Universidade Federal do Ceará

Towards Design Principles for Optimal Transport Networks

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Topics

- Optimal path cracks.
- Navigation in complex networks.

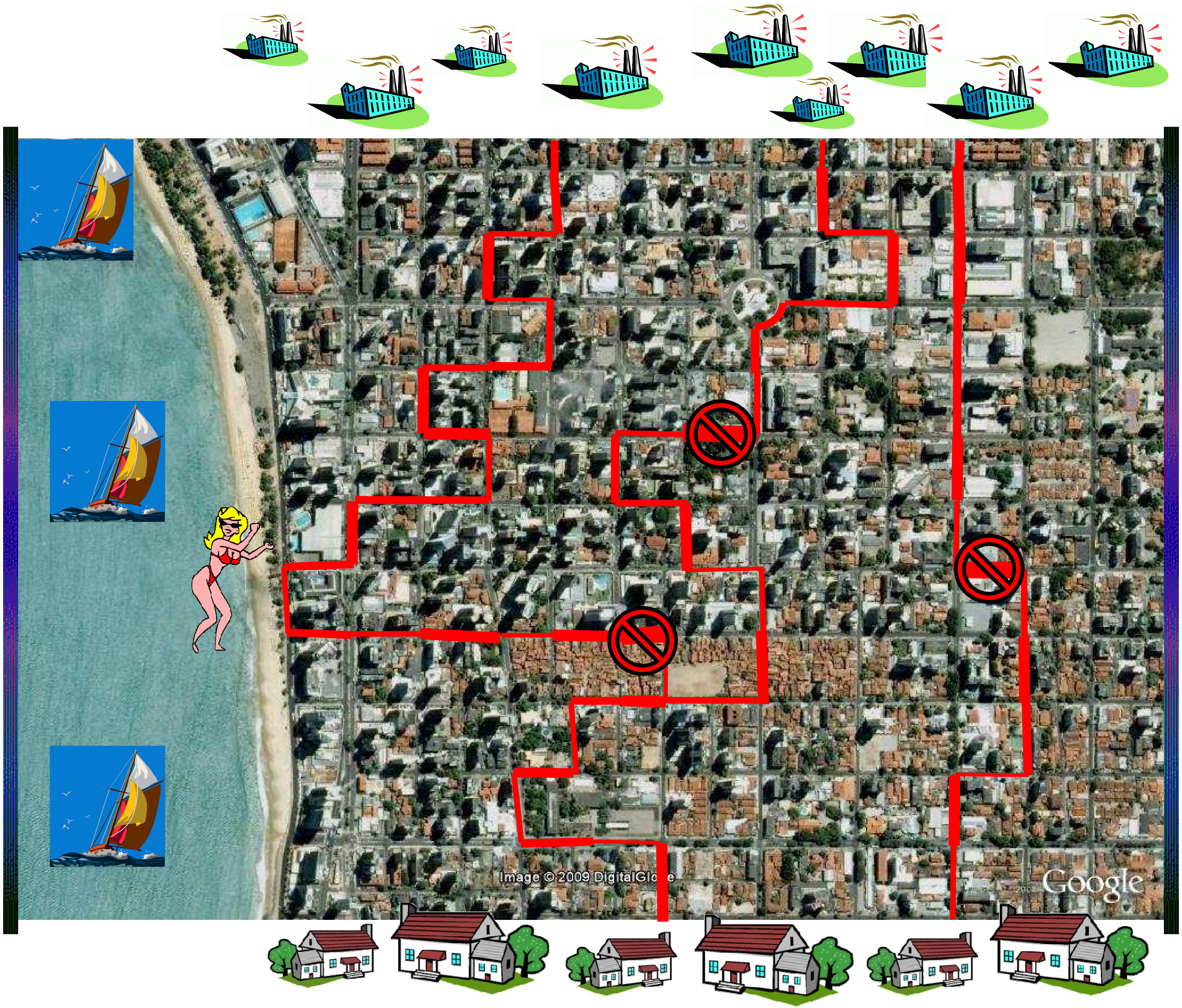
Optimal path in disordered media: some definitions and previous related studies

1) On an n-dimensional lattice, we assign to each site i a given "energy" value ε_i according to a given probability distribution $P(\varepsilon)$. The energy of a path is the sum of all energies of its sites.

2) The optimal path (OP) is defined here as the one among all paths connecting the bottom to the top of the lattice that has the smallest energy [Kirkpatrick & Toulouse, *J. Phys. Lett.* (1985); Kertesz, Horvath & Weber, *Fractals* (1992); Barabasi, *Phys. Rev. Lett.* (1996)].

3) Optimal paths extracted from energy landscapes generated with weak disorder are self-affine and belong to the same universality class of directed polymers [Schwartz, Nazaryev & Havlin, *Phys. Rev. E* (1998)].

4) In the strong disorder limit, optimal paths are self-similar with fractal dimensions given by $D_f \approx 1.22$ and 1.43 in two and three-dimensions, respectively [Cieplak, Maritan & Banavar, *Phys. Rev. Lett.* (1994), (1996); Porto, Havlin, Schwarzer & Bunde, *Phys. Rev. Lett.* (1997)].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Two important questions arise:

➢How and when will the transportation network collapse?

➢What is the role of disorder on the performance of the transportation network?

We perform numerical simulations:

➢Square lattices of size L with fixed BC's at the top and bottom and periodic BC's in the transversal direction.

➢Disorder is introduced by assigning to each site i an energy ε given by:

$$\varepsilon_i = \exp[\beta(p_i - 1)]$$

where p_i is a random variable uniformly distributed in [0,1].

This is equivalent to choose ε_i from a power-law distribution,

$$P(\varepsilon_i) \sim 1/\varepsilon_i$$ (now normalizable)

with maximum cutoff $\varepsilon_{\max} = e^{\beta}$.

Algorithm

JSA, Oliveira, Moreira & Herrmann, submitted (2009)

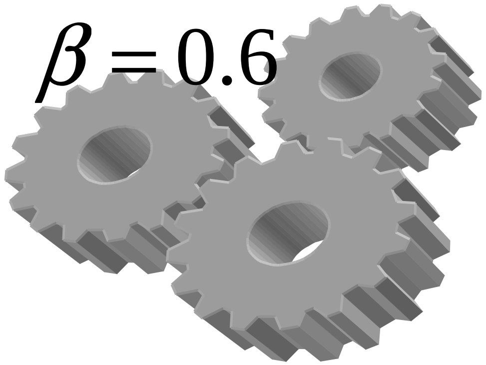
1) The Dijkstra algorithm [Dijkstra, *Num. Math.* (1959)] is used to calculate the first OP connecting the bottom to the top of the network;

2) The site in the OP having the highest energy is permanently blocked (i.e., an irreversible "micro-crack" is formed);

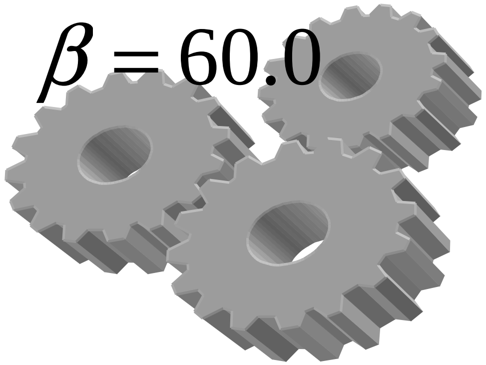
3) The next OP is calculated, from which the highest energy site is again removed and so on, and so forth;

4) The process continues iteratively until the system is disrupted, i.e., we can no longer find any path connecting bottom to top.

