COMPLEXIDADE EM ECONOMIA

Instituto Nacional de Ciência e Tecnologia – Sistemas Complexos

Correção exata para os coeficientes empíricos de tendência e de difusão : aplicação a séries temporais financeiras

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CBPF, Rio de Janeiro

1-5 março de 2010

Accessibility of Mathematical Limits

arbitrary low temperatures

$T \rightarrow 0$

thermodynamic limit

$N \rightarrow \infty$

asymptotic long time behavior

$n, t \rightarrow \infty$

arbitrary small time intervals

$\Delta t \rightarrow 0$

INTRODUCTION

The dynamics of many complex systems exhibits an interplay of processes with different spatio-temporal scales :

large-scale slow modes - deterministic forcing

&

small-scale fast modes - stochastic forcing

Modeling fluctuating phenomena as a Ito-Stochastic differential equation:

$$dX_t = D_1(X_t)dt + \sqrt{2D_2(X_t)}\, dW_t$$

where W_t is a standardized Wiener process: $< dW_t >= 0 \quad < dW_t^2 >= dt$

INTRODUCTION

The time evolution of the Probability Density Function (PDF)

$P(x,t) \equiv P(X_t = x\ ,\ t)$

can be described by the associated Fokker-Planck Equation:

$$\partial_t P(x,t) = -\partial_x [D_1(x) P(x,t)] + \partial_{xx} [D_2(x) P(x,t)]$$

For ideal time series X_t , the coefficients can be perfectly reconstructed by:

$$D_k\ (x) = \lim_{\tau \to 0} \widetilde{D}_k\ (x,\tau)$$

This limit can be directly estimated from the finite-time conditional moments:

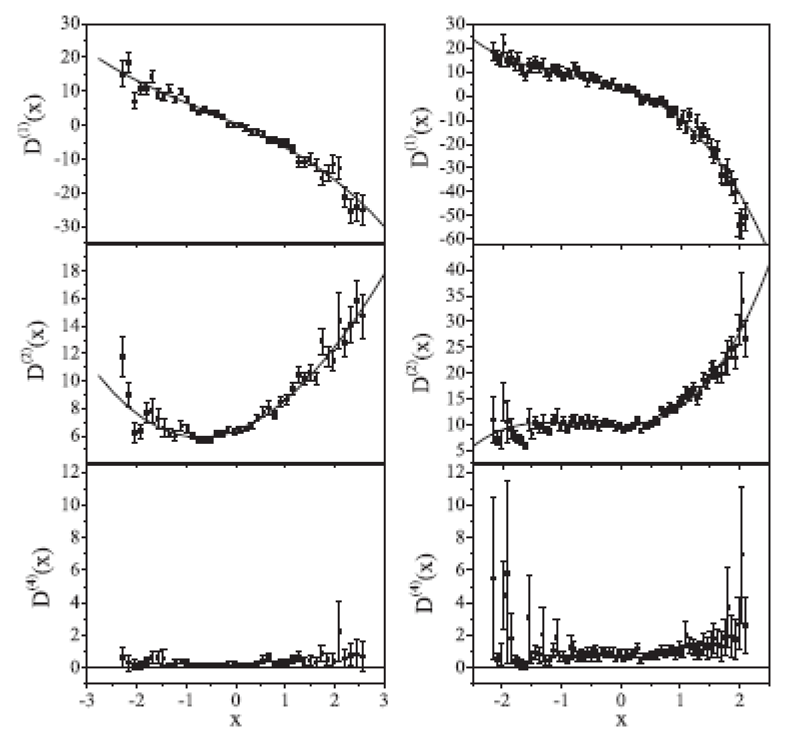
$$\widetilde{D}_k\ (x,\tau) = \frac{1}{\tau k!} < [X_{t+\tau} - X_t]^k > \Big|_{X_t = x}$$

CHARACTERIZATION OF DRIFT AND DIFFUSION COEFFICIENTS : MOTIVATION

stochastic qualifiers of brain dynamics:

discrimination of physiological and pathological activities

Prusseit &Lenhertz, PRL 98, 138103 (2007)



Electroencephalographic recordings:
normal (left) epileptic (right)

