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Time-Evolving Statistics of Chaotic Orbits of Conservative Maps in the Context of the Central Limit Theorem

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Central Limit Theorems

N random variables: $X = X_1 + X_2 + \cdots + X_N$

a) Independent (IID):

$$\sigma < \infty : \; X \xrightarrow[N \to \infty]{\text{Standard CLT}} p(x) = \frac{1}{\sqrt{2\pi N\sigma}} e^{-\frac{x^2}{2N\sigma^2}}$$

$$\sigma = \infty : \; X \xrightarrow[N \to \infty]{\text{Lévy-Gnedenko CLT}} L_\gamma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos kx \, e^{-\alpha|k|^\gamma} dk \sim \frac{1}{|x|^{1+\gamma}}$$

$$(0 < \gamma < 2)$$

b) Global correlations (*q*-independence; exchangeability):

$$X \xrightarrow[N \to \infty]{q\text{-CLT}} p_q(x) \sim e_q^{-\beta x^2} \equiv \left(1 + (q-1)\beta x^2\right)^{1/(1-q)} \sim \frac{1}{|x|^{\frac{2}{q-1}}}$$

S. Umarov, C.Tsallis. & S. Steinberg, Milan J Math **76**, 307 (2008)
S. Umarov, C.Tsallis., M. Gell-Mann & S. Steinberg, J Math Phys. **51** 033502 (2010)
M. G. Hahn, Xi. Jiang, & S. Umarov, J. Phys. A: Math. Teor. **43** (16), 165208 (2010)

Central Limit Theorem

for deterministic systems

$$x_{t+1} = f\left(x_t\right), t = 0, 1, 2 \ldots \infty$$

- **Strong chaos**

  (IID→ASI)

$$s = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (x_i - \langle x \rangle) \xrightarrow[N \to \infty]{\text{CLT}} \rho(s) \sim e^{-\beta s^2}$$

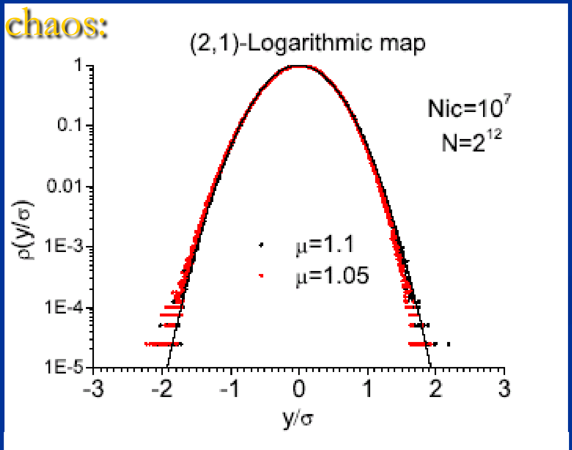
M.C. Mackey & M. Tyran-Kaminska, Phys. Rep. **422**, 167 (2006)
U. Tirnakli, C. Beck & C. Tsallis, Phys. Rev. E **75**, 040106(R) (2007)
G. Ruiz & C. Tsallis, Eur. Phys. J. B **64** (2009) 577

- **Weak chaos**

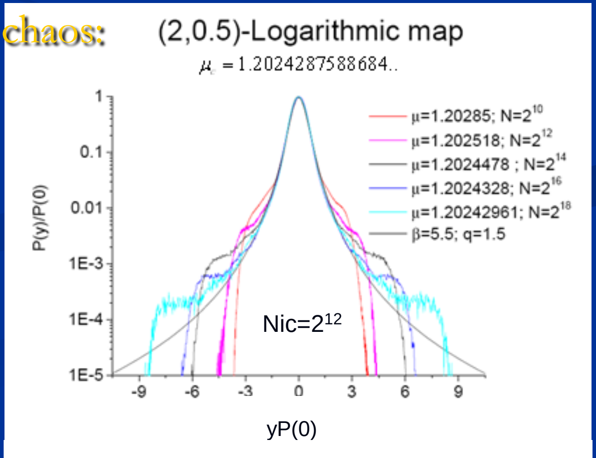
$$s = N^{\gamma} \sum_{i=1}^{N} (x_i - \langle x \rangle) \xrightarrow[N \to \infty]{q\text{-CLT}} \rho(s) \sim e_q^{-\beta s^2}$$

