q-generalization of the inverse Fourier transform

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q-moments and the q-Fourier transform

Let $Q \in \mathbb{R}$ and f be a probability density function (p.d.f.) such that

$$\boxed{\nu_Q[f] = \int_{-\infty}^{\infty} [f(x)]^Q \, dx < \infty \, .}$$

We can define an **escort p.d.f.**, namely $f_Q(x) = [f(x)]^Q / \nu_Q[f]$. Using this p.d.f. we can construct the Q-**moments**, $\Pi_Q^{(n)}[f]$:

$$\Pi_Q^{(n)} = \int_{-\infty}^{\infty} x^n f_Q(x) \, dx \quad (n \in \mathbb{N}) \, ,$$

being the unnormalized ones, $\mu_Q^{(n)}[f]$:

$$\boxed{\mu_Q^{(n)}[f] = \int_{-\infty}^{\infty} x^n [f(x)]^Q \, dx \, .}$$

The q-**Fourier transform** (q-FT) of any non-negative measurable function f is defined by

$$\boxed{F_q[f](\xi) = \int_{\mathrm{supp}\, f} f(x) e_q^{i\xi x[f(x)]^{q-1}}\, dx} \quad (1 \leq q < 3)\,,$$

where e_q^{ix} represents the principal value of $[1 + (1-q)ix]^{1/(1-q)}$. Let $q_n = nq - (n-1)$. It was proven [1] that the nth unnormalized q_n-moment of a p.d.f. f is related to $F_q[f]$ by the equation

$$\boxed{\left.\frac{d^n F_q[f](\xi)}{d\xi^n}\right|_{\xi=0} = i^n \prod_{j=1}^{n-1} \{[1 + j(q-1)]\}\, \mu_{q_n}^{(n)}[f]\,.}$$

[1] C. Tsallis, A.R. Plastino and R.F. Alvarez-Estrada, J. Math. Phys. 50 (2009) 043303.

q-Gaussian

Let $q < 3$ and $\beta > 0$. We define a q-**Gaussian** as a mapping $x \mapsto G_{q,\beta}(x)$ such that

$$\boxed{G_{q,\beta}(x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}\,,}$$

where C_q is a constant that can be obtained from the normalization condition

$$\int_{-\infty}^{\infty} G_{q,\beta}(x)\,dx = 1\,.$$

The q-FT of a q-Gaussian with $q \geq 1$ is a q_1-Gaussian with $q_1 = (q+1)/(3-q)$. This implies that the q-FT is invertible on the space of q-Gaussians [2].

[2] S. Umarov, C. Tsallis and S. Steinberg, Milan J. Math. 76 (2008) 307.

Hilhorst example

Being $A \geq 0$, $1 < q < 2$, and $\alpha(q) = (2-q)/[2(q-1)]$, we define the mapping $x \mapsto f_{q,A}(x)$ such that

$$\boxed{f_{q,A}(x) = \frac{[1 - A|x|^{2\alpha(q)}]^{1/(q-2)}}{C_q\{1 + (q-1)x^2[1 - A|x|^{2\alpha(q)}]^{-1/\alpha(q)}\}^{1/(q-1)}}}$$

if $A \leq |x|^{-1/[2\alpha(q)]}$, and $f_{q,A} = 0$ otherwise. This mapping can be considered as a p.d.f. since $f_{q,A}(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f_{q,A}(x)\,dx = 1.$$

We can easily notice that $f_{q,0}(x) = G_{q,1}(x)$. However, by no means it is trivial to notice that $F_q[f_{q,A}](\xi) = F_q[G_{q,1}](\xi)$. It follows from this fact that the q-FT is not invertible on the space of p.d.f's [3].

[3] H.J. Hilhorst, J. Stat. Mech. (2010) P10023.

