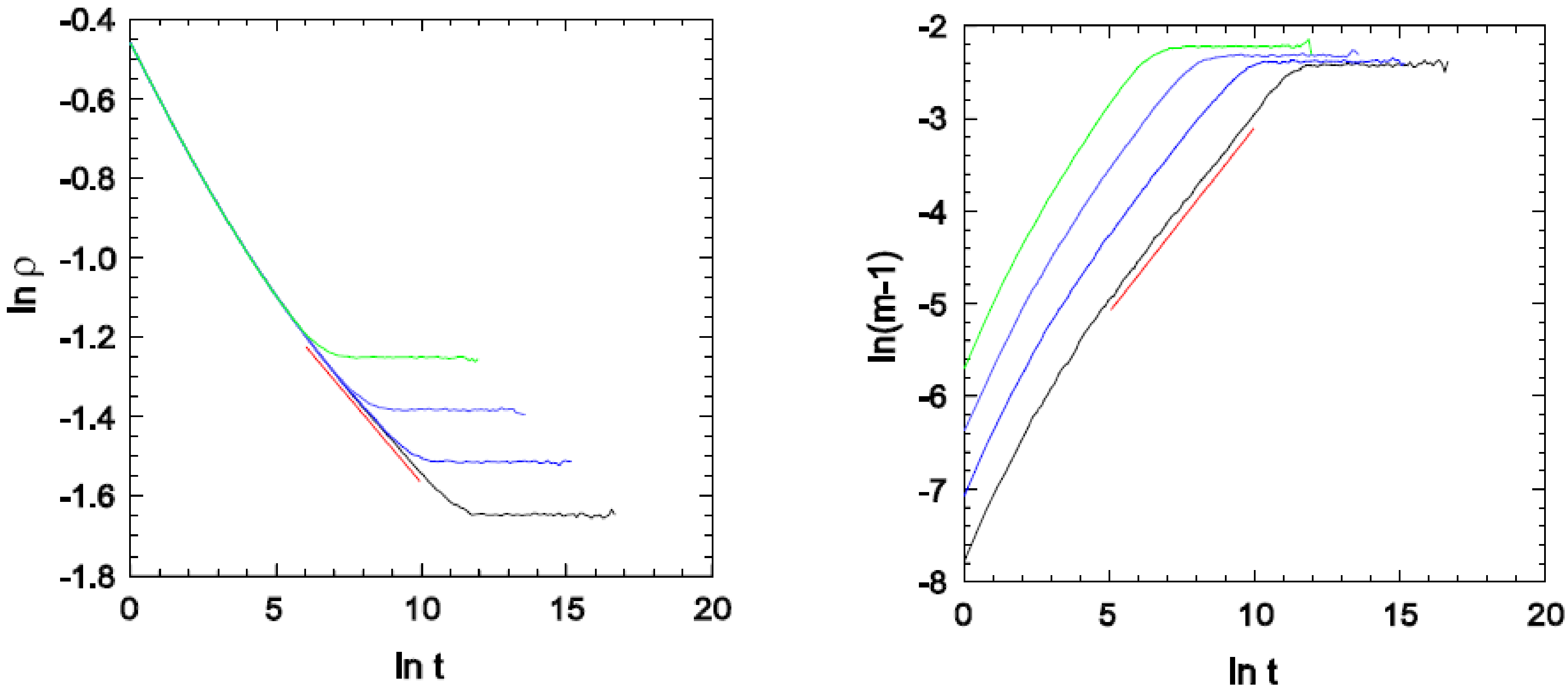
Finite-size scaling: at the critical point, $\boldsymbol{\rho_{QS}} \sim L^{-\beta/\nu_\perp}$,

$$\tau \sim L^{z} \quad \text{and} \quad m \to m_c \quad \text{(a universal quantity)}$$