U. Tirnakli, C. Beck & C. Tsallis, Phys. Rev. E **75**, 040106(R) (2007)
U. Tirnakli, C. Beck & C. Tsallis, Phys. Rev E **79**, 056209 (2009)
G. Ruiz & C. Tsallis, Eur. Phys. J. B **64** (2009) 577
O. Afsar & U. Tirnakli, preprint (2010), 1001.2689 [cond-mat.stat-mech]

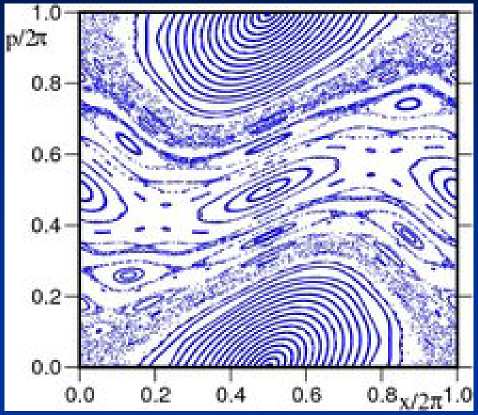
Strong chaos:



Weak chaos:



Chaotic orbits of conservative maps



[Poincare section of standard map]

Numerical analysis

- Out of edge of chaos $\left(\lambda_{\max} \neq 0\right)$
- PDF of rescaled sums of N iterates $Z_N^{(j)} \equiv \frac{1}{N^\gamma}\sum_{i=1}^{N}\left(x_i^{(j)} - \left\langle x_i^{(j)}\right\rangle\right);\ \left(j = 1 \ldots Nic\right) \Rightarrow P\left(Z_N^{(j)}\right)$

$$z_N^{(j)} \equiv \sum_{i=1}^{N}\left(x_i^{(j)} - \left\langle x_i^{(j)}\right\rangle\right) \Rightarrow P\left(z_N^{(j)} / \sigma_N\right)$$

- Phase space dynamics & long-lived QSS

q-Gaussians $\longrightarrow$ *Triangular* (logarithmic scale) $\longrightarrow$ Gaussians

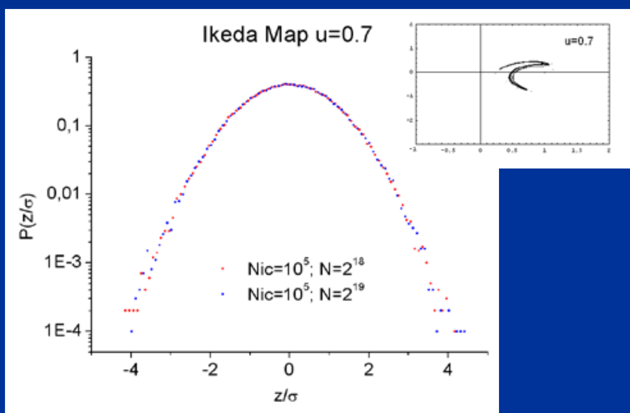
Ikeda Map

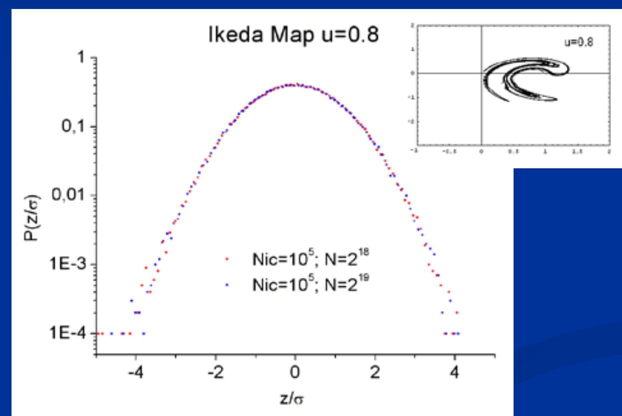
$$\begin{cases} x_{n+1} = 1 + u\left(x_n \cos\tau - y_n \sin\tau\right) \\ y_{n+1} = u\left(x_n \sin\tau + y_n \cos\tau\right) \end{cases}, \quad \tau \equiv 0.4 - \frac{6}{(1 + x_n^2 + y_n^2)}; \quad \left[\text{Jacobian: } J(u,\tau) = u^2\right]$$

