Generalized Logarithmic and Exponential Functions and Growth Models

Alexandre Souto Martinez
Rodrigo Silva González and
Aquino Lauri Espíndola

Department of Physics and Mathematics
Ribeirão Preto School of Philosophy, Sciences and Literature
University of São Paulo,
Avenida Bandeirantes, 3900
Ribeirão Preto, São Paulo
14040-901, Brazil

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Contents

Introduction

- One-parameter logarithmic and exponential functions have been proposed in the context of:
  - non-extensive statistical mechanics $\Rightarrow S(A+B) \neq S(A) + S(B)$;
  - relativistic statistical mechanics;
  - quantum group theory.
- Two and three-parameter generalization of these functions have also been proposed.
- One parameter generalization of these functions can be obtained using simple geometrical arguments.

Non-Symmetrical Hyperboles and Power Laws

- Consider $f_{\tilde{q}}(t) = \frac{1}{t^{1-\tilde{q}}}$

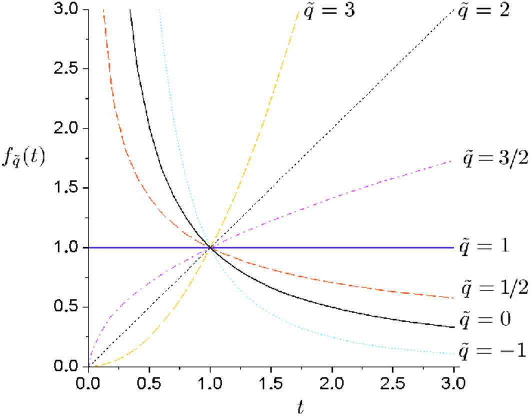
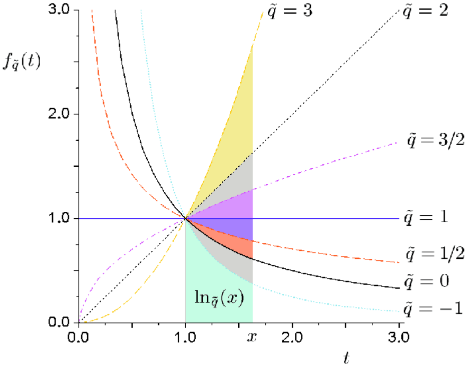


Figure: Behaviors of equation $1/t^{1-\tilde{q}}$ as a function of $\tilde{q}$.

$\tilde{q}$-Logarithm Function

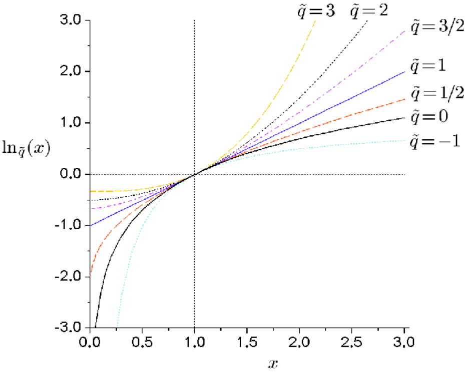
- $\ln_{\tilde{q}}(x)$ is defined as the value of the area underneath the non-symmetric hyperbole in the interval $t \in [1, x]$:



$$\ln_{\tilde{q}}(x) = \int_1^x \frac{dt}{t^{1-\tilde{q}}}$$

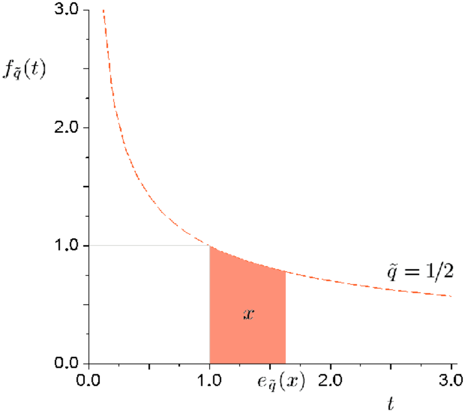
$$\ln_{\tilde{q}}(x) = \begin{cases} \frac{x^{\tilde{q}}-1}{\tilde{q}}, & \text{if } \tilde{q} \neq 0, \\ \ln(x), & \text{if } \tilde{q} = 0. \end{cases}$$