Phase boundary, v=0.1



Anomalous behavior: m(t) and ρ (t) cannot be collapsed



Summary of Results for v=0.1

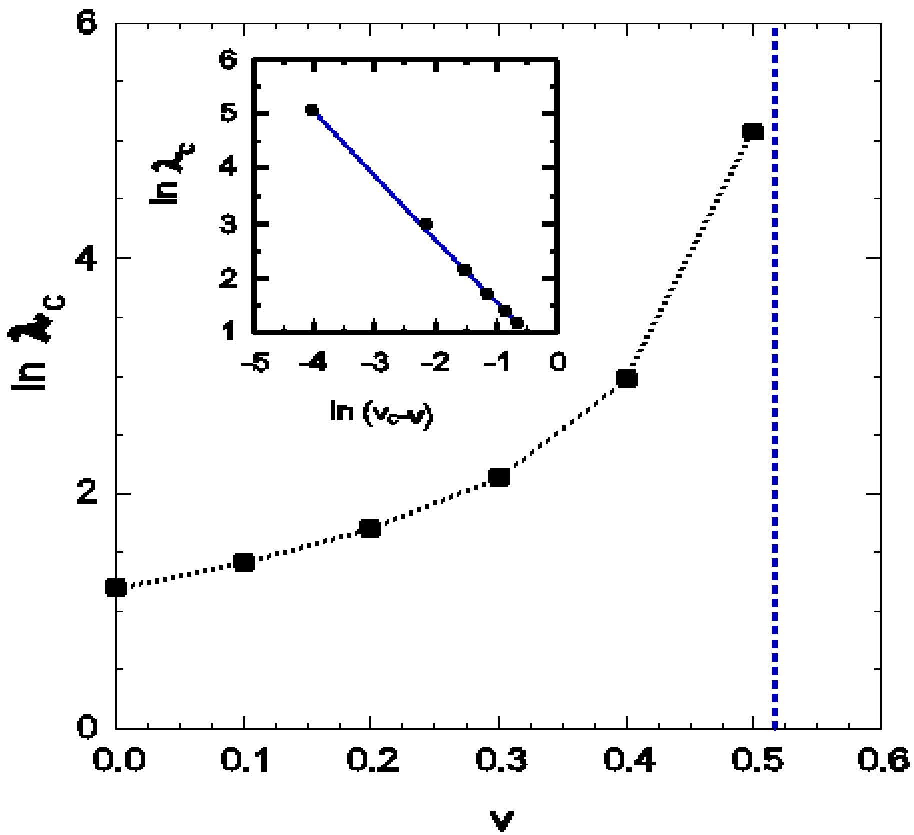
Critical exponents z, δ, $\beta/\nu_\perp$, and moment ratio m_c appear to vary continuously with vacancy diffusion rate d, and approach DP-class values as d increases

Spreading simulations confirm scaling of survival probability, $P \sim t^{-\delta}$ but other quantities show anomalous scaling

The lifetime τ grows more slowly than a power law at the critical point, for small D

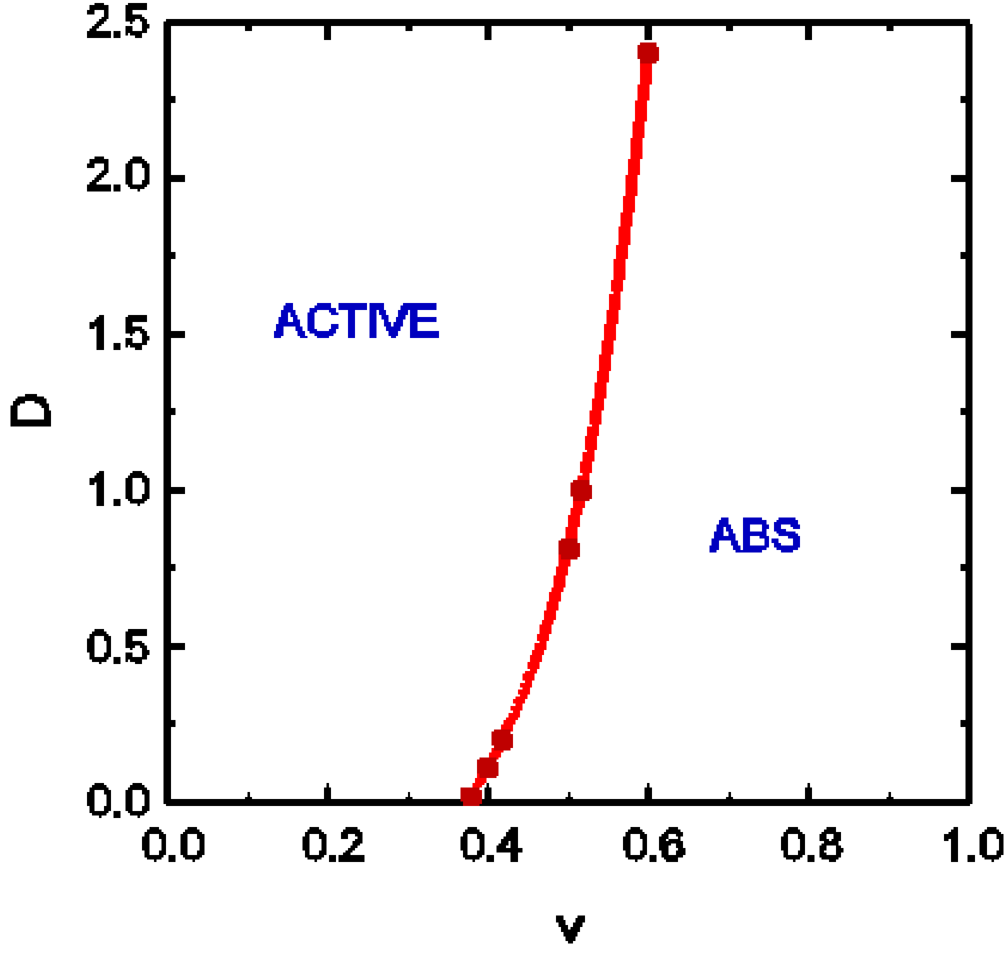
Summing up, static scaling is observed, but certain aspects of time-dependent behavior are anomalous.

