Universidade Estadual de Maringá

Diffusion Equations, Solutions and Applications

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Comb – Model and Extensions

Equation to be considered

$$\frac{\partial}{\partial t}\rho(x,y;t) = \int_0^t dt'\,\mathcal{D}_y(t-t')\frac{\partial^2}{\partial y^2}\rho(x,y;t') + \delta(y)\int_0^t dt'\,\mathcal{D}_x(t-t')\frac{\partial^\mu}{\partial |x|^\mu}\rho(x,y;t') \,.$$

It is subjected to the boundary and initial conditions

$$\rho(\pm\infty,y,t) = \rho(x,\pm\infty,t) = 0 \quad , \quad \rho(x,y,0) = \hat{\rho}(x,y)$$

First case

$$\mathcal{D}_x(t) = \mathcal{D}_x\delta(t),\ \mathcal{D}_y(t) = \mathcal{D}_y\delta(t),\ \mu = 2$$

For this case, aftet applying the Fourier - Laplace transforms, we obtain

$$\mathcal{D}_y\frac{d^2}{dy^2}\rho(k_x,y,s) - \left(s + \mathcal{D}_x k_x^2\delta(y)\right)\rho(k_x,y,s) = -\rho(k_x,y,0)$$

which has as solution

$$\rho(k_x,y,s) = -\int_{-\infty}^{\infty} d\overline{y}\,\hat{\rho}(k_x,y)\mathcal{G}(k_x,y,\overline{y},s)$$

$$\mathcal{G}(k_x,y,\overline{y},s) = -\frac{1}{2\sqrt{s\mathcal{D}_y}}\left(e^{-\sqrt{\frac{s}{\mathcal{D}_y}}|y-\overline{y}|} - e^{-\sqrt{\frac{s}{\mathcal{D}_y}}(|y|+|\overline{y}|)}\right) - \frac{e^{\sqrt{\frac{s}{\mathcal{D}_y}}(|y|+|\overline{y}|)}}{2\sqrt{s\mathcal{D}_y} - k_x^2}\,.$$

Performing the inverve Fourier - Laplace transforms,

$$\mathcal{G}(x,y,\overline{y},t) = -\frac{\delta(x)}{\sqrt{4\pi\mathcal{D}_y t}}\left(e^{-\frac{(y-\overline{y})^2}{4\mathcal{D}_y t}} - e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y t}}\right) - \sqrt{\frac{\mathcal{D}_x}{4\pi\sqrt{\mathcal{D}_y}}}\left(\frac{|y|+|\overline{y}|}{\sqrt{4\pi\mathcal{D}_y}}\right)\int_0^t d\overline{t}\,\frac{e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y(t-\overline{t})}}}{\sqrt{(t-\overline{t})}\,\overline{t}^{\frac{3}{2}}}\,\mathrm{H}^{1,0}_{1,1}\left[\sqrt{\frac{2\sqrt{\mathcal{D}_y}}{\mathcal{D}_x\sqrt{\overline{t}}}}\,|x|\,\middle|^{(\frac{1}{4},\frac{1}{4})}_{(0,1)}\right].$$

Second Case

$$\mathcal{D}_y(t) = \mathcal{D}_y t^{\gamma_y-2}/\Gamma(\gamma_y-1),\ \ \mathcal{D}_x(t) = \mathcal{D}_x t^{\gamma_x-2}/\Gamma(\gamma_x-1),\quad \mu\neq 2$$