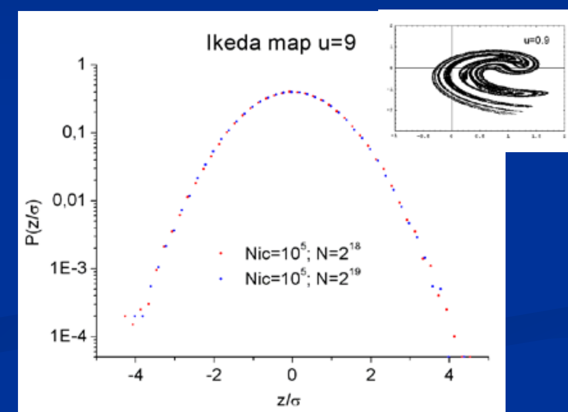
Out of the edge of chaos regime:

u	0.7	0.8	0.9	1.0
L_{max}	0.334	0.344	0.5076	0.118

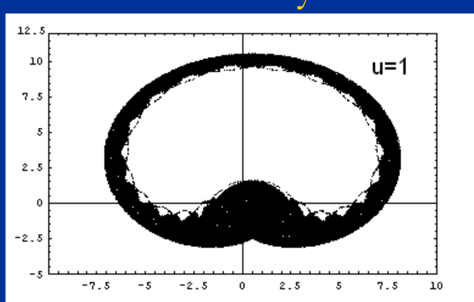
- Dissipative dynamics, u<1:

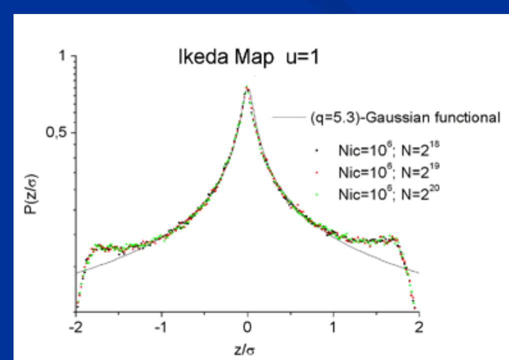


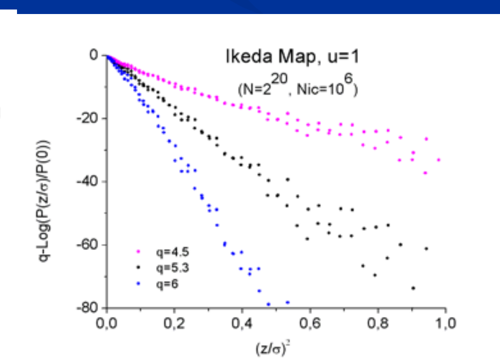




- Hamiltonian dynamics (u=1):







a) Non ergodicity
b) λ_{max} not large enough to preclude "edge of chaos"

$$p_q(x) \sim e_q^{-\beta x^2} = \left[1 - (1-q)\beta x^2\right]^{\frac{1}{1-q}} \quad (-\infty < q < 3)$$

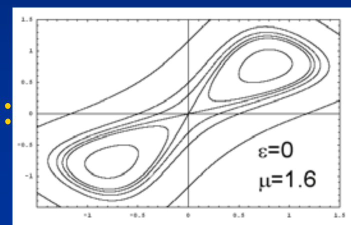
$$\ln_q(x) \equiv \frac{x^{1-q} - 1}{1-q} = f^{-1}\left(e_q^{\,x}\right)$$

MacMillan Map

$$\begin{cases} x_{n+1} = y_n \\ y_{n+1} = -x_n + 2\mu \dfrac{y_n}{1+y_n^2} + \varepsilon\left(y_n + \beta x_n\right) \end{cases}; \quad \left[\textbf{Jacobian: } J(\varepsilon,\beta) = 1 - \varepsilon\beta\right]$$