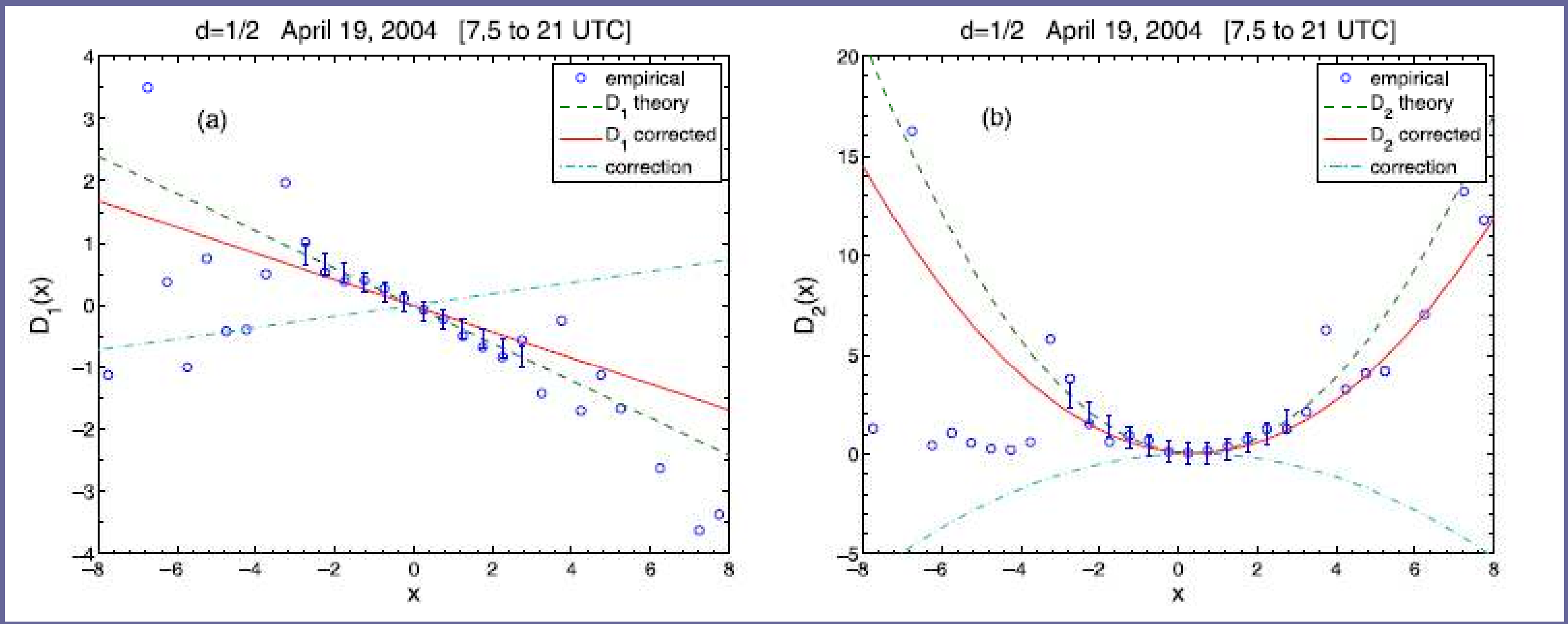
CHARACTERIZATION OF DRIFT AND DIFFUSION COEFFICIENTS: MOTIVATION

Earth climate-atmosphere components interplay for meteorological predictions:

slow large scale synoptic conditions & small fast scale iced crystal production

in cirrus clouds

Ivanova & Ackerman, J. Geophys. Res. 114 , D06113 (2009)



reflectivity measurements of cirrus clouds

CHARACTERIZATION OF DRIFT AND DIFFUSION COEFFICIENTS: MOTIVATION

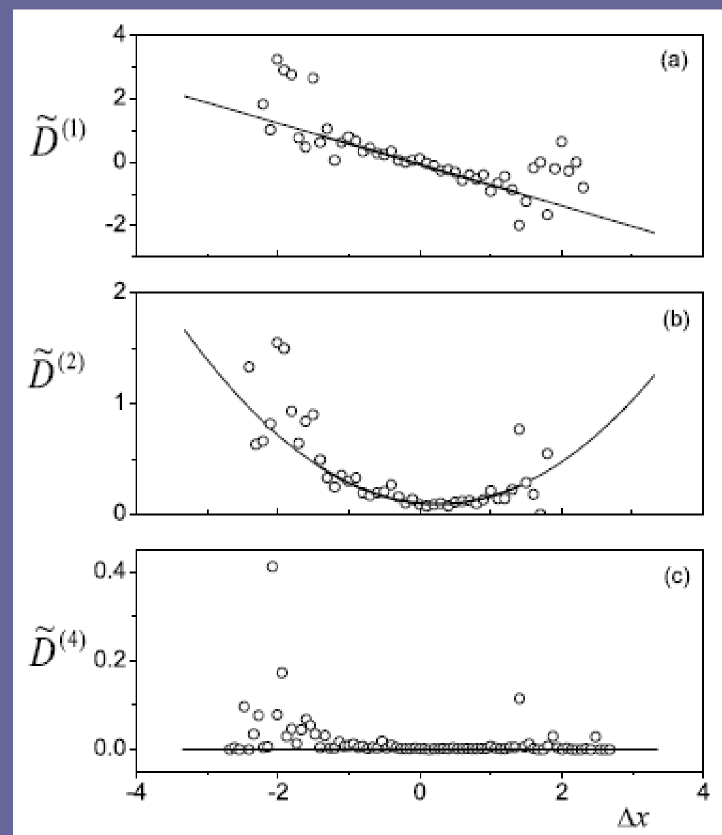
stock index dynamics worldwide – price formation driven by:

slow varying large-scale aggregated information & unforeseen fast private information

A.A.G.Cortines, C.Anteneodo & R.Riera , EPJB **65,** 289 (2008)

Table 1. World indices and corresponding labels, country ar
number of trading days in calendar period Jan. 1997–Jul. 200