- For any $\tilde{q}$, the area is for:
  - $0 < x < 1$, negative
  - $x = 1$, null
  - $x > 1$, positive
- Convergence and divergence:
  - $(\tilde{q} < 0)$:
    - $x \to 0$, $\ln_{\tilde{q}}(x) \to -\infty$
    - $x \to \infty$, $\ln_{\tilde{q}}(x) \to -1/\tilde{q}$
  - $(\tilde{q} = 0)$:
    - $x \to 0$, $\ln_{\tilde{q}}(x) \to -\infty$
    - $x \to \infty$, $\ln_{\tilde{q}}(x) \to +\infty$
  - $(\tilde{q} > 0)$:
    - $x \to 0$, $\ln_{\tilde{q}}(x) \to -1/\tilde{q}$
    - $x \to \infty$, $\ln_{\tilde{q}}(x) \to +\infty$



$\tilde{q}$-Exponential Function

- Let x be the area underneath the curve $f_{\tilde{q}}(t)$, for $t \in [0, e_{\tilde{q}}(x)]$:



$$e_{\tilde{q}}[\ln_{\tilde{q}}(x)] = \ln_{\tilde{q}}[e_{\tilde{q}}(x)] = x$$

$$e_{\tilde{q}}(x) = \lim_{\tilde{q}' \to \tilde{q}} [1 + \tilde{q}' x]_{+}^{1/\tilde{q}'}$$

- $[a]_+ = \max(a, 0)$ guarantees $t > 0$ in $f_{\tilde{q}}(t) = \frac{1}{t^{1-\tilde{q}}}$.

- $e_{\tilde{q}}(x) > 0$ (non-negative function):
  - $e_{\tilde{q}}(0) = 1$
  - $\tilde{q} \to \pm\infty$, $e_{\pm\infty}(x) = 1$

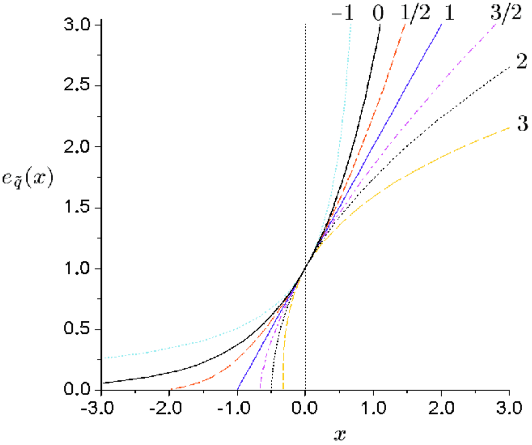
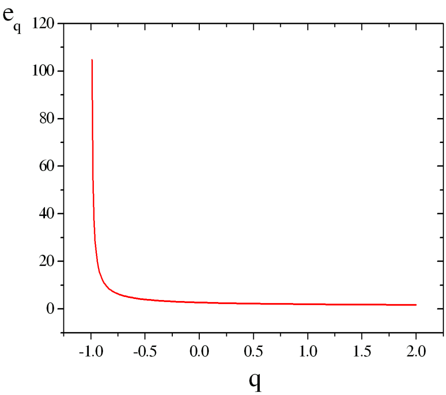


Figure: Behaviors of equation $\lim_{\tilde{q}' \to \tilde{q}} [1 + \tilde{q}' x]_+^{1/\tilde{q}'}$ as a function of $\tilde{q}$.

Generalized Euler's number



- The Euler's number:

$$\lim_{n \to 0} (1+n)^{1/n} = e = 2.718\ldots$$

- The generalized the Euler's number, for $x = 1$:

$$e_{\tilde{q}} = e_{\tilde{q}}(1) = (1+\tilde{q})^{1/\tilde{q}}.$$

Richards' Model (arXiv:0803.2635)

$$\frac{d \ln p(t)}{dt} = -\kappa \ln_{\tilde{q}} p(t),$$

- $p(t) = N(t)/N_\infty$, with:
  - $N(t)$ is the population size at time t
  - N_∞ is the carrying capacity
- κ is the intrinsic growth rate
- Limiting cases for:
  - $(\tilde{q} = 0) \Rightarrow$ Gompertz model, $d \ln p/dt = -\kappa \ln p$
  - $(\tilde{q} = 1) \Rightarrow$ Verhulst model, $d \ln p/dt = -\kappa \ \ln_1(p) = \kappa\ (1 - p)$
- The model solution is the $\tilde{q}$-generalized logistic equation:

$$p(t) = \frac{1}{e_{\tilde{q}}[\ln_{\tilde{q}}(p_0^{-1})e^{-\kappa t}]} = e_{-\tilde{q}}[-\ln_{\tilde{q}}(p_0^{-1})e^{-\kappa t}].$$

- Notice that: $1/e_{\tilde{q}}(x) = e_{-\tilde{q}}(-x)$.