The Green function for this case is

$$\mathcal{G}(x,y,\overline{y},t) = -\frac{\delta(x)}{\sqrt{4\pi\mathcal{D}_y t^{\gamma_y}}}\left(\mathrm{H}^{1,0}_{1,1}\left[\frac{(y-\overline{y})^2}{\sqrt{\mathcal{D}_y t^{\gamma_y}}}\middle|^{(1-\frac{\gamma_y}{2},\frac{\gamma_y}{2})}_{(0,1)}\right] - \mathrm{H}^{1,0}_{1,1}\left[\frac{(|y|+|\overline{y}|)^2}{\sqrt{\mathcal{D}_y t^{\gamma_y}}}\middle|^{(1-\frac{\gamma_y}{2},\frac{\gamma_y}{2})}_{(0,1)}\right]\right) - \frac{1}{\pi|x|}\int_0^t \frac{d\overline{t}}{(t-\overline{t})\overline{t}^{\frac{\gamma_y}{2}}}\,\mathrm{H}^{2,1}_{2,3}\left[\frac{2\sqrt{\mathcal{D}_y\overline{t}^{\gamma_y}}}{\mathcal{D}_x\overline{t}^{\gamma_x}}|x|^\mu\middle|^{(1,1),(\overline{\beta},\overline{\xi})}_{(\frac{1}{2},\frac{\mu}{2}),(1,1),(1,\frac{\mu}{2})}\right]\mathrm{H}^{1,0}_{1,1}\left[\frac{|y|+|\overline{y}|}{\sqrt{\mathcal{D}_y(t-\overline{t})^{\gamma_y}}}\middle|^{(0,\frac{\gamma_y}{2})}_{(0,1)}\right]$$

with

$$\overline{\beta} = 1-\gamma_y/2 \quad , \quad \overline{\xi} = \gamma_x - \gamma_y/2 \quad .$$

Impedance and Fractional Diffusion Equation

Equations of the usual case

$$\frac{\partial}{\partial t}\delta n_\pm(z,t) = -\frac{\partial}{\partial z}j_\pm(z,t)$$

$$j_\pm(z,t) = -\mathcal{K}\left(\frac{\partial}{\partial z}\delta n_\pm(z,t) \pm \frac{Nq}{k_B T}\frac{\partial}{\partial z}V(z,t)\right)$$

$$\frac{\partial^2}{\partial z^2}V(z,t) = -\frac{q}{\epsilon}\left(\delta n_+(z,t) - \delta n_-(z,t)\right)$$

These equations are subjected to the conditions

$$\int_{-d/2}^{d/2}\delta n_+(z,t)dz = \int_{-d/2}^{d/2}\delta n_-(z,t)dz = 0$$

$$V\left(\pm\frac{d}{2},t\right) = \pm\frac{V_0}{2}e^{i\omega t} \quad , \quad j_\pm\left(\pm\frac{d}{2},t\right) = 0$$