Label	Index	Country	# days
1	AEX	Netherlands	2678
2	ATX	Austria	2597
3	BEL 20	Belgium	2669
4	CAC 40	France	2669
5	DAX	Germany	2662
6	FTSE 100	UK	2652
7	SMI	Switzerland	2647
8	BSE 30	India	2477
9	HSI	Hong Kong	2599
10	JSXC	Indonesia	2428
11	KLSEC	Malaysia	2591
12	KOSPI	South Korea	2490
13	Nikkei 225	Japan	2581
14	STI	Singapore	2643
15	TWI	Taiwan	2462
16	DJIA	USA	2666
17	Nasdaq	USA	2652
18	NYSE	USA	2652
19	S&P 500	USA	2645
20	Ibovespa	Brazil	2596
21	IPC	Mexico	2631
22	Merval	Argentina	2597
23	CMA	Egypt	1964
24	AOX	Australia	2666



German index DAX at weekly timescales

EMPIRICAL ACCESS TO THE TRUE DRIFT AND DIFFUSION COEFFICIENTS

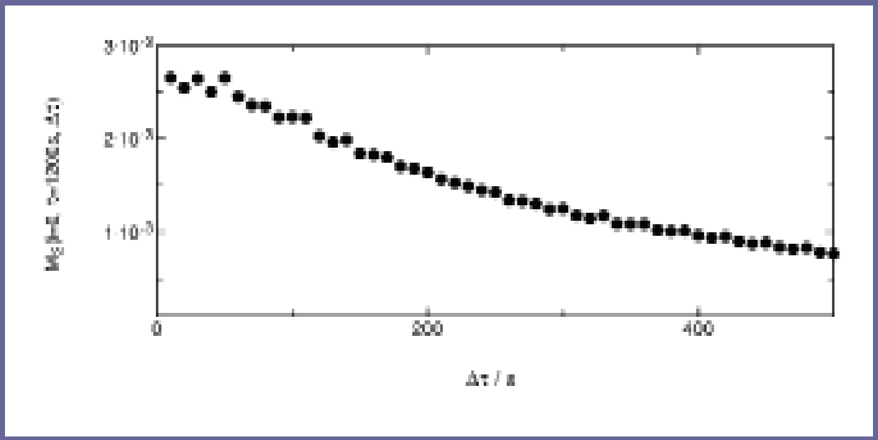
However, due to the finite sampling rate of real data X_t ,

one access only the finite-τ estimation $\widetilde{D}_k\ (x,\tau)$

which may significantly differ from the true limit.

To achieve $D_k\ (x) = \lim\limits_{\tau \to 0} \widetilde{D}_k\ (x,\tau)$

some authors have proposed extrapolation schemes:



METHODOLOGY: ITO-TAYLOR EXPANSION

The error in the finite-τ coefficients can be derived from

$$dX_t = D_1(X_t)dt + \sqrt{2D_2(X_t)}\, dW_t$$

in its integrated form :

$$X_{t+\tau} = X_t + \int_t^{t+\tau} D_1(X_{t'})dt' + \int_t^{t+\tau} \sqrt{2D_2(X_{t'})}\, dW_{t'}$$

Let us consider the stochastic Ito-expansion for a given function $F(X_t)$:

$$dF = (D_1\partial_x F + D_2\partial_{xx}F)dt + (\sqrt{2D_2}\,\partial_x F)dW \equiv L^0 F dt + L^1 F dW$$

and its integrated form:

$$F(X_{t'}) = F(X_t) + \int_t^{t'} L^0 F(X_{t''})dt'' + \int_t^{t'} L^1 F(X_{t''})dW_{t''}$$

METHODOLOGY: ITO-TAYLOR EXPANSION

By applying Ito formula to the functions $D_1(X_{t'})$ and $\sqrt{2D_2(X_{t'})}$:

$$X_{t+\tau} = X_t + \int_t^{t+\tau} \left\{ D_1(X_t) + \int_t^{t'} L^0 D_1(X_{t''}) dt'' + + \int_t^{t'} L^1 D_1(X_{t''}) dW_{t''} \right\} dt'$$
$$+ \int_t^{t+\tau} \left\{ \sqrt{2D_2(X_t)} + \int_t^{t'} L^0 \sqrt{2D_2(X_{t''})}\, dt'' + \int_t^{t'} L^1 \sqrt{2D_2(X_{t''})}\, dW_{t''} \right\} dW_{t'}$$

After iterated applications of Ito formula one gets an expression in terms of multiple stochastic integrals:

$$X_{t+\tau} - X_t = \sum_{\alpha_k} c_{\alpha_k}(D_1, D_2)\, I_{\alpha_k}$$

where $\alpha_k = (j_1, j_2, \ldots, j_k)$, with $j_i = 0, 1$ for all i, $c_{\alpha_k}(A, B) = L^{j_1} L^{j_2} \ldots L^{j_{k-1}} L^{j_k}$ and I_{α_k} are multiple stochastic integrals of the form $I_{\alpha_k} = \int_0^{\tau} \int_0^{t_k} \int_0^{t_{k-1}} \ldots \int_0^{t_2} dt_1^{j_1} \ldots dt_{k-1}^{j_{k-1}} dt_k^{j_k}$, being $dt_i^0 \equiv dt_i$ and $dt_i^1 \equiv dW_i$.