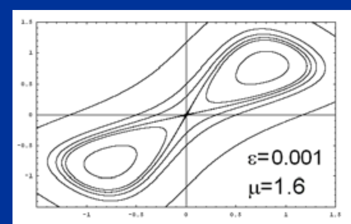
Conservative (β=0) → (μ, ε)

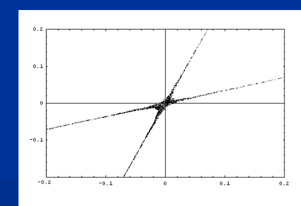
- Unperturbed map (μ, ε = 0):



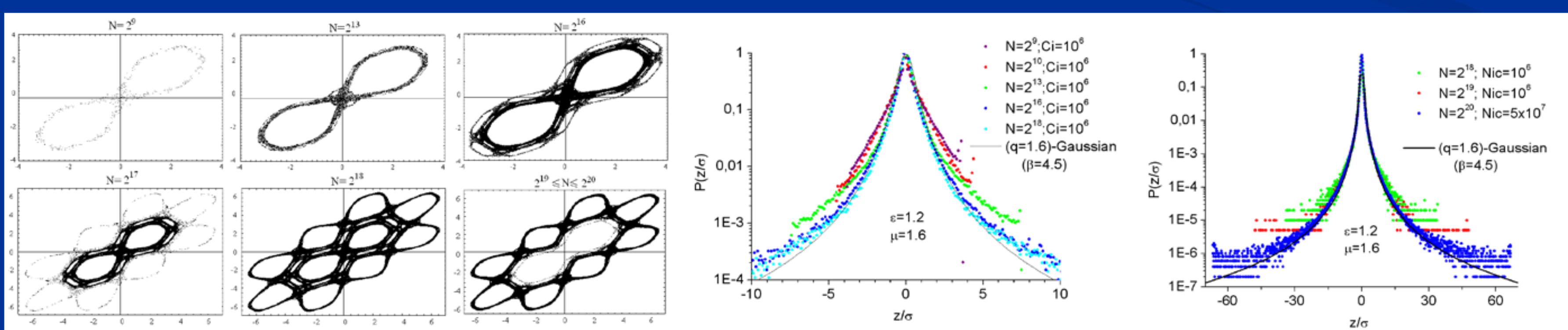
$$x^2y^2 + x^2 + y^2 - 2\mu xy = C; \quad C(x_0, y_0)$$
$$\Downarrow C = 0$$
$$x^2y^2 + x^2 + y^2 - 2\mu xy = 0 \quad (\mu > 1)$$

- Perturbed map (μ, $\varepsilon \neq 0$):

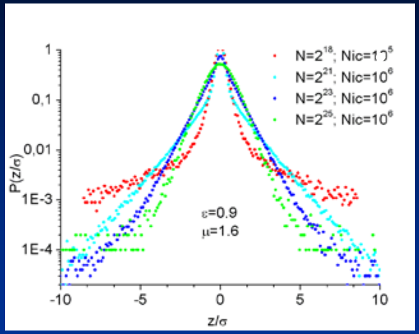




A) $(\varepsilon=1.2,\ \mu=1.6)$-**MacMillan Map** ($\lambda_{max}=0.0513$)**:** Persisting q-Gaussians



B) $(\varepsilon=0.9,\ \mu=1.6)$-**MacMillan Map** $(\lambda_{max}=0.0875)$**:** Time evolving pdfs



***q*-Gaussians:**

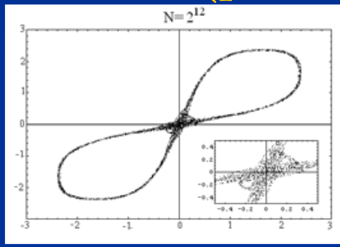
$$p_q(z) \sim e_q^{-\beta z^2} = \left[1-(1-q)\beta z^2\right]^{\frac{1}{1-q}},\ (-\infty < q < 3)$$