A second look: CPMV at the Critical Vacancy Density



For fixed diffusion rate D, critical reproduction rate λ_c grows with vacancy density v and *diverges* at v_c(D)

Critical vacancy density line in the v-D plane (simulation)

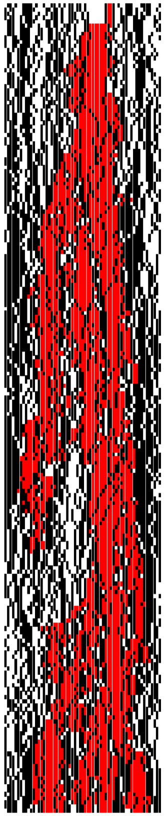


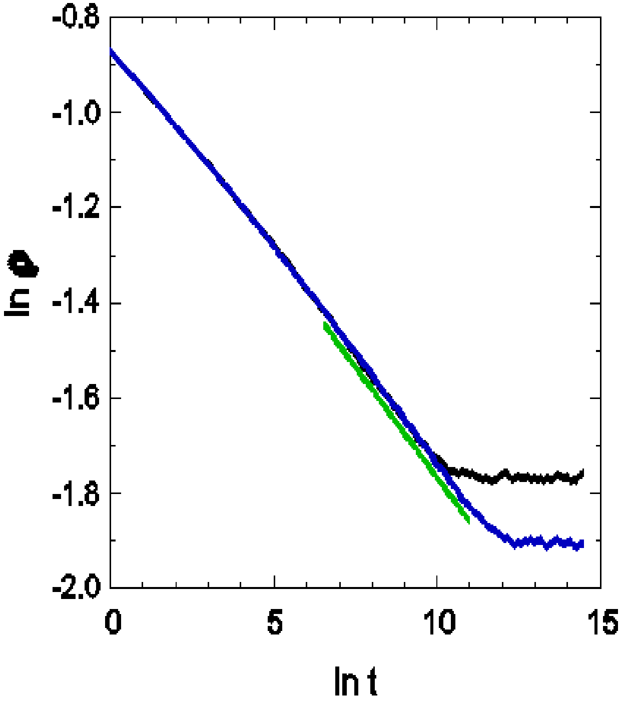
For v < 0.38, λ_c diverges only when D $\longrightarrow$ 0

Simulation with $\lambda = \infty$: allow only *isolated* active sites to become inactive (at a rate of unity), and activate any nondiluted site the instant it gains an active neighbor

Typical evolution starting from a single active site

D=1, v=0.515





Simpler scaling behavior at v_c than for smaller v

Critical properties along the critical vacancy density line

D	v_c	$\beta/\nu_\perp$	m_c	z_m	δ_c	δ_s	η	z_s
0.2	0.4182(5)	0.174(6)	1.083(3)	1.95(4)	0.087(2)	0.086(2)	0.303(3)	0.95(1)
1	0.517(1)	0.184(20)	1.084(11)	1.98(3)	0.091(4)	0.086(2)	0.307(1)	0.965(10)
DP	—	0.2521(1)	1.1736(1)	1.580 74(4)	0.159 47(3)	$(=\delta_c)$	0.313 68(4)	1.265 23(3)

Similar results are found for v=0.4, 0.5, and 0.6

The hyperscaling relation $4\delta + 2\eta = dz$ is satisfied to within uncertainty

These results suggest that critical exponents are ***independent*** of D along the critical line v_c

Contact process with mobile vacancies - Summary

Simple scaling behavior at critical vacancy density, with clearly non-DP critical exponents, possible connection to DEP

For smaller v, apparently variable exponents: Is this a crossover between DP and a new fixed point?