In short

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Table 1
Growth models which can be obtained from Eq. (29)

Model	$\tilde{q}'$	$\tilde{q}$	γ	κ	Equation
Malthus (exponential)	0	*	0	r	$d \ln p/dt = r$
Verhulst (logistic)	0	1	1	r	$d \ln p/dt = r(1-p)$
Gompertz	0	0	1	*	$d \ln p/dt = -\kappa \ln p$
Hyper-Gompertz	0	0	*	*	$d \ln p/dt = \kappa(-\ln p)^{\gamma}$
Richards	0	*	1	$r\tilde{q}$	$d \ln p/dt = r(1-p^{\tilde{q}})$
Tsoularis and Wallace	*	*	*	*	$d \ln_{\tilde{q}'} p/dt = \kappa(-\ln_{\tilde{q}} p)^{\gamma}$
Marusic and Bajzer	*	*	1	*	$d \ln_{\tilde{q}'} p/dt = -\kappa \ln_{\tilde{q}} p$
Mitscherlich (monomolecular)	1	1	1	*	$\ln_1 dp/dt = dp/dt = \kappa(1-p)$
Blumberg	*	1	*	*	$d \ln_{\tilde{q}'} p/dt = \kappa(1-p)^{\gamma}$
Turner et al.	$1+\tilde{q}(1-\gamma)$	*	*	*	$d \ln p/dt = \kappa p^{\tilde{q}(1-\gamma)}(-\ln_{\tilde{q}} p)^{\gamma}$
Specialized von Bertalanffy	1/3	1/3	1	*	$d \ln_{1/3} p/dt = -\kappa \ln_{1/3} p$
Generalized von Bertalanffy	$\tilde{q}$	*	1	*	$d \ln_{\tilde{q}} p/dt = -\kappa \ln_{\tilde{q}} p$
Smith	$1-0.473$	1	1	*	Approximation

The asterisk means that the considered parameter can take an arbitrary value within its domain.

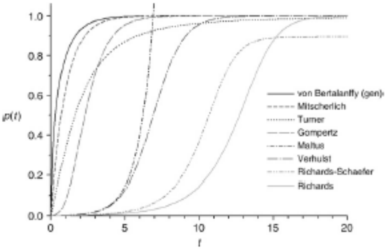


Fig. 1. Models which present analytical solution from Eq. (29). The curves have been obtained with the following values: $p_0 = 0.001$, $\alpha = 0.1$ and

Microscopic Model

3.2.1.2. Microscopic model. The competition between cell drive to replicate and inhibitory interactions, that are modeled by long range interaction among the cell, furnishes an interesting microscopic mechanism to obtain Richards' model [27]. The long range interaction is dependent on the distance r between two cells as a power law r^{γ} and the cells have a fractal structure characterized by a fractal dimension D_f. Using Eq. (1), one can write their Eq. (7) of Ref. [27] as: $\mathrm{d}\ln n(t)/\mathrm{d}t = \langle G \rangle - J\omega \ln_{\tilde{q}}[D_f n(t)/\omega]$, where ω is a constant related to geometry of the problem and $\tilde{q} = 1 - \gamma/D_f$. Here the parameter $\tilde{q}$ acquires a physical meaning related to the interaction range γ and fractal dimension of the cellular structure D_f. This physical interpretation of $\tilde{q}$ has only been possible due to Richards' model underlying microscopic description.