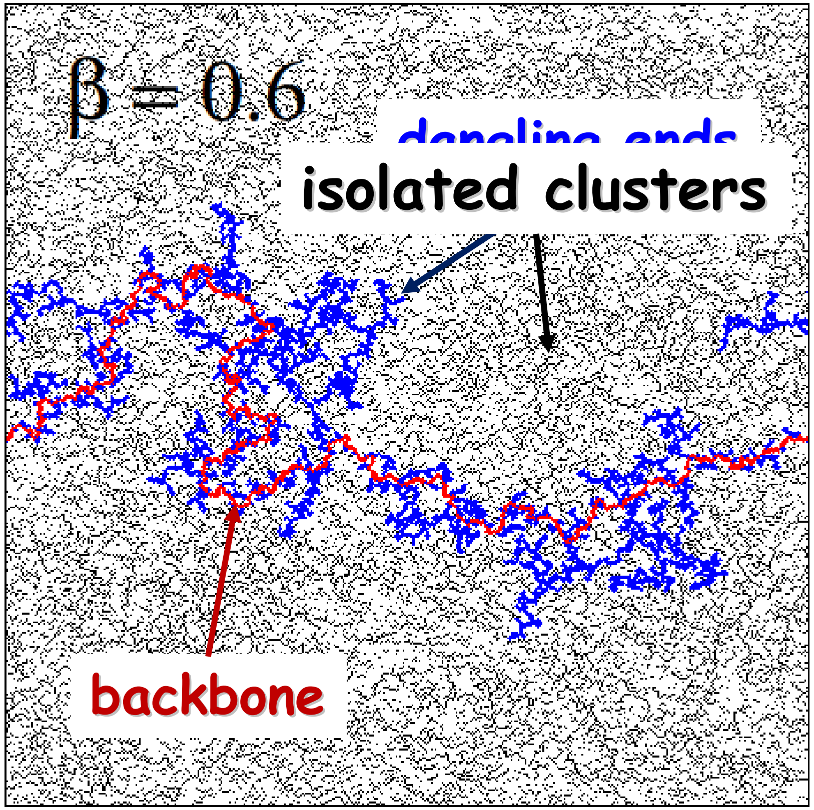
$\beta = 0.6$

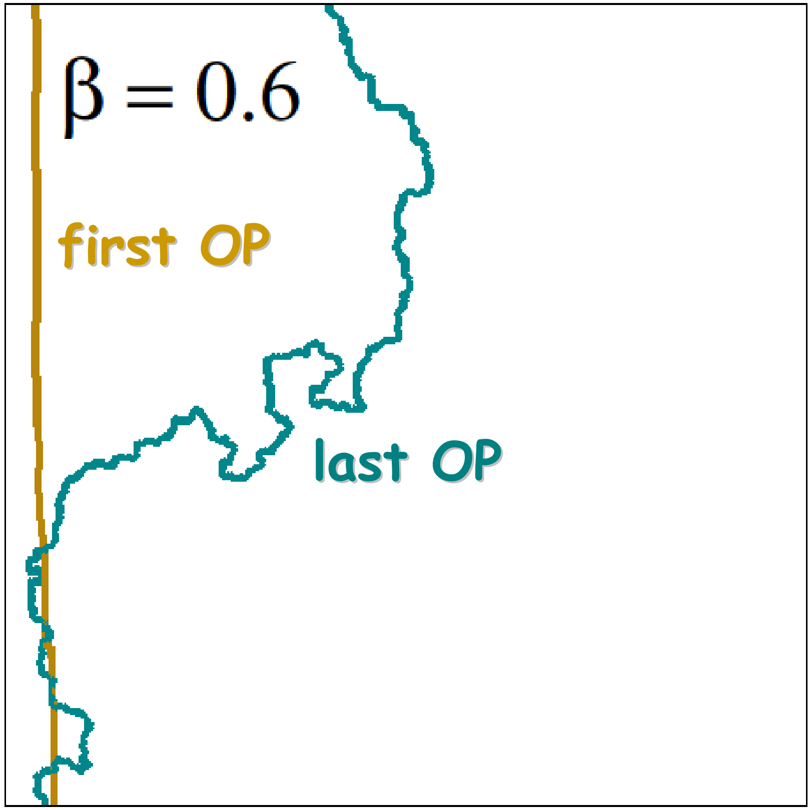


$\beta = 60.0$



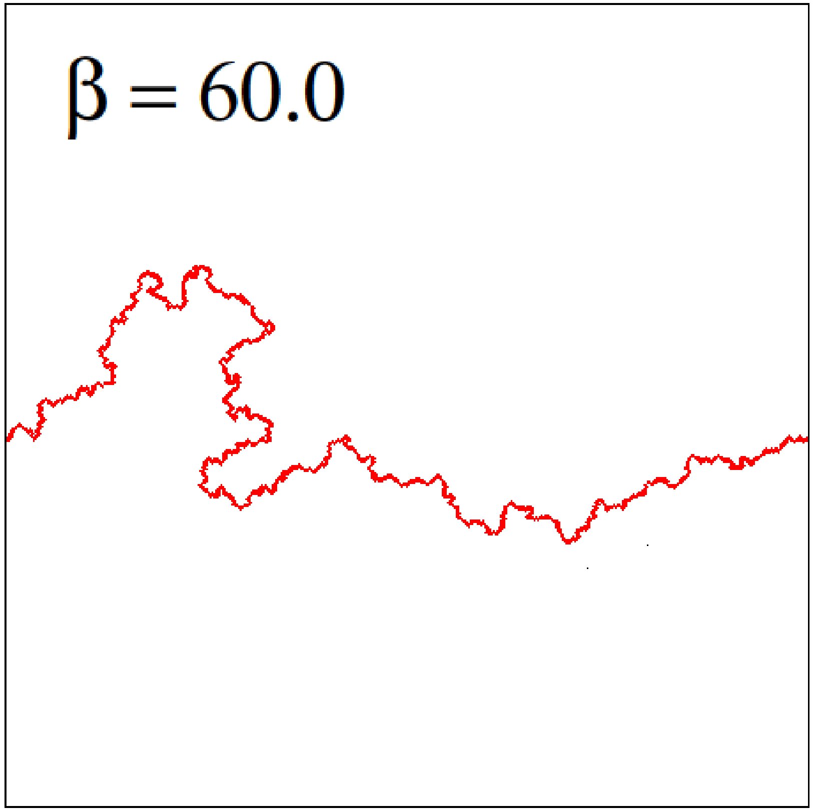
Results: weak disorder

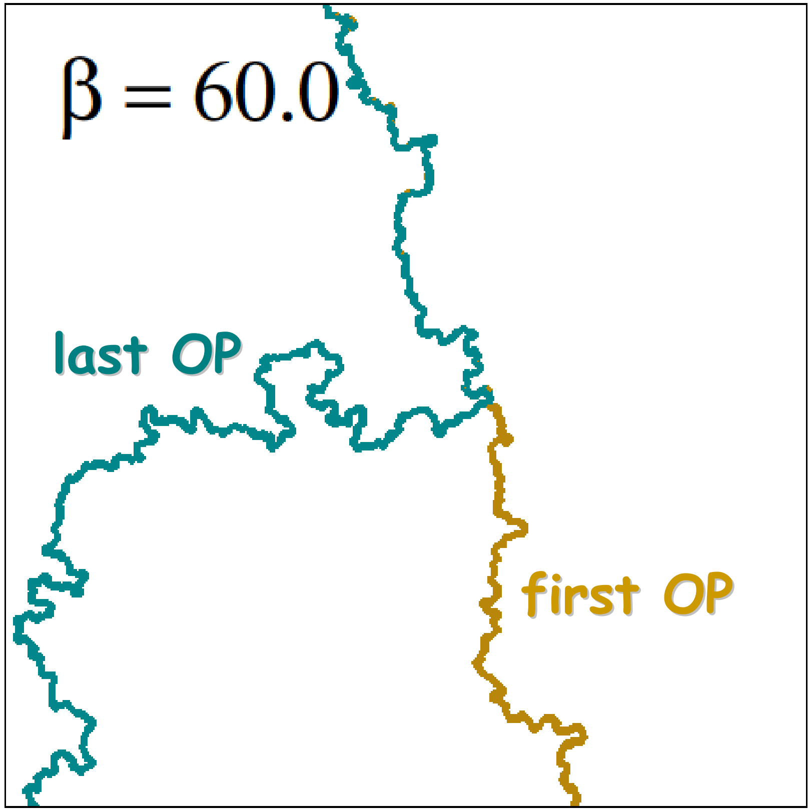




OPC elements { OPC fracture {

Results: strong disorder



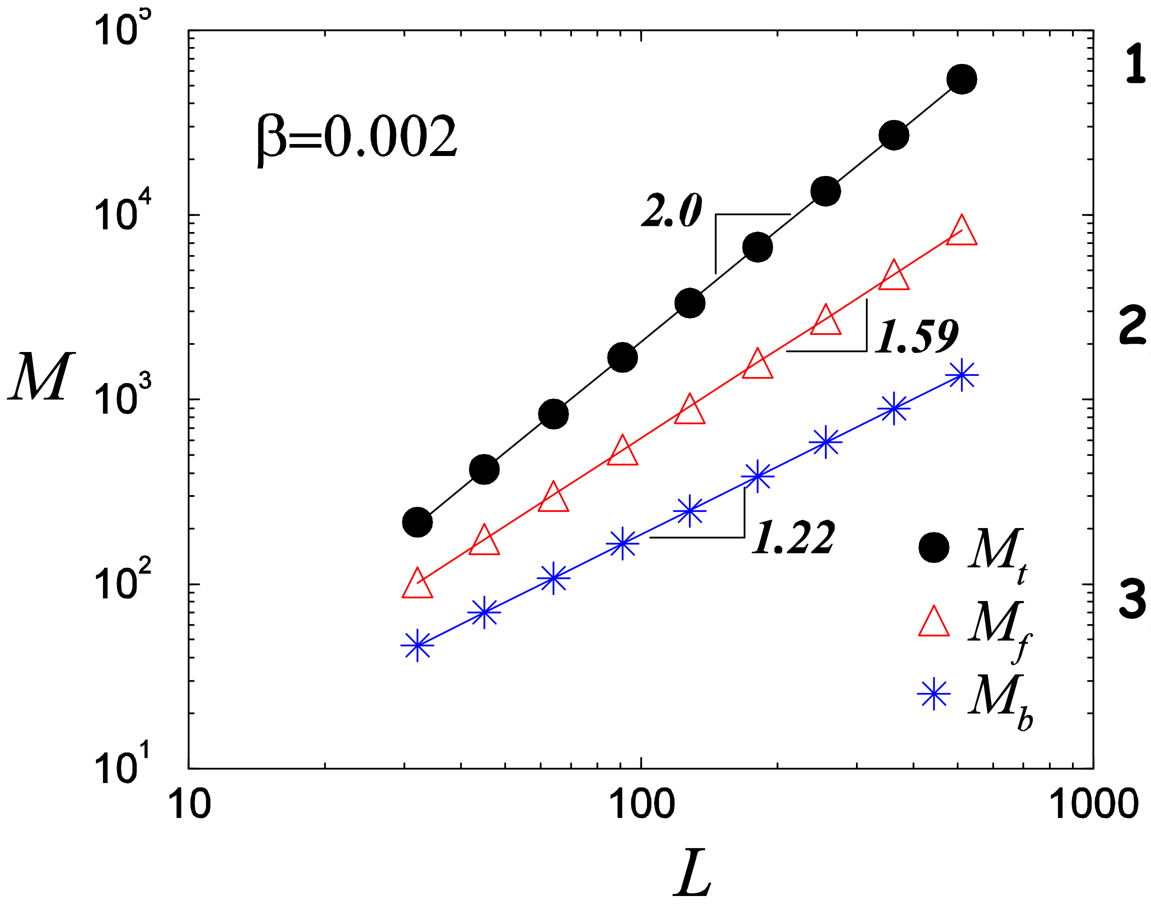


OPC elements
- OPC fracture
  - backbone (preserved!!)
  - dangling ends (disappeared)
- isolated clusters (practically disappeared)

Quantitative Results

JSA, Oliveira, Moreira & Herrman, PRL (2009)

- Simulations with 1000 realizations of lattices for each different size 32≤L≤512 and distinct values of the disorder parameter β.
- Weak disorder $\Rightarrow$ clear scaling laws.



1) $M_b \sim L^{D_b}$ (backbone)

with $D_b = 1.22 \pm 0.02$

2) $M_f \sim L^{D_f}$ (OPC fracture)