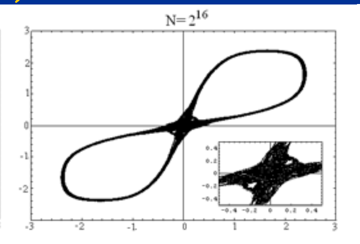
***Triangular*:**

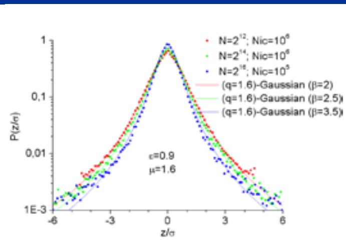
$$p_q(z) \sim e^{-k|z|}$$

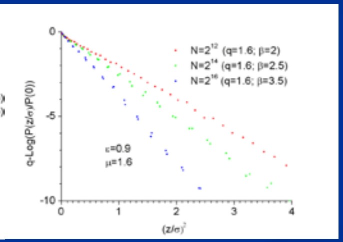
Superposition of them

I) $N < 2^{16}$: $(q=1.6)$-Gaussian



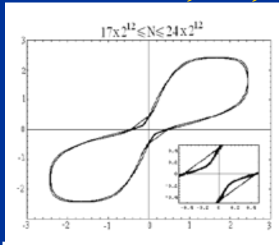


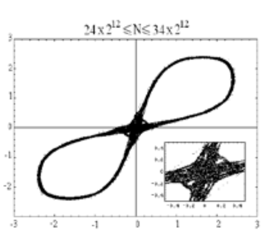


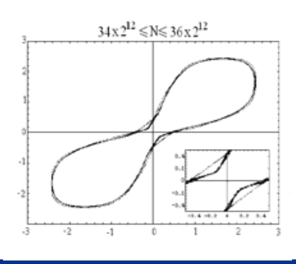


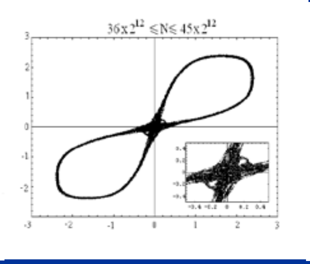
QSS

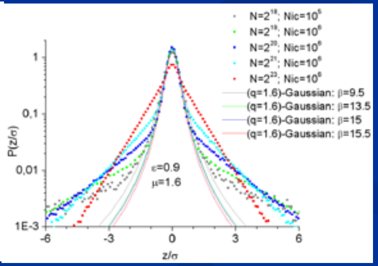
II) $N = 2^{16},\ldots,2^{23}$: Superposition QSSs





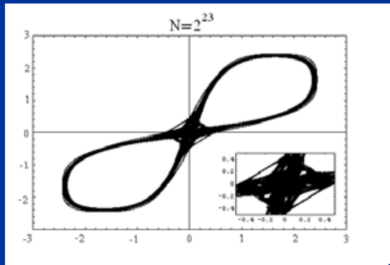


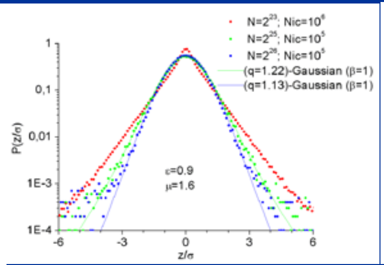


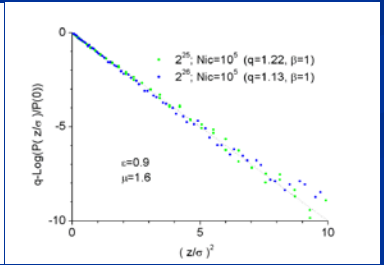


Sequence of QSSs

III) $N > 2^{23}$: $(q\rightarrow 1)$-Gaussian







$N \rightarrow \infty$

Gaussian?