METHODOLOGY: ITO-TAYLOR EXPANSION

Inserting a simpler notation: $D_1(X_{t'}) \equiv A(t')$ and $\sqrt{2D_2(X_{t'})} \equiv B(t')$

$$X_{t+\tau} - X_t =$$

$$\int_t^{t+\tau} \left\{ A(t) + \int_t^{t'} L^0 A(t'')dt'' + + \int_t^{t'} L^1 A(t'')dW_{t''} \right\} dt' \qquad (I)$$

$$+$$

$$\int_t^{t+\tau} \left\{ B(t) + \int_t^{t'} L^0 B(t'')\,dt'' + \int_t^{t'} L^1 B(t'')\;dW_{t''} \right\} dW_{t'} \qquad (II)$$

$$I = \int_t^{t+\tau} A(t')dt' = \int_t^{t+\tau} dt' A(t) + \int_t^{t+\tau} dt' \int_t^{t'} L_0 A(t'')dt'' + $$
$$+ \int_t^{t+\tau} dt' \int_t^{t'} L_1 A(t'')dW(t'') \,.$$

$$II = \int_t^{t+\tau} B(t')dW(t'') = \int_t^{t+\tau} dW(t')B(t) + \int_t^{t+\tau} dW(t') \int_t^{t'} L_0 B(t'')dt'' + $$
$$+ \int_t^{t+\tau} dW(t') \int_t^{t'} L_1 B(t'')dW(t'') \,.$$

METHODOLOGY: ITO-TAYLOR EXPANSION

Using the operator definitions:

$$L^0 \equiv A\partial_x + \frac{1}{2}B^2\partial_{xx} \qquad L^1 \equiv B\partial_x$$

$$\begin{aligned} I = \; & \int_t^{t+\tau} dt' A(t) + \int_t^{t+\tau} dt' \int_t^{t'} A(t'')\partial_{X_{t''}} A(t'') dt'' \\ + \; & \int_t^{t+\tau} dt' \int_t^{t'} \frac{1}{2} B^2(t'') \partial^2_{X_{t''}X_{t''}} A(t'') dt'' \\ + \; & \int_t^{t+\tau} dt' \int_t^{t'} B(t'')\partial_{X_{t''}} A(t'') dW(t'') \, . \end{aligned}$$

$$\begin{aligned} II = \; & \int_t^{t+\tau} dW(t') B(t) + \int_t^{t+\tau} dW(t') \int_t^{t'} A(t'')\partial_{X_{t''}} B(t'') dt'' \\ + \; & \int_t^{t+\tau} dW(t') \int_t^{t'} \frac{1}{2} B^2(t'') \partial^2_{X_{t''}X_{t''}} B(t'') dt'' \\ + \; & \int_t^{t+\tau} dW(t') \int_t^{t'} B(t'')\partial_{X_{t''}} B(t'') dW(t'') \, . \end{aligned}$$

METHODOLOGY: ITO-TAYLOR EXPANSION

Truncating the expansion, the operations in *t''* are performed in *t* , leading to:

$$I = \quad A(t)\int_t^{t+\tau} dt' + A(t)\partial_{X_t}A(t)\int_t^{t+\tau} dt' \int_t^{t'} dt'' + \frac{1}{2}B^2(t)\partial^2_{X_tX_t}A(t) \times$$
$$\times \int_t^{t+\tau} dt' \int_t^{t'} dt'' + B(t)\partial_{X_t}A(t)\int_t^{t+\tau} dt' \int_t^{t'} dW(t'')\,,$$

$$II \quad = B(t)\int_t^{t+\tau} dW(t') + A(t)\partial_{X_t}B(t)\int_t^{t+\tau} dW(t') \int_t^{t'} dt'' + \frac{1}{2}B^2(t)\partial^2_{X_tX_t}B(t)$$
$$\times \int_t^{t+\tau} dW(t') \int_t^{t'} dt'' + B(t)\partial_{X_t}B(t)\int_t^{t+\tau} dW(t') \int_t^{t'} dW(t'')\,.$$

METHODOLOGY: ITO-TAYLOR EXPANSION

Defining the multiple stochastic integrals:

$$I_0 \equiv \int_t^{t+\tau} dt' \qquad I_{00} \equiv \int_t^{t+\tau} dt' \int_t^{t'} dt'' \qquad I_{01} \equiv \int_t^{t+\tau} dt' \int_t^{t'} dW_{t''}$$

$$I_1 \equiv \int_t^{t+\tau} dW_{t'} \qquad I_{10} \equiv \int_t^{t+\tau} dW_{t'} \int_t^{t'} dt'' \qquad I_{11} \equiv \int_t^{t+\tau} dW_{t'} \int_t^{t'} dW_{t''}$$

Averaging::

$$< I_0 > = \tau \qquad < I_{00} > = \tau^2/2 \qquad < I_{01} > = 0$$

$$< I_1 > = 0 \qquad < I_{10} > = 0 \qquad < I_{11} > = 0$$

METHODOLOGY: ITO-TAYLOR EXPANSION

Therefore, for the first conditional moment one has:

$$\widetilde{D}_1\ (x,\tau) = \frac{1}{\tau} < [X_{t+\tau} - X_t]\ >\Big|_{X_t = x}$$

$$\widetilde{D}_1\ (x,\tau) = A + (A\partial_x A + \frac{1}{2} B\partial_{xx} A)\frac{\tau}{2}$$