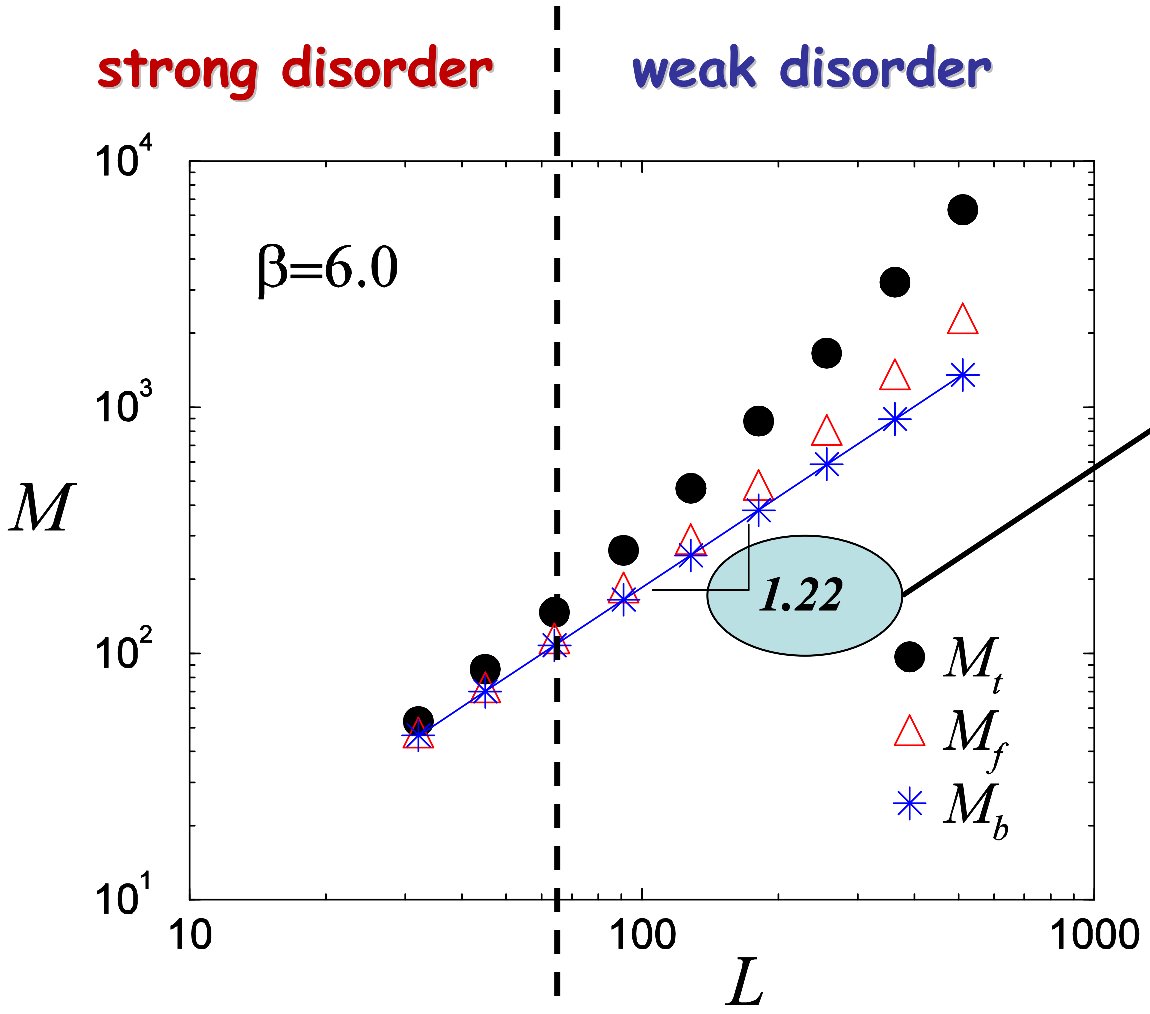
with $D_f = 1.59 \pm 0.02$

3) $M_t \sim L^{D_t}$ (all cracks)

with $D_t = 2.00 \pm 0.01$

Quantitative Results

- Transition from weak to strong disorder.
- The stronger the disorder (small L or high β), the smaller is the number of final blocked sites ⇒ more localized in a singly-connected crack line.



1) OP's under strong disorder
2) Disordered Polymers
3) Strands in IP
4) paths on MST's

Conclusions

- The backbone of the fracture constituted of OPC's is apparently (not proved) disorder independent. It is also a self-similar object with fractal dimension $D_b \approx 1.22$.

- This dimension is (statistically) similar to the ones obtained for OP's under strong disorder [Schwartz et al., *PRE* (1998)], Disordered Polymers [Cieplak et al., *PRL* (1994)], strands in Invasion Percolation [Cieplak et al., *PRL* (1996)], and paths on Minimum Spanning Trees [Dobrin et al., *PRL* (2001)].

- The role of disorder is to dramatically reduce the total number of blocked sites before the system collapses:

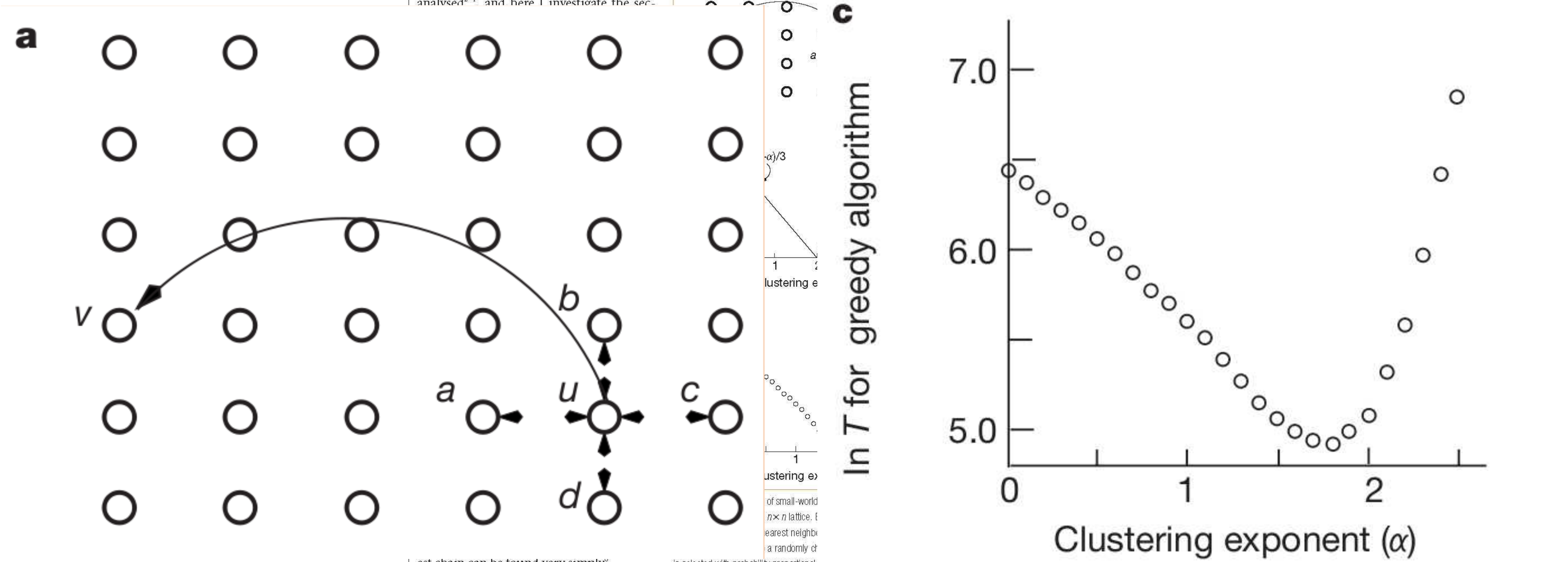
weak disorder $\Longrightarrow \quad M_t \sim L^2$

strong disorder $\Longrightarrow \quad M_t \rightarrow M_b \sim L^{1.22}$

- This information can be used to improve a given transportation network or in the design of systems with enhanced performance.

Navigation on complex networks: the conjecture of Kleinberg

1) On an d-dimensional lattice, we assign to a site i a long-range connection with a site j with probability $P_{ij} \sim r_{ij}^{-\alpha}$, where r_{ij} is the "Manhattan distance" between sites i e j.



2) The optimal navigation in the presence of long-range connections with local information is achieved with α=d [J. M. Kleinberg, *Nature* (2000)].