4-D model of accelerator dynamics

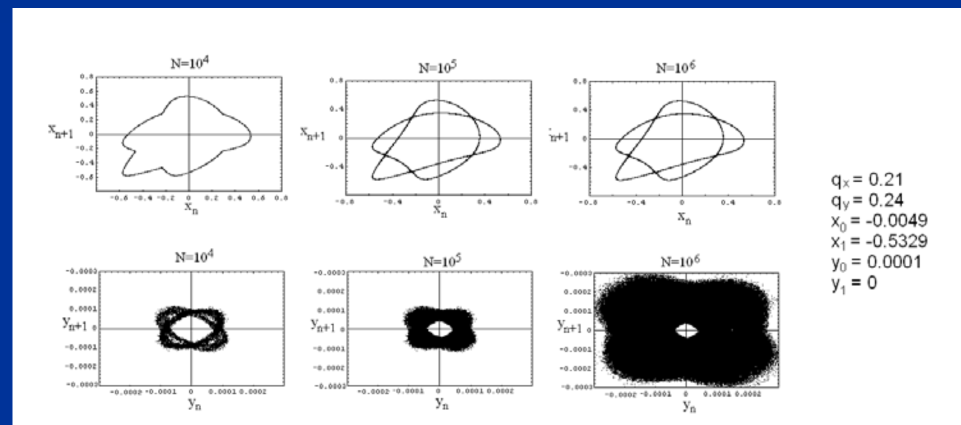
$$\begin{cases} x_{n+1} = 2c_x x_n - x_{n-1} - \rho x_n^2 + y_n^2 \\ y_{n+1} = 2c_y y_n - y_{n-1} + 2x_n y_n \end{cases}; \quad \text{where} \quad \begin{pmatrix} \rho = \beta_x s_x / \beta_y s_y; \ \left[\text{betatron functs: } \beta_{xy} \propto q_{x,y}^{-1}\right] \\ c_{x,y} \equiv \cos 2\pi q_{x,y} \\ s_{x,y} \equiv \sin 2\pi q_{x,y} \end{pmatrix}$$

T. Bountis, M. Kollmann, Phys. D **71**, 122 (1994).

$$q_x = 0.21; \quad q_y = 0.24$$

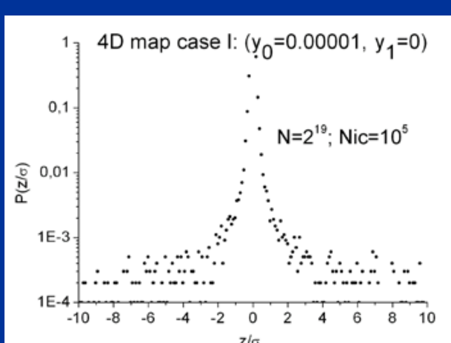
$$x_0 = -0.0049, x_1 = -0.5329, (y_0, y_1) \text{ near } (0,0)$$

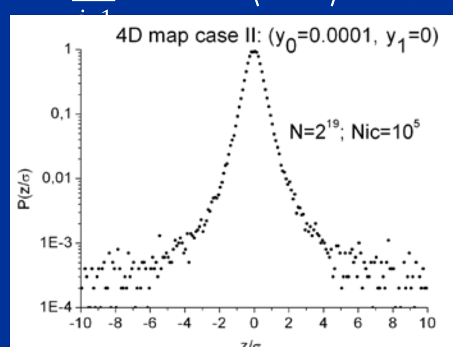
Arnold diffusion (2-dim projections):

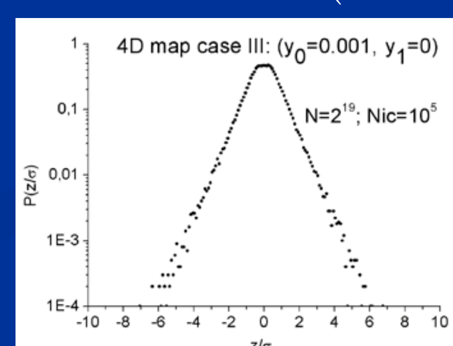


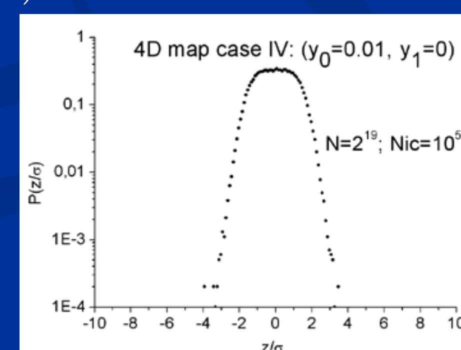
Sums of *y*-variable iterates:

$$z_n^{(j)} \equiv \sum_{i=1}^{N} \left(y_i^{(j)} - \left\langle y_i^{(j)} \right\rangle\right); \ \left(j = 1 \ldots Nic\right) \quad \Rightarrow \quad P\left(z_n^{(j)} / \sigma_N\right)$$