Future work:

Map out $v_c(D)$ and associated exponents with higher precision, verify universality along this line of critical points

Apply exact QSD analysis, series expansions

Two and three dimensions

Investigate other forms of slowly evolving disorder, and effect of mobile vacancies on other classes of absorbing-state phase transitions

Contact process with sublattice symmetry breaking
(Marcelo Martins de Oliveira & RD)

Motivation: can the CP exhibit a patterned phase

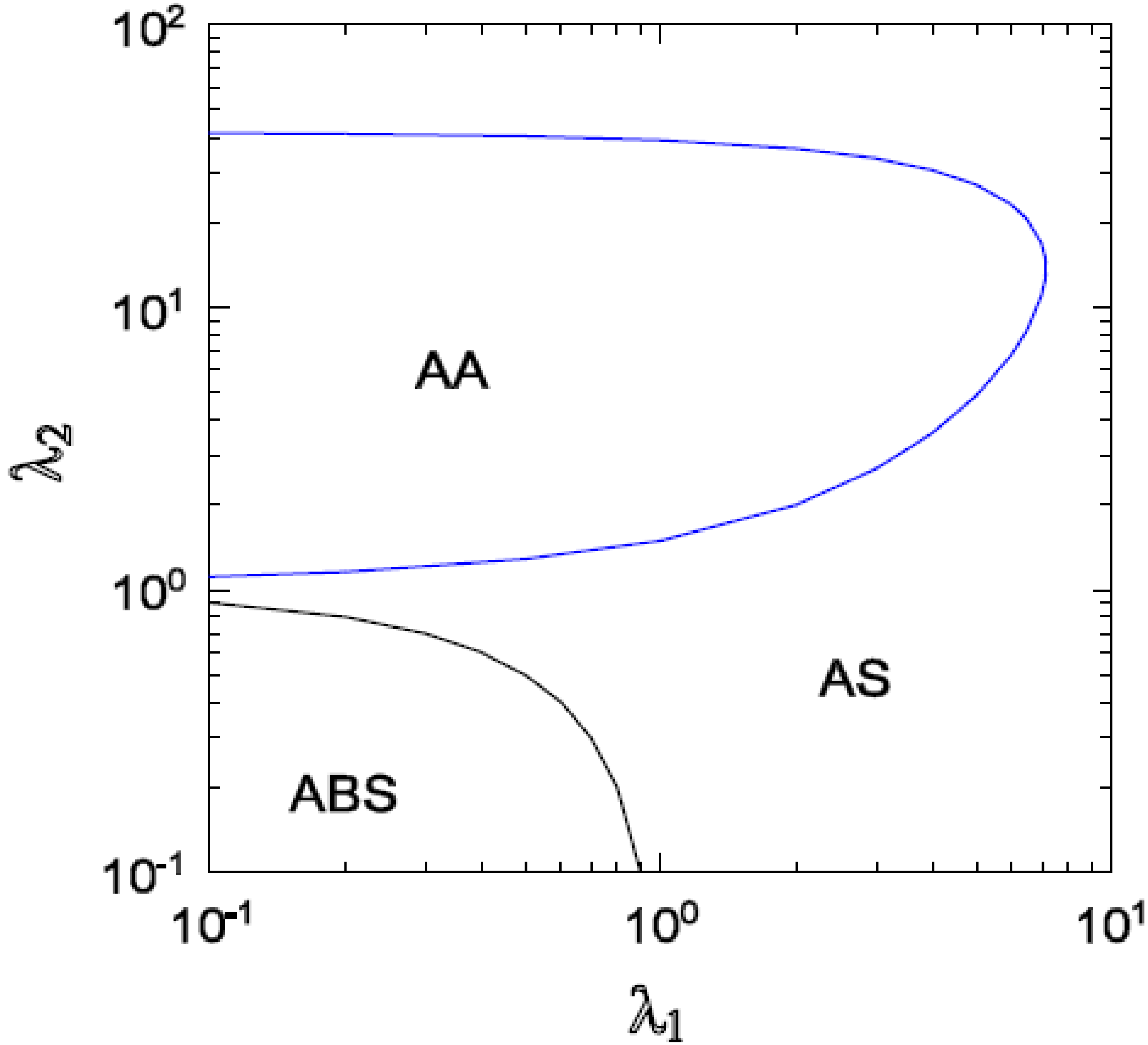
Model: CP with creation rates λ_1 and λ_2 at first and second neighbors, resp.