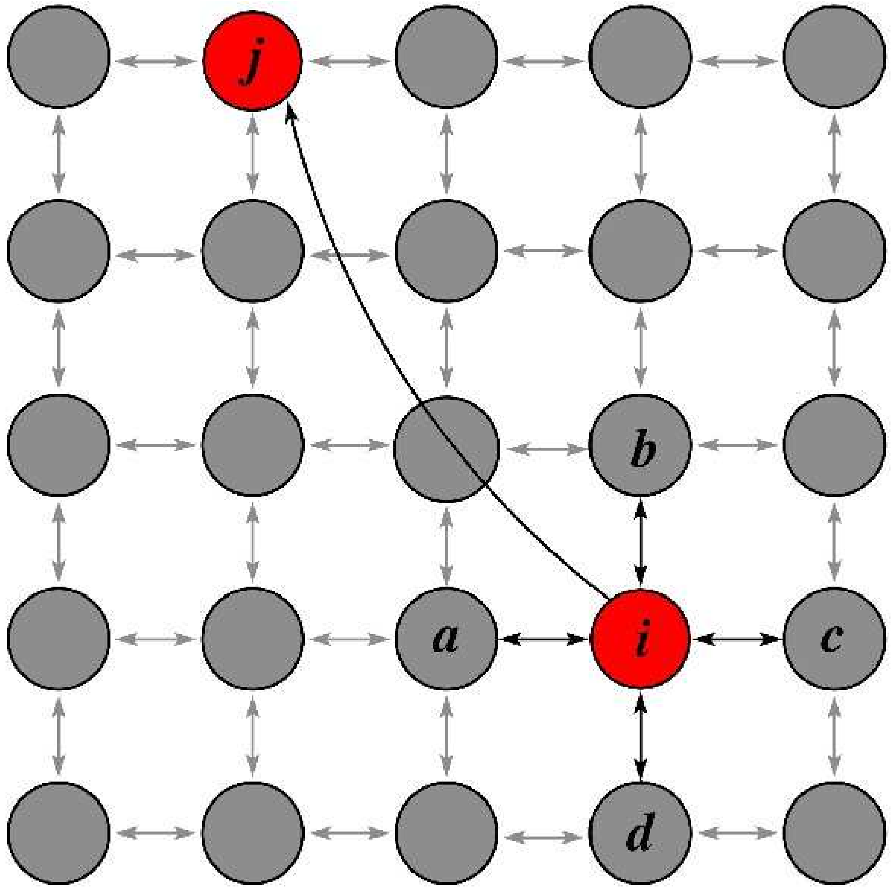
Can a cost constraint affect the optimal?

1) We consider a two-dimensional square lattice with sites connected with their nearest neighbor.

2) Shortcuts introduced between pairs i and j of sites with probability $P_{ij} \sim r_{ij}^{-a}$, where r_{ij} is the "Manhattan distance".

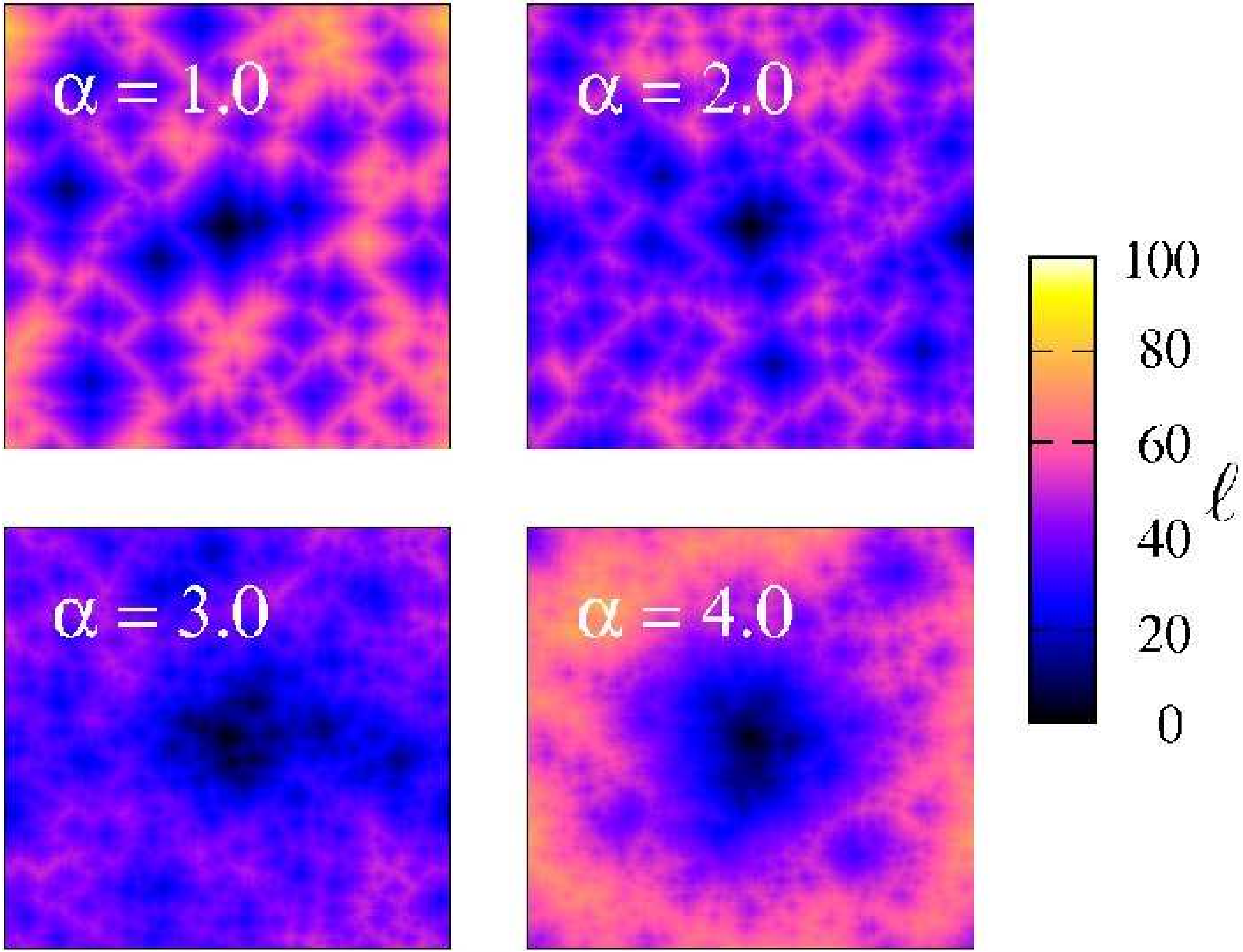
3) The addition of shortcuts stops when their total length (cost) reaches a given value