For the second conditional moment one has:

$$\widetilde{D}_2\ (x,\tau) = \frac{1}{2!\tau} < [X_{t+\tau} - X_t]^2 >\Big|_{X_t = x}$$

$$2\tau\tilde{D}_2 = \langle(\Delta X)^2\rangle = \sum_{\alpha_n,\beta_m} c_{\alpha_n} c_{\beta_m} \langle I_{\alpha_n} I_{\beta_m}\rangle.$$

LINEAR-DRIFT AND QUADRATIC-DIFFUSION COEFFICIENTS

We consider a representative class of diffusion models described by:

$$dX_t = D_1(X_t)dt + \sqrt{2D_2(X_t)}\, dW_t$$

$$D_1(x) = -a_1 x$$

$$D_2(x) = b_0 + b_2 x^2$$

By inserting the expressions $X_{t+\tau} - X_t = \sum_{\alpha_k} c_{\alpha_k}(D_1, D_2)\, I_{\alpha_k}$

and performing the average in $\widetilde{D}_k\ (x,\tau) = \frac{1}{\tau k!} < [X_{t+\tau} - X_t]^k > \Big|_{X_t = x}$

the resulting expressions preserve the linear and quadratic x-dependence:

$$\widetilde{D}_1(x,\tau) = -\widetilde{a}_1(\tau)\, x \qquad \widetilde{D}_2(x,\tau) = \widetilde{b}_0(\tau) + \widetilde{b}_2(\tau) x^2$$

ARBITRARY-ORDER CORRECTIONS

R. Riera & C. Anteneodo PRE 80, 031103 (2009)

Hence, we are led

to the theoretical relation

between finite-τ coefficients

and the true ones:

$$\tilde{a}_1 = a_1 \sum_{j\geq 0} \frac{[-a_1]^j}{(j+1)!}\tau^j$$

$$\tilde{b}_0 = b_0 \sum_{j\geq 0} \frac{[(-2(a_1-b_2)]^j}{(j+1)!}\tau^j$$

$$\tilde{b}_2 = b_2 \sum_{j\geq 0} \frac{\frac{1}{2}[2(a_1-b_2)]^{j+1}-[a_1]^{j+1}}{(j+1)!}\tau^j$$

Our results generalizes previous ones in literature:

Sura & Barsugli , Phys. Lett. A 305, 304 (2002)

Ragwitz & Kant , Phys. Rev. Lett. 87, 254501 (2001)

Gottschall & Peinke , NJP 10, 083034 (2008)

ARBITRARY-ORDER CORRECTIONS

Summing the series we find the exact finite-τ expressions:

$$\tilde{a}_1 = \frac{1-Z}{\tau},$$

$$\tilde{b}_0 = \frac{b_0}{a_1 - b_2}\,\frac{1-Z^2W}{2\tau},$$

$$\tilde{b}_2 = \frac{1-Z}{\tau} - \frac{1-Z^2W}{2\tau}.$$

Defining: $Z \equiv \exp(-a_1\tau)$ $W \equiv \exp(2b_2\tau)$

KEY-NOTES

- The stationary PDF $$P^*(x) = P_o/[1 + \frac{b_2}{b_0}x^2]^{\frac{a_1}{2b_2}+1},$$

  has finite variance for (a_1- b_2)>0 : $$\sigma^2 = b_0/(a_1 - b_2).$$

- There is a invariant relation among estimated and true parameters: $$\frac{\tilde{a}_1 - \tilde{b}_2}{\tilde{b}_0} = \frac{a_1 - b_2}{b_0}.$$

representing the uphold of data variance under changes of sampling intervals.

- Normalized data only implies the rescaling: $$b_0 \to b_0/\sigma^2.$$

  leading to $$a_1 = b_0 + b_2 .$$

EXACT FINITE-τ EXPRESSIONS

Summarizing, for normalized data one gets:

$$\tilde{a}_1 = \frac{1 - \exp(-a_1\tau)}{\tau}$$

$$\tilde{b}_0 = \frac{1 - \exp(-2b_0\tau)}{2\tau}$$

with the constraints:

$$b_2 = a_1 - b_0$$

$$\tilde{b}_2 = \tilde{a}_1 - \tilde{b}_0$$

Extracting the true parameters from the finite-τ estimates:

$$a_1 = \frac{\ln(1 - \tilde{a}_1\tau))}{-\tau},$$

$$b_0 = \frac{\ln(1 - 2\tilde{b}_0\tau))}{-2\tau}.$$

KEY-NOTES

❖ The relevant quantities are τa_1, τb_0 and τb_2

meaning the invariance of laws on the chosen temporal units

In what follows we fix the time scale $\tau = 1$.