Loquistic Map

- To discretize $d\ln p(t)/dt = -\kappa \ln_{\tilde{q}} p(t) \Rightarrow dp/dt = -\kappa p \ln_{\tilde{q}} p$:

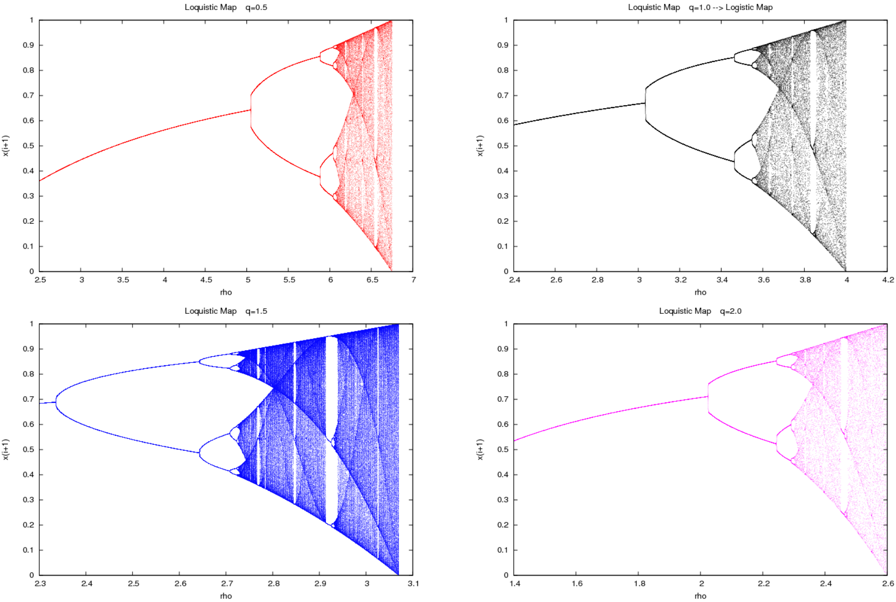
$$\frac{(p_{i+1}-p_i)}{\Delta t} = -\frac{\kappa p_i (p_i^{\tilde{q}}-1)}{\tilde{q}}, \quad \rho'_{\tilde{q}} = \frac{1+\kappa\Delta t}{\tilde{q}}, \quad x_i = p_i\left[\frac{\rho_{\tilde{q}}-1}{\rho_{\tilde{q}}}\right]^{\tilde{q}}.$$

- This leads to the *loquistic* map:

$$x_{i+1} = \rho'_{\tilde{q}} x_i (1 - x_i^{\tilde{q}}) = -\rho_{\tilde{q}} x_i \ln_{\tilde{q}}(x_i) \ ,$$

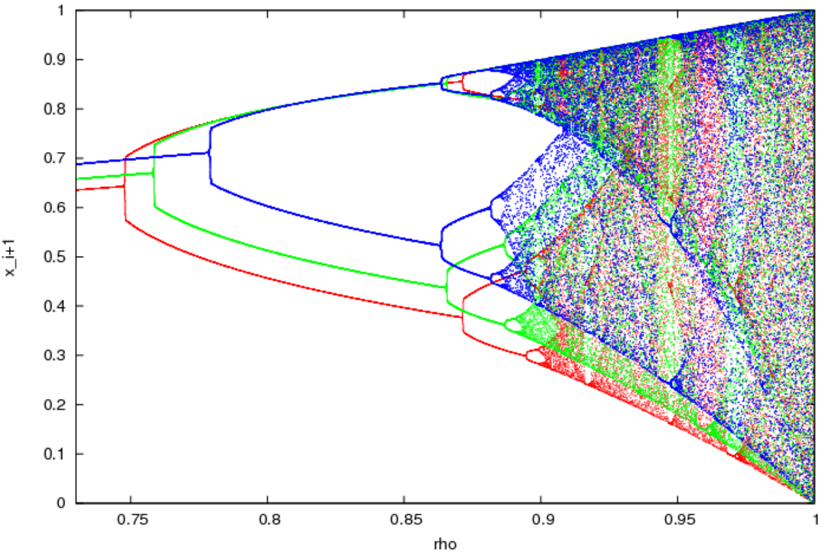
  where $\rho_{\tilde{q}} = \tilde{q}\rho'_{\tilde{q}}$.

- If $(\tilde{q} = 1) \Rightarrow$ logistic map obtained from the discretization of Verhulst model.



Loquistic Map q=0.5
Loquistic Map q=1.0 --> Logistic Map
Loquistic Map q=1.5
Loquistic Map q=2.0
x(i+1)
rho

Normalized Maps



θ-Ricker

- *θ-Ricker* model:

$$x_{i+1} = x_i e^{r[1-(x_i/\kappa)^\theta]}.$$

  - $\kappa_1 = e^{\,r}$,
  - $\tilde{x} = (r^{1/\theta} x)/\kappa$, relevant variable.

$$\tilde{x}_{i+1} = \kappa_1 \tilde{x}_i e^{-\tilde{x}_i^\theta}.$$

- Limiting cases, for:
  - $\theta = 1$:
    - standard Ricker model.
    - expanding the exp. to the first order $\Rightarrow$ logistic map.
  - $\theta \neq 1$, expanding the exp. to the first order:
    - loquistic map
- All are scramble models.