$$\Lambda = \sum r_{ij}$$



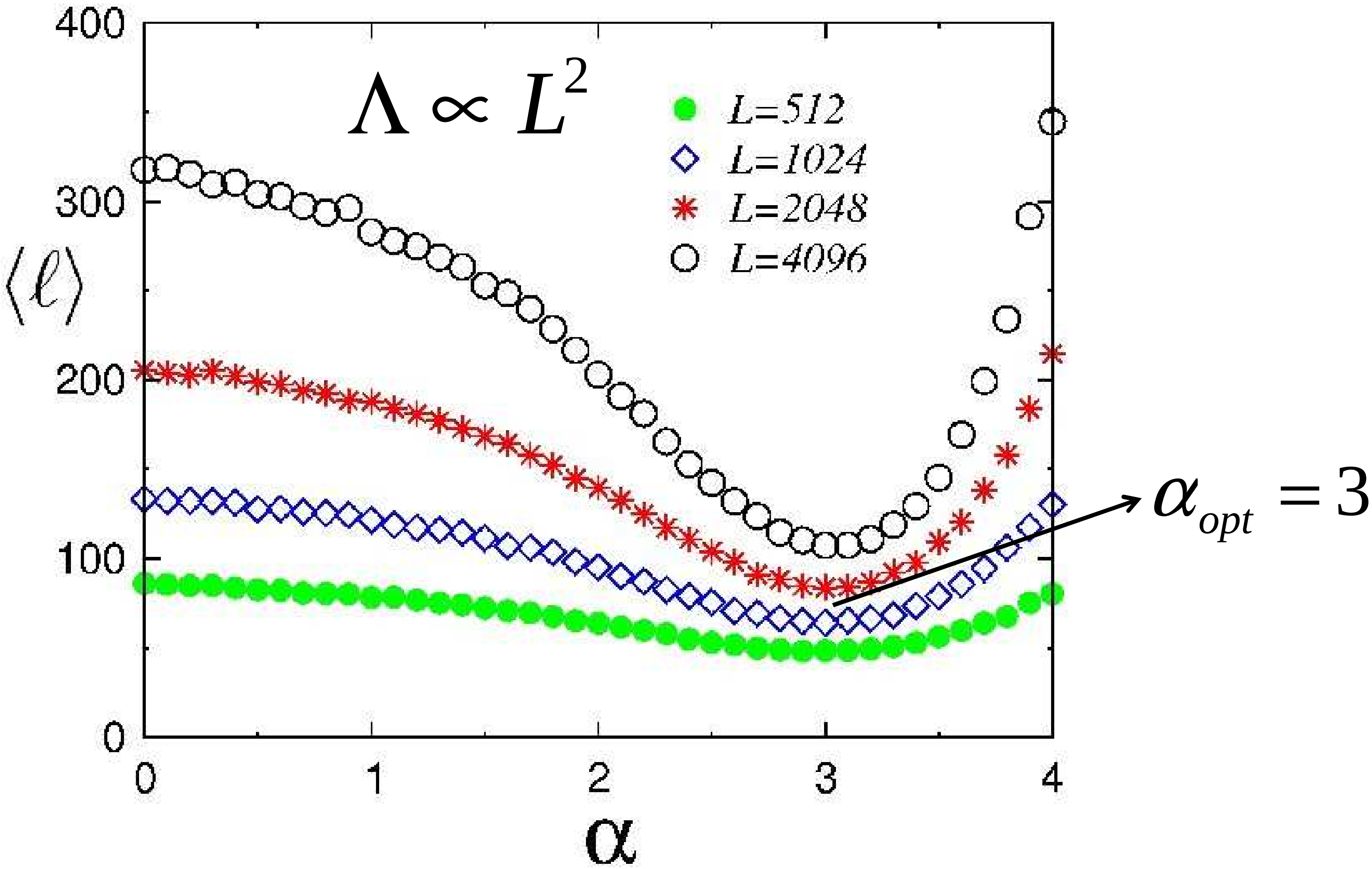
Shortest Paths

$$Cost = \Lambda \propto L^2$$



Global Information with Cost

Li, Reis, Moreira, Stanley, Havlin & JSA, PRL (2010)



Some scaling analysis…

By arbitrarily fixing the cost parameter to $\Lambda \propto L^2$,
we obtain,

$$\rho \sim \langle r \rangle^{-1} \qquad (1)$$

Where ρ is the expected density and $\langle r \rangle$ is the average length of the shortcuts. Since,

$$\langle r \rangle \sim \int_1^L r^{d-\alpha} dr \qquad (2)$$

we obtain,

$$\rho \sim \begin{cases} L^{-1} & , 0 \leqslant \alpha \leqslant d; \\ L^{-(d-\alpha+1)} & , d \leqslant \alpha \leqslant d+1; \\ L^0 & , d+1 \leqslant \alpha. \end{cases} \qquad (3)$$

Thus for $\alpha<3$, the density of shortcuts added decreases as a power law with L.

To see it, we argued that, the average shortest path length (ASPL) is bounded by the relation

$$\langle \ell \rangle > \rho^{1/d} \quad (4)$$

Where the right-hand side of the equation appears at the small-world scenario [M. Barthélémy and L. A. N. Amaral, Phys. Rev. Lett. (1999)]. Since, for $\alpha<3$ the bound,

$$\langle \ell \rangle \sim L^{(3-\alpha)/d} \quad (5)$$

is rigorous for our case and the ASPL must scale as a power of L. Thus, from Eqs.(3) and (5), it follows that only for $\alpha=3$, the ASPL scales logarithmically with L.

The global navigation scheme can be considered as a lower bound to any other transport navigation process.