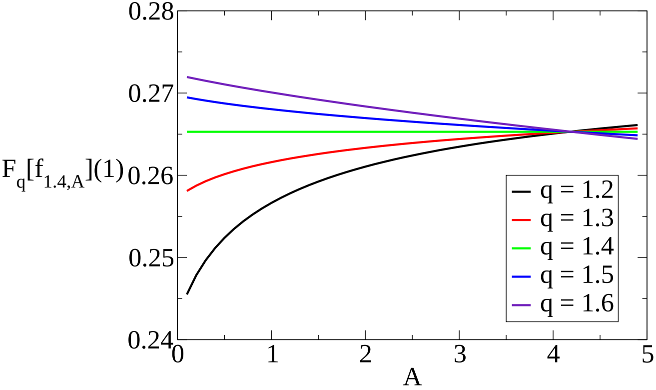

Figure: The dependence on A of $F_q[f_{1.4,A}](1)$ for diferent values of q. We can notice that $F_q[f_{1.4,A}](1)$ depends monotonically on A for any $q \neq 1.4$.

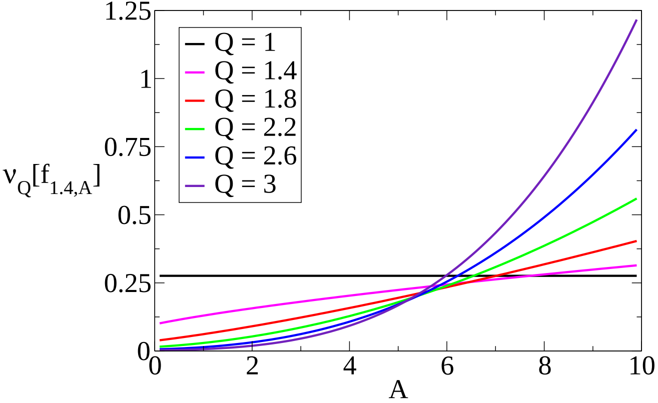

Figure: The dependence on A of the quantity $\nu_Q[f_{1.4,A}]$ for diferent values of Q. We can notice that $\nu_Q[f_{1.4,A}]$ depends monotonically on A for any $Q \neq 1$. So, if we knew the q-FT of $f_{1.4,A}$ and the value of $\nu_Q[f_{1.4,A}]$ for some $Q \neq 1$, we would be able to determine the parameter A, and, consequently, the p.d.f. $f_{1.4,A}$.

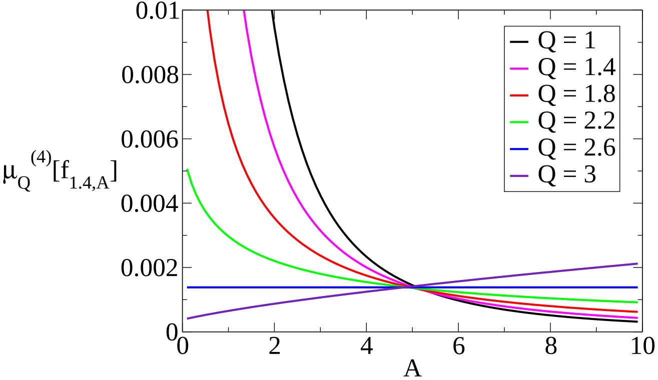

Figure: The dependence on A of the 4th unnormalized Q-moment of $f_{1.4,A}$ for diferent values of Q. We can notice that $\mu_Q^{(4)}[f_{1.4,A}]$ depends monotonically on A for any $Q \neq 2.6$ $(2.6 = 4 \cdot (1.4) - 3)$.

q-generalization of the inverse Fourier transform

Let f be a non-negative measurable piecewise continuous function. For each $y \in \operatorname{supp} f$, we define $f^{(y)}(x) = f(x+y)$. Then,

$$F_q[f^{(y)}](\xi, y) = \int_{\operatorname{supp} f^{(y)}} f(x+y) e_q^{i\xi x [f(x+y)]^{q-1}} \, dx \, .$$