Basic annihilation rate of unity, enhanced by μn^2 at a site with n occupied first neighbors

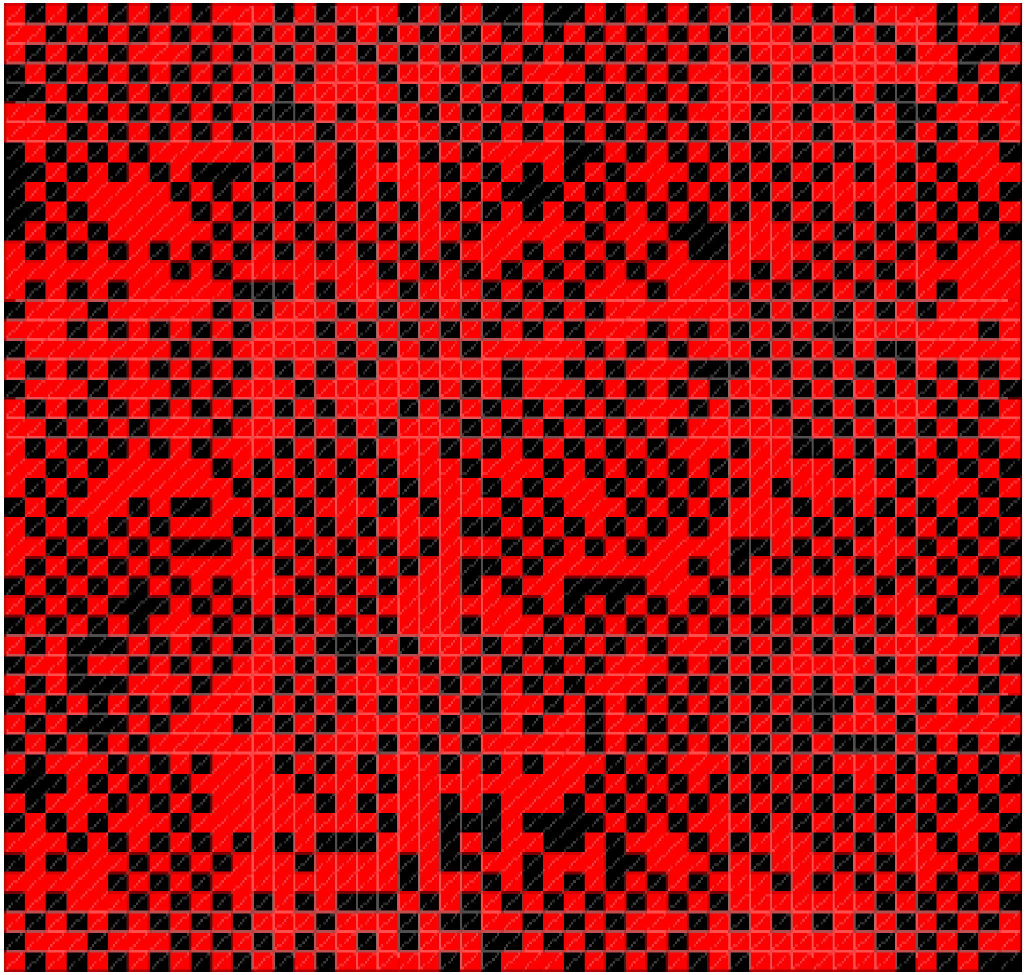
Mean-field theory: three phases, absorbing, active-symmetric and active-asymmetric, separated by continuous phase transitions; re-entrant phase diagram

AS/AA transitions should be Ising-like

CP with sublattice ordering: MFT, μ= 2



Simulation yields a qualitatively similar phase diagram



Typical configuration in the AA phase (simulation, μ =0.2)

On the line λ_1=0, direct transition from ABS to AA phase, followed by a transition to the AS phase at higher λ_2

For μ > 0 this line represents competeing species, with $\lambda_{2,c} = \lambda_{c,CP}$

μ < 0 corresponds to symbiosis, $\lambda_{2,c} < \lambda_{c,CP}$, and transition appears to belong to a new universality class

SUMMARY

A number of universality classes for absorbing-state phase transitions are now well characterized

Experimental confirmation of DP scaling, and possibly of CDP and self-organized criticality

It is likely that many more classes remain to be discovered, as one introduces new components, symmetries and/or conserved quantities

Relevance of diffusive disorder in the contact process

Possibility of patterned activity