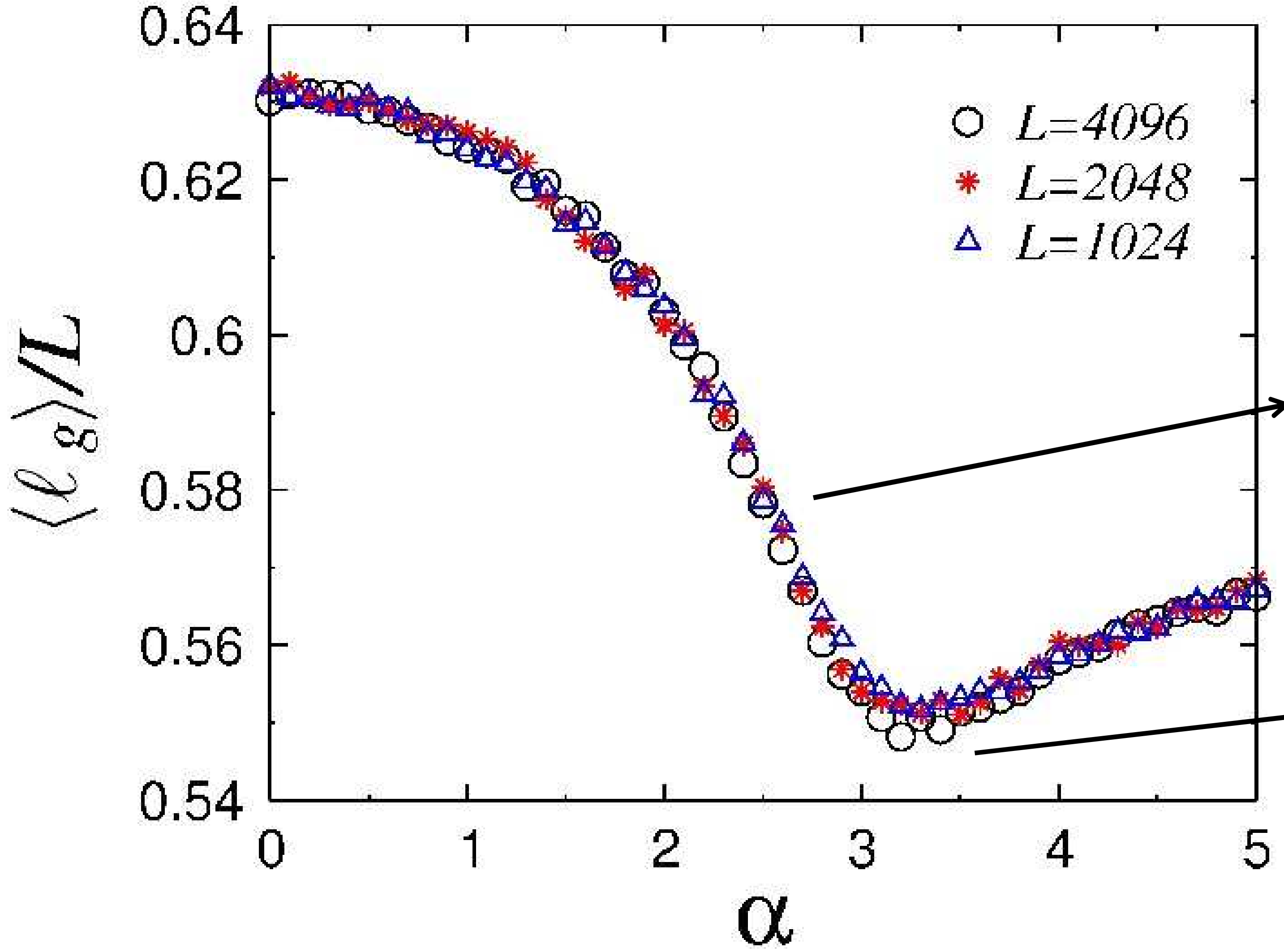
The next question is…

What would be the optimal condition when only local information is available?

In order of simulate navigation with local information, we use the greedy algorithm [J. M. Kleinberg, *Nature* (2000); *Proc. 32nd ACM Symposium on Theory of Computing* (2000)].

Local Information with Cost

$$\Lambda \propto L^2$$

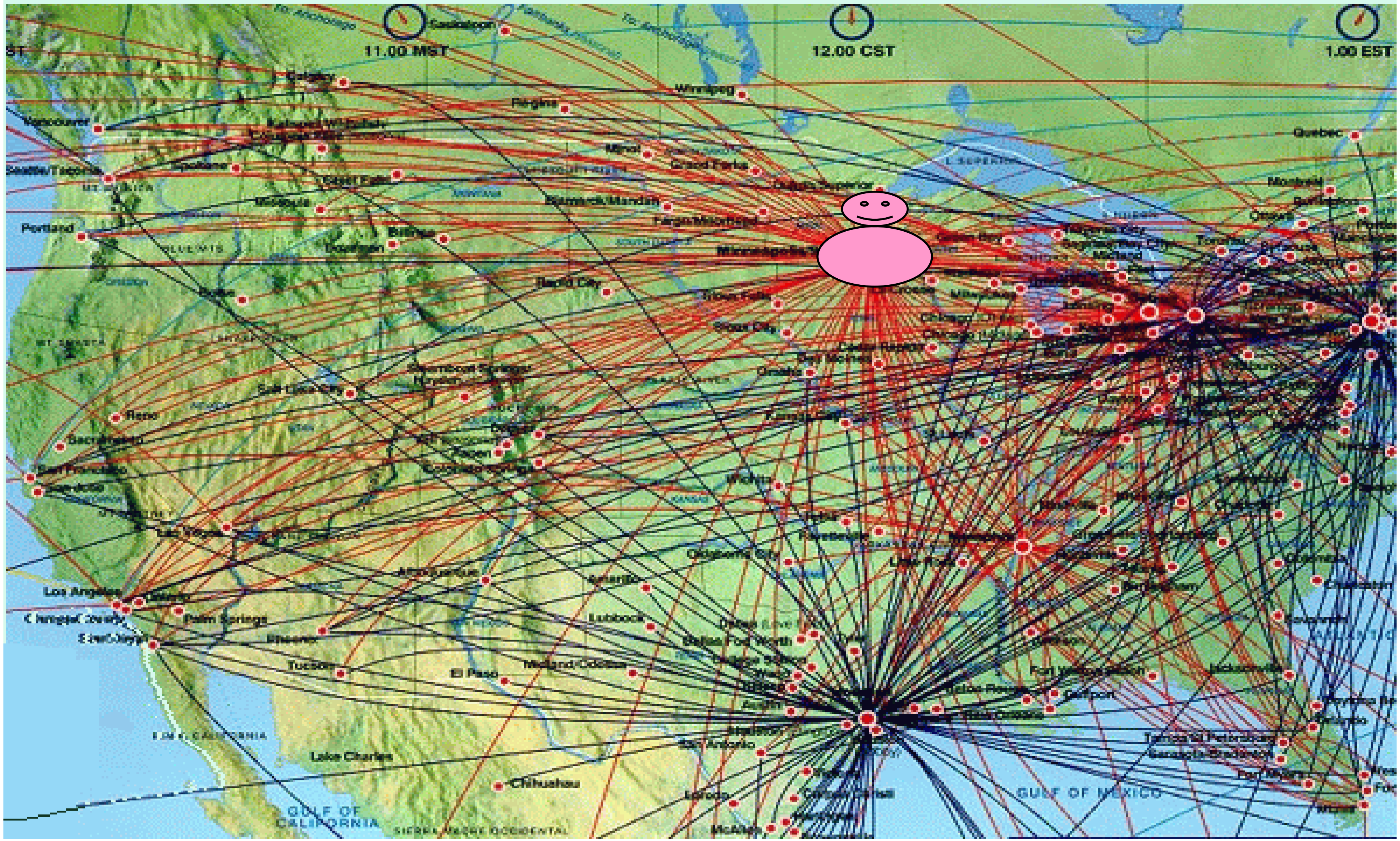


$\langle \ell_g \rangle \sim L$

$\alpha_{opt} \approx 3$

USA Airport Network

$\alpha = 3.0 \pm 0.2$ **[Bianconi *et al.*, PNAS (2009)]**



Conclusions

1) Our results suggest that, regardless of the strategy used by the traveler, based on local or global knowledge of the network structure, the best transportation condition is obtained with an exponent $\alpha=d+1$, where d is the topological dimension of the underlying lattice.