By extending the previous results, we obtain

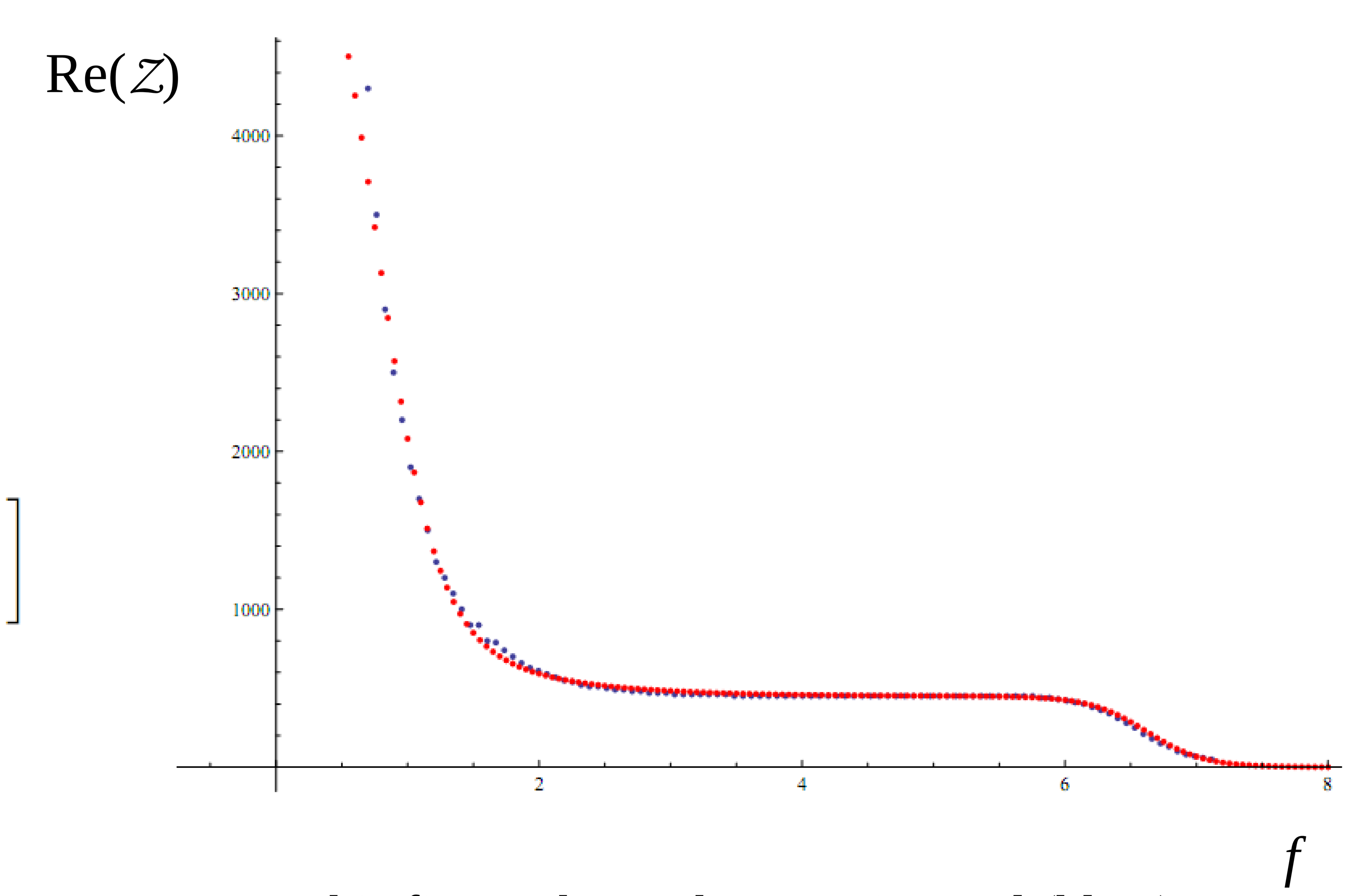
$${}_{-\infty}\mathcal{D}_t^\gamma\,\delta n_\pm(z,t) = \mathcal{K}_\gamma\frac{\partial^2}{\partial z^2}\delta n_\pm(z,t) \pm \frac{Nq\mathcal{K}_\gamma}{k_B T}\frac{\partial^2}{\partial z^2}V(z,t)$$

$${}_{t_0}\mathcal{D}_t^\gamma\,\delta n_\pm(z,t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_{t_0}^t d\overline{t}\,\frac{\delta n_\pm(z,\overline{t}\,)}{(t-\overline{t})^{\gamma+1-n}}$$

$$\beta = \sqrt{\frac{1}{\lambda_D^2} + \frac{(i\omega)^\gamma}{\mathcal{K}_\gamma}},\quad \lambda_D = \sqrt{\frac{k_B\varepsilon T}{2Nq^2}}\quad .$$

The impedance for this case is given by

$$\mathcal{Z} = \frac{2}{i\omega\varepsilon S\beta^2}\left(\frac{1}{\lambda_D^2\beta}\tanh\left(\frac{\beta d}{2}\right) + \frac{(i\omega)^\gamma d}{2\mathcal{K}_\gamma}\right).$$



This figure shows the experimental (blue) and the theoretical results (red).