Generalized θ-Ricker

- In Eq. $x_{i+1} = \kappa_1 x_i e^{-r(x_i/\kappa)^\theta}$: $e(-x) \Rightarrow e_{-\tilde{q}}(-x)$

$$x_{i+1} = \kappa_1 x_i e_{-\tilde{q}}[-r(x_i/\kappa)^\theta] = \frac{\kappa_1 x_i}{\left[1 + \tilde{q} r \left(\frac{x_i}{\kappa}\right)^\theta\right]^{1/\tilde{q}}} . \qquad (1)$$

- To obtain standard notation of combined models[1] write:
  - $c = 1/\tilde{q}$,
  - $\kappa_2 = r/(\kappa c)$,

$$x_{i+1} = \frac{\kappa_1 x_i}{(1 + \kappa_2 x_i^\theta)^c}$$

[1] A. Brannstrom and D. J. T. Sumpter. The role of competition and clustering in population dynamics. *Proc. R. Soc. B* **272** (2005) 2065-2072.

Limiting Models

$$x_{i+1} = \kappa_1 x_i e_{-\tilde{q}}[-r(x_i/\kappa)^\theta]$$

- For $\theta = 1 \Rightarrow$ Hassel model.
  - $\tilde{q} = -1, \Rightarrow$ logistic model.
  - $\tilde{q} = 0, \Rightarrow$ Ricker model.
  - $\tilde{q} = 1, \Rightarrow$ Beverton-Holt model.
- For $\theta \neq 1$:
  - $\tilde{q} = 0, \Rightarrow$ θ-Ricker model.
  - $\tilde{q} = 1, \Rightarrow$ Maynard-Smith-Slatkin model,.
  - $\tilde{q} = -1, \Rightarrow$ loquistic model.
- For $\tilde{q} \to -\infty$, the trivial linear model.
- These models are either scramble or contest.

Generalized Skellan Model

- All the models generalized are power-law-like models for $\tilde{q} \neq 0$.
- The Skellan (contest) model, $x_{i+1} = \kappa(1 - e^{-rx_i})$, **could not be** retrieved.
- However, if $e(-x) \Rightarrow e_{-\tilde{q}}(-x)$:

$$x_{i+1} = k\left[1 - e_{-\tilde{q}}(-rx_i)\right] \ . \qquad (2)$$

  - $\tilde{q} = -\infty$, $\Rightarrow$ constant model.
  - $\tilde{q} = -1$, $\Rightarrow$ the trivial linear.
  - $\tilde{q} = 0$, one retrieves the Skellan model.
  - $\tilde{q} = 1 \Rightarrow$ the Beverton-Holt contest model.

In Short

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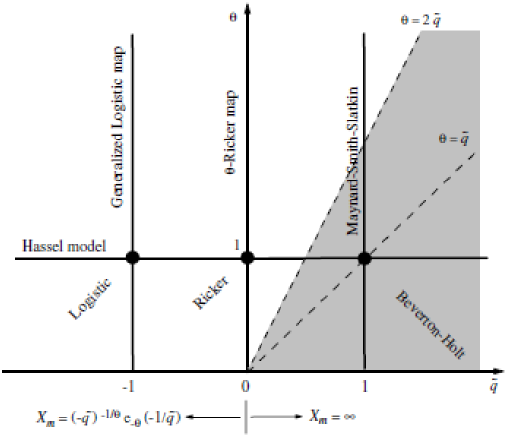

Fig. 6. Diagram distinguishing the several types of behavior of the generalized θ-Ricker model [Eq. (14)]. For $\bar{q} < 0$, one has the map $\bar{x}_{i+1} = \kappa_1 \bar{x}_i (1 - |\bar{q}|\bar{x}_i^{\theta})^{1/|\bar{q}|}$ and $0 \leq \bar{x} \leq \bar{x}_m$, with x_m given by Eq. (16). This system represents scramble models and achieves chaos, period doubling. For $\bar{q} \geq 0$,