Other choices only implies the rescaling:

$$(a_1, b_0, b_2) \to (\tau a_1, \tau b_0, \tau b_2)\,.$$

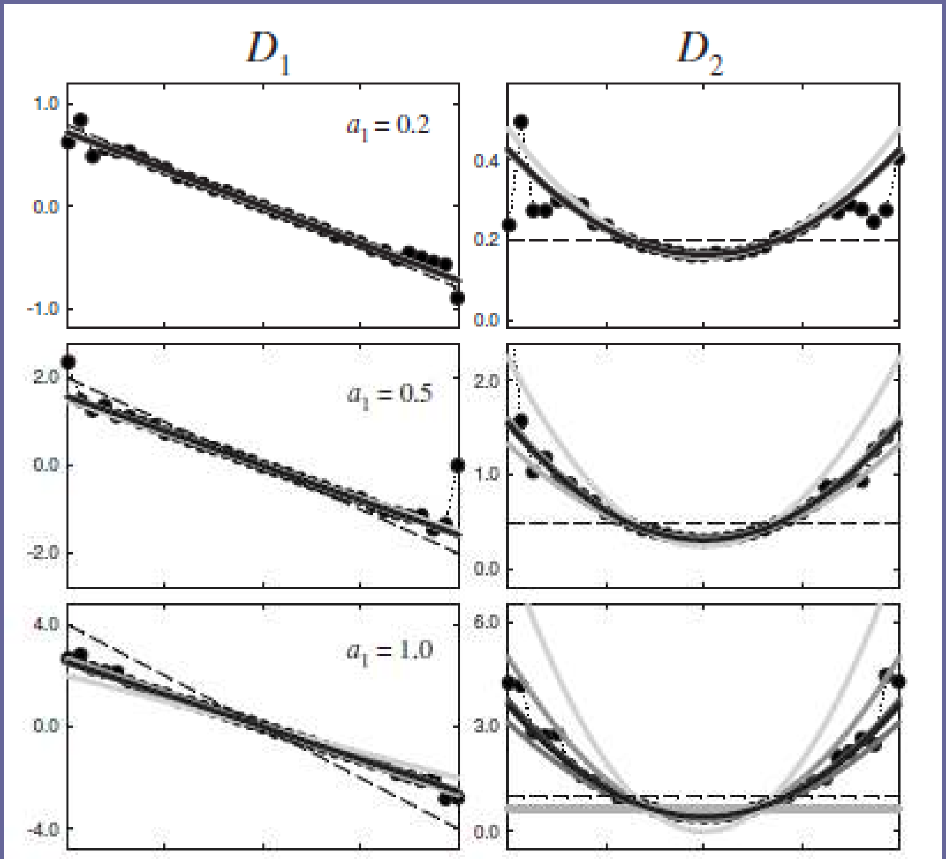
FITNESS OF LOW-ORDER APPROXIMATIONS

O-U Processes: $a_1 = b_0 \ (b_2 = 0)$

$$\tilde{a}_1 = a_1 \sum_{j \geq 0} \frac{[-a_1]^j}{(j+1)!} \tau^j$$

$$\tilde{b}_0 = b_0 \sum_{j \geq 0} \frac{[(-2(a_1 - b_2)]^j}{(j+1)!} \tau^j$$

$$\tilde{b}_2 = \tilde{a}_1 - \tilde{b}_0$$



$\tau = 1.$

Numerical computation for artificial series

Theoretical results for different orders of truncation (darker colors for higher orders);

The infinite order (exact) is in thick black lines

The zero order (true values) is in dashed lines

FITNESS OF LOW-ORDER APPROXIMATIONS

General Processes: $a_1 = b_0 + b_2\,.$

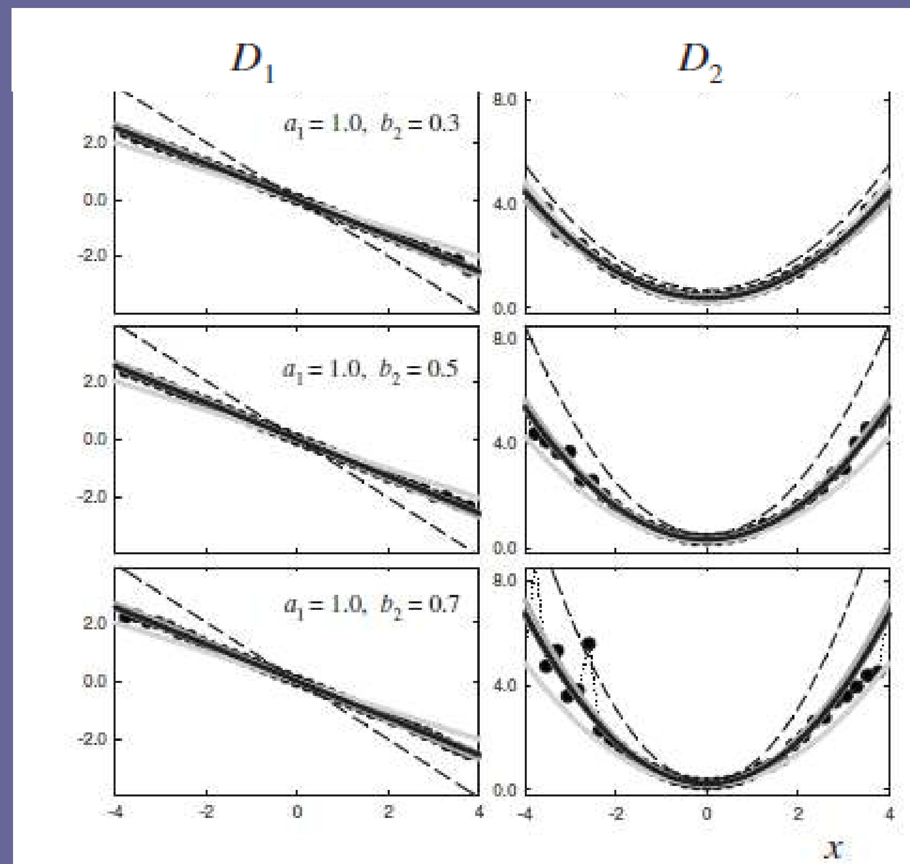
$$\tilde{a}_1 = \quad a_1 \sum_{j \geq 0} \frac{[-a_1]^j}{(j+1)!}\tau^j$$