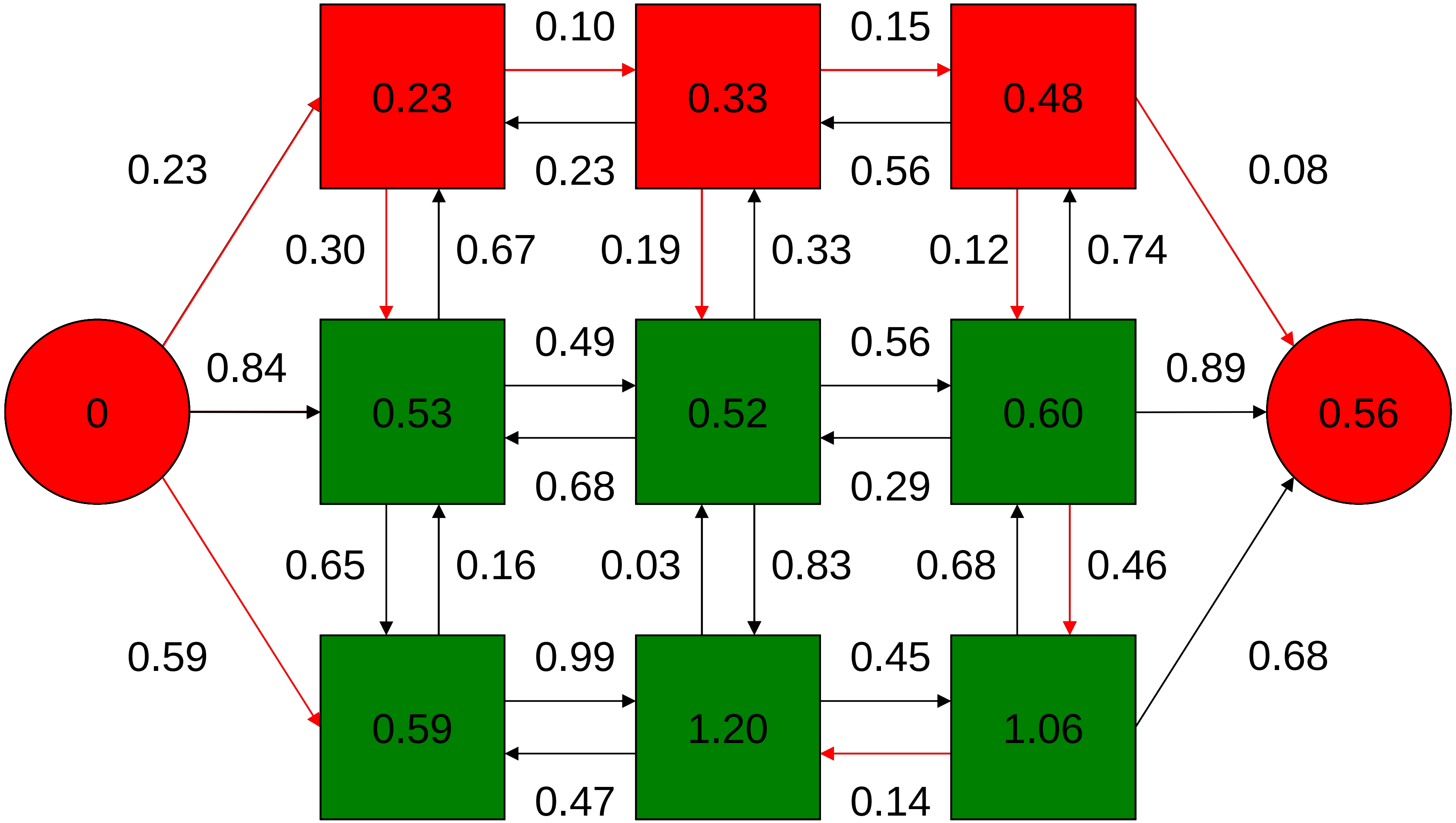
2) In the case where the traveler has global knowledge of the network, and is able to identify the shortest path for navigation, we obtain a slow (logarithmic) growth with size for the transit time at the optimal condition.

3) In the case where the transportation path is decided based on the "Manhattan distance" to the target (local knowledge), we obtain a linear increase of the transit time with system size, for all values of α.

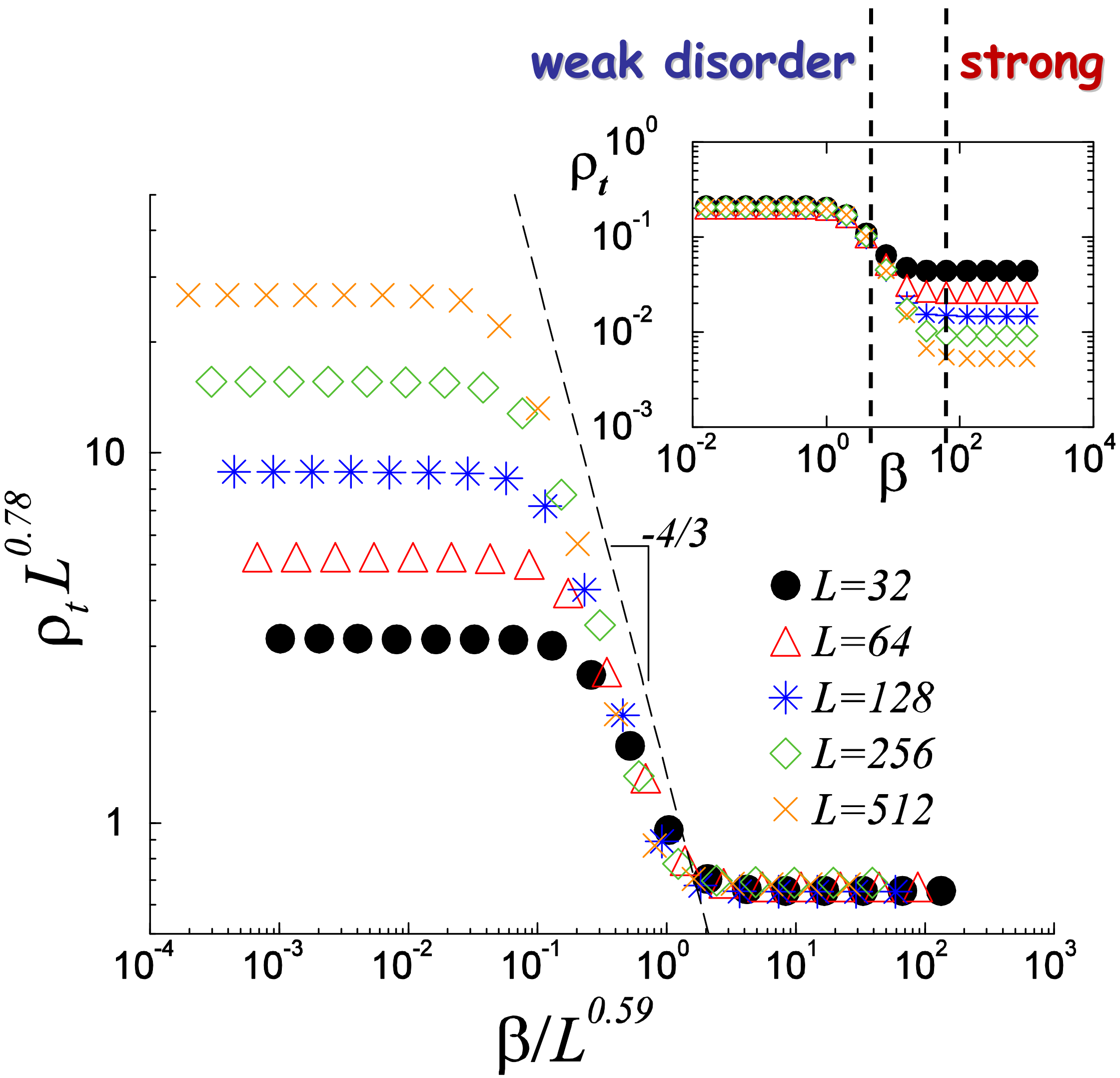
Thank you!

The Dijkstra Algorithm

Dijkstra, *Num. Math.* (1959)



Transition from weak to strong disorder



Three Regimes

i) small β

$$\rho_t = const$$

ii) intermediate β

$$\rho_t(\beta) \sim \beta^{-\theta}$$

with $\theta \approx 4/3$

iii) large β

$$\rho_t(L) \sim L^{D_b - 2}$$

with $D_b \approx 1.22$

At the crossover $\beta_\times$ we obtain,

$$\beta_\times^{-\theta} \sim L^{D_b - 2} \quad \Rightarrow \quad \beta_\times \sim L^{(2-D_b)/\theta}$$