Using the change of variables $z = x + y$, we have that

$$\int_{-\infty}^{\infty} F_q[f^{(y)}](\xi, y) \, d\xi = \int_{-\infty}^{\infty} \int_{\operatorname{supp} f} f(z) e_q^{i\xi(z-y)[f(z)]^{q-1}} \, dz \, d\xi \, .$$

Assuming that f is such that we are allowed to permute the integral signs, we have that

$$\int_{-\infty}^{\infty} F_q[f^{(y)}](\xi, y) \, d\xi = \frac{2-q}{2\pi} \int_{\operatorname{supp} f} f(z) \delta_q(\xi(z-y)[f(z)]^{q-1}) \, dz \, ,$$

where

$$\boxed{\delta_q(x) = \frac{2-q}{2\pi}\int_{-\infty}^{\infty} e_q^{i\xi x}\, d\xi} \quad (1 \leq q < 2)$$

is called the δ_q distribution. It results that $\delta_q(x) = \delta(x)$ for a certain family of functions [4-7]. Then, if f belongs to this family, the following property of the q-FT is obtained [8]:

$$\boxed{f(y) = \left[\frac{2-q}{\gamma\pi}\int_{-\infty}^{\infty} F_q[f^{(y)}](\xi, y)\, d\xi\right]^{1/(2-q)}} \quad (1 \leq q < 2)\,,$$

where $\gamma = 2$ if y is an interior point of $\operatorname{supp} f$, and $\gamma = 1$ if y is a boundary point of $\operatorname{supp} f$.

When $q \to 1$, the Eq. above yields the expression of the inverse Fourier transform.

[4] M. Jauregui and C. Tsallis, J. Math. Phys. 51 (2010) 063304.

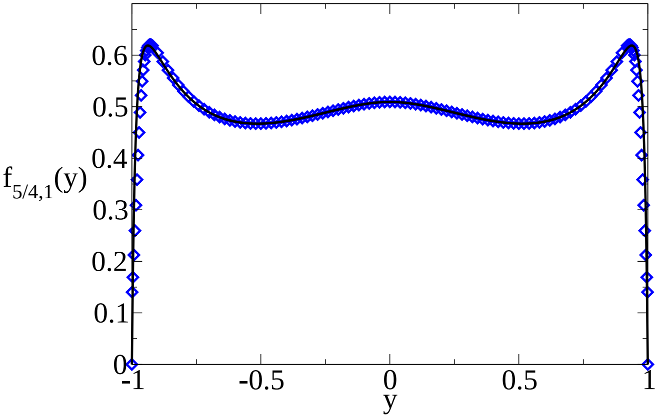

Figure: The Hilhorst function $f_{5/4,1}(x)$. The dots were obtained numerically using the last equation since $F_{5/4}[f^{(y)}_{5/4,1}]$ could not be obtained analytically. We used $\gamma = 2$ for all the points within the interval $(-1, 1)$, and $\gamma = 1$ for $x = \pm 1$.

Conclusions

It is possible to determine a p.d.f. f from the knowledge of its q-FT and the value of $\nu_Q[f]$ for some $Q \neq 1$. When $Q = 1$, this extra information becomes trivial since $\nu_1[f] = 1$.

If we know the q-FT of an arbitrary translation of a p.d.f. f, then we can determine it using the equation

$$f(y) = \left[\frac{2-q}{\gamma\pi} \int_{-\infty}^{\infty} F_q[f^{(y)}](\xi, y)\, d\xi \right]^{1/(2-q)} .$$

This is a remarkable property of the q-FT, which can make it be useful in engineering and other applied areas.

[5] A. Chevreuil, A. Plastino and C. Vignat, J. Math. Phys. 51 (2010) 093502.

[6] M. Mamode, J. Math. Phys. 51 (2010) 123509.

[7] A. Plastino and M.C. Rocca, arXiv:1012.1223 [math-ph]

[8] M. Jauregui and C. Tsallis, Phys. Lett. A (2011), in press, 1010.6275 [cond-mat.stat-mech]