$$\tilde{b}_0 = \quad b_0 \sum_{j \geq 0} \frac{[(-2(a_1 - b_2)]^j}{(j+1)!}\tau^j$$

$$\tilde{a}_1 = \quad \frac{1 - \exp(-a_1\tau)}{\tau}$$

$$\tilde{b}_0 = \quad \frac{1 - \exp(-2b_0\tau)}{2\tau}$$

$$\tilde{b}_2 = \tilde{a}_1 - \tilde{b}_0$$



$\tau = 1.$

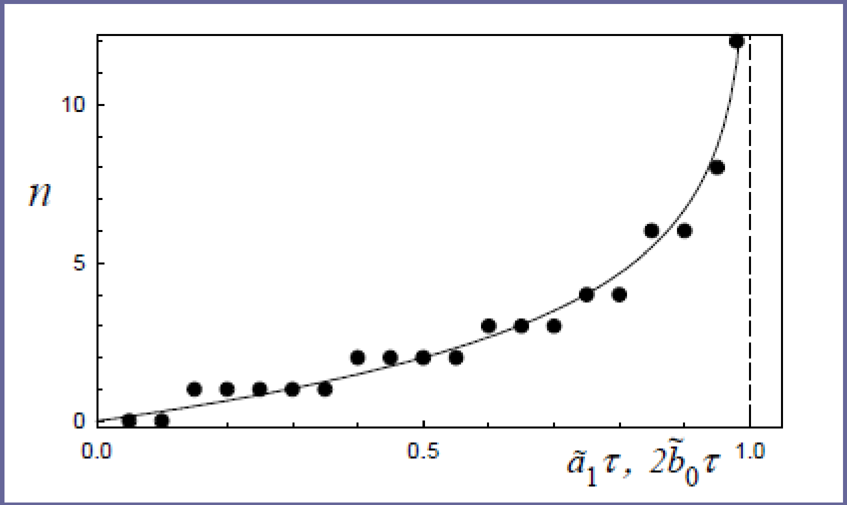
FITNESS OF LOW-ORDER APPROXIMATIONS

Consider the series expansion for $\tilde{a}_1$ truncated at order n.

$$\tilde{a}_1 = a_1 \sum_{j\geq 0} \frac{[-a_1]^j}{(j+1)!}\tau^j$$

By inversion of the series we obtain the n-th order correction for a_1 from the finite -τ estimate $\tilde{a}_1$.

The order necessary to achieve the true value within 5% error increases as



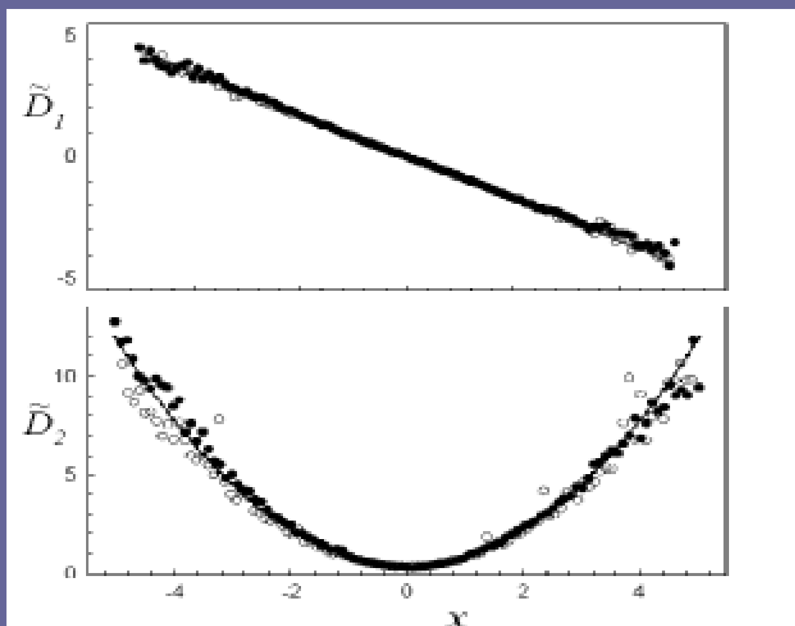
Conclusions:

- the value of a_1 sets the rate of convergence of D_1 and D_2
- convergence is slower as a_1 increases
- one should be careful when applying low-order finite- τ corrections for diffusion models
- our results provide a criterion up to which order n , or, up to which value of τ the approximation is reliable
- order larger than 2 is required to attain the true value (within 5% error) when $\tilde{a}_1 > 0.5$

CORRECT ESTIMATES OF DRIFT AND DIFFUSION COEFFICIENTS

For real time series X_t , the coefficients can be perfectly reconstructed :