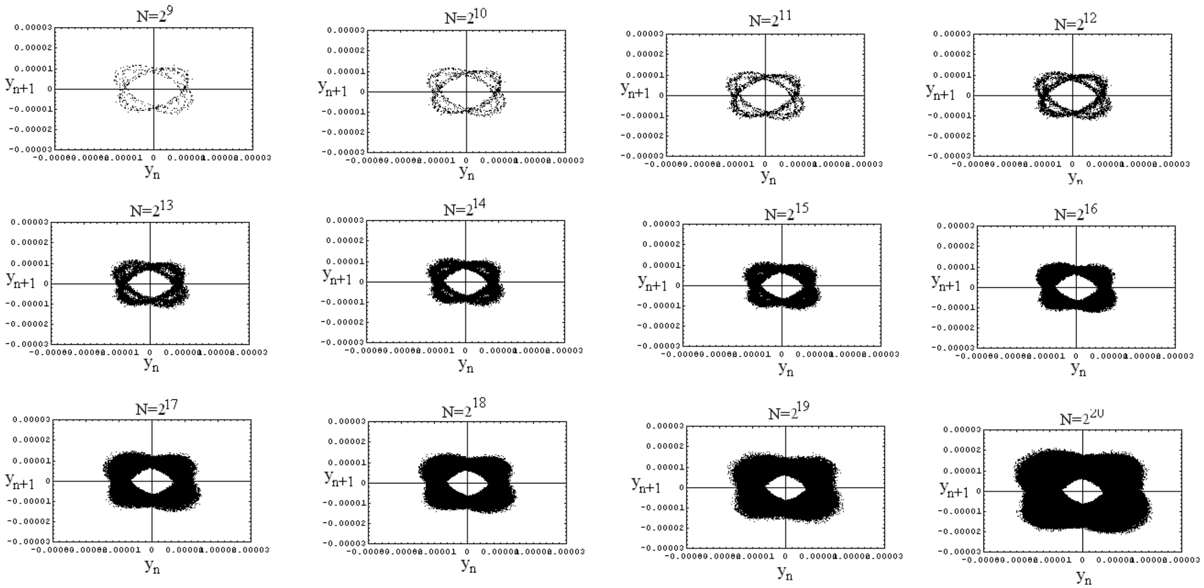


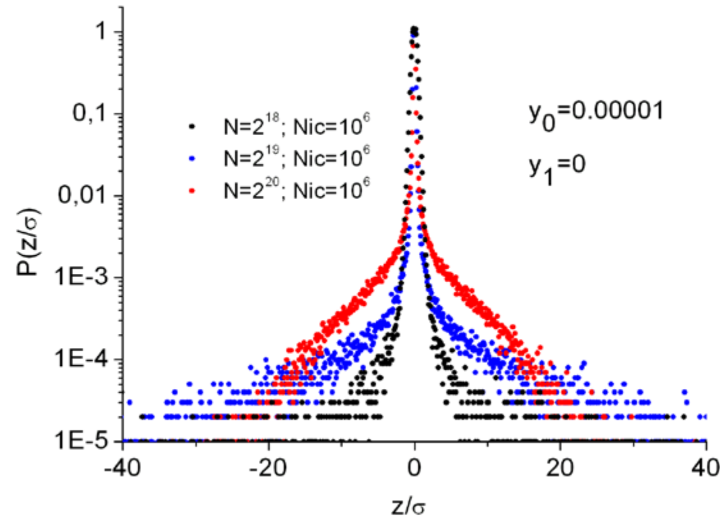


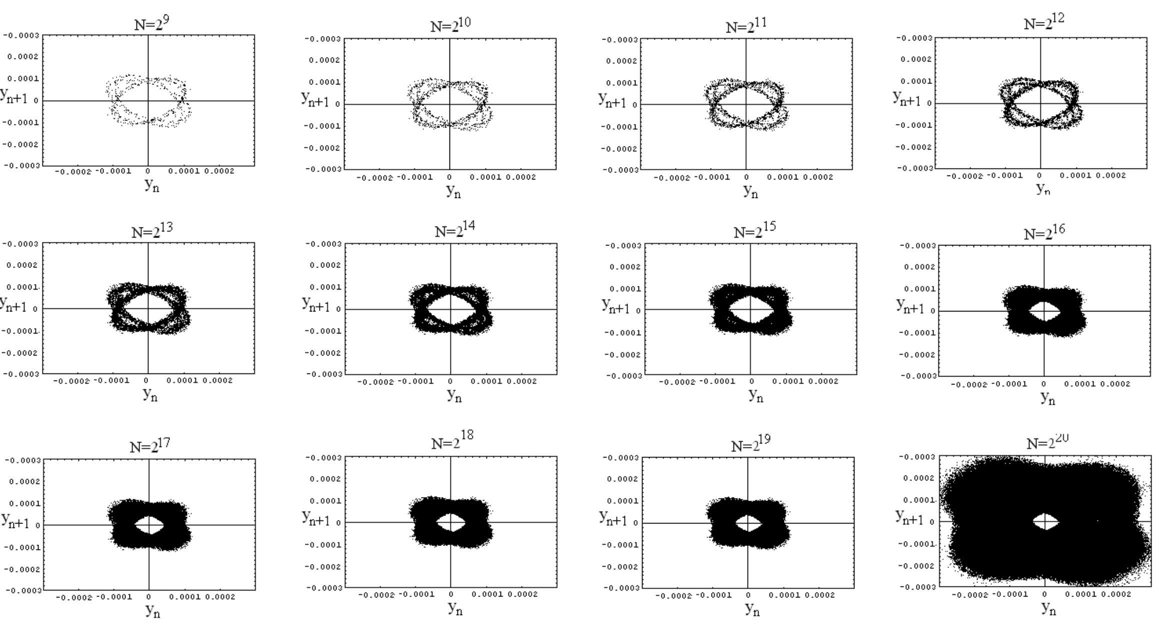


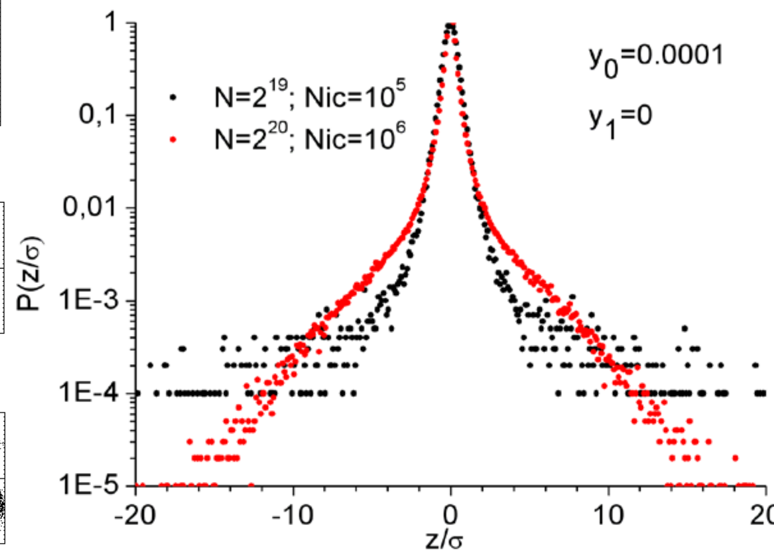


Time-evolving features:









Conclusions

- Our work serves to connect different types of distributions with different phase space dynamics.

- In some Hamiltonian maps, where the chaotic layers are thin and the (positive) maximal Lyapunov exponent small, long-lasting quasi-stationary states (QSS) are formed whose pdfs appear to converge to *q*-Gaussians associated with Nonextensive Statistical Mechanics.

- Detailed structure of chaotic regions in Hamiltonian systems (network islands and invariant sets of cantori) is the responsible for obtaining QSS with long-lived *q*-Gaussian distribution in conservative systems.

- More generally, the pdfs describe a sequence of QSS that pass from a *q*-Gaussian to a triangular shape and ultimately to a Gaussian, as orbits diffuse to larger chaotic domains (Lyapunov exponents attain larger values and the phase space dynamics becomes closer to ergodicity).

- Higher complexity of the dynamics may be the reason why QSS with *q*-Gaussian looking distributions persist longer.