Epidemic in a Population with Mobile Immune Individuals

Ronald Dickman

Universidade Federal de Minas Gerais

Support: CNPq & Fapemig

OUTLINE

Absorbing-State Phase Transitions

- Basic Examples
- Universality

Contact process with diffusing vacancies

- Is mobile disorder relevant?
- Weak dilution: apparent nonuniversality
- Critical vacancy concentration: universality regained?

Absorbing state of a Markov process:

Consider a population of organisms, population size N(t)

N evolves via a stochastic dynamics with transitions from N to N+1 (reproduction), and to N-1 (death)

N=0 is an *absorbing state*: if N=0 at some time t, then N(t') = 0 for all times t' > t

Systems