Brazilian index IBOVESPA at intraday timescales



A.A.G.Cortines, C.Anteneodo & R.Riera , in preparation

$$(\tilde{a}_1, \tilde{b}_2, \tilde{b}_0) = (0.84,\ 0.38,\ 0.46)$$

$$a_1 = \frac{\ln(1 - \tilde{a}_1 \tau))}{-\tau},$$

$$b_0 = \frac{\ln(1 - 2\tilde{b}_0 \tau))}{-2\tau}.$$

$$(a_1, b_2, b_0) = (1.83,\ 0.71,\ 1.12)$$

HIGHER-ORDER COEFFICIENTS

Markovian processes are governed by the Kramers-Moyal expansion:

$$\partial_t P(x,t) = \sum_{k\geq 1} \left(-\partial_x\right)^k [D_k(x) P(x,t)]$$

For consistency, diffusion processes requires vanishing D_k for $k \geq 3$.

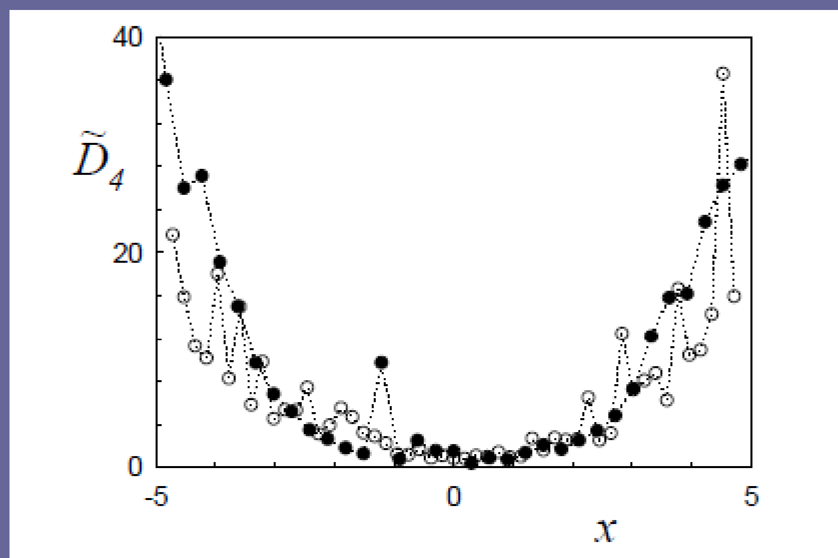
Pawula theorem simplifies our task: we only need to check D_4

However the estimates of these coefficients also presents finite- τ effects

⇨ It is necessary to check if the observed deviations of D_4 are due to the inite sampling rate of real data

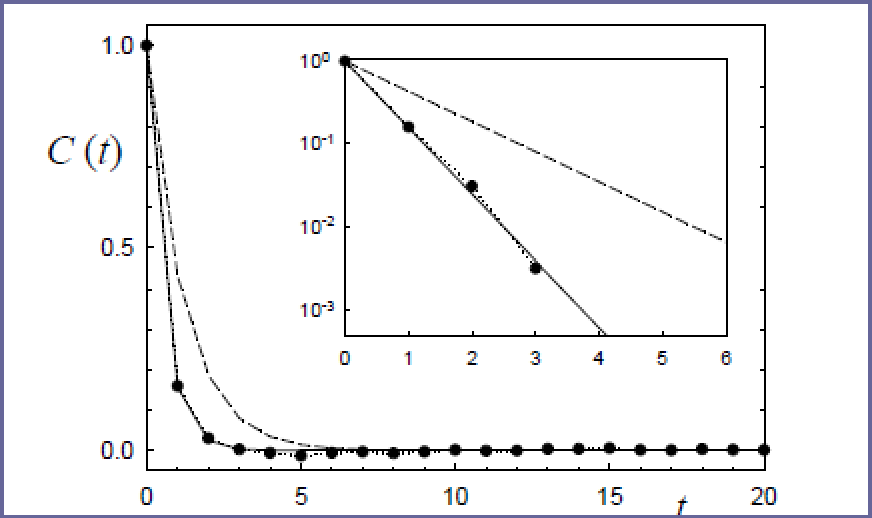
EXTRA CHECKS

The outcomes of an artificial time series generated with the inferred true parameter (●) reproduces the empirical results (○) of IBOVESPA, confirming that the observed deviations are due to the finite sampling rate.



EXTRA CHECKS

- Excellent agreement between the linear autocorrelation of returns predicted by the theoretical model $C(t) = \exp(-a_1 t)$ (full line) and the empirical results (●).
- Linear autocorrelation consistent with finite- τ parameter $C(t) = \exp(-\tilde{a}_1 t)$ (dashed line) overestimates the empirical results



EXACT CORRETIONS FOR FINITE-τ LINEAR D_1 AND QUADRATIC D_2

Conclusions:

- we presented the exact corrections that one should apply to the empirical finite-τ coefficients to find the true hidden ones

- for the exemplary financial time series , the coefficients D_1 and D_2 can be perfectly reconstructed

- as a test of consistency of the diffusion modeling, one should check if the non-null character of D_4 is due to the finite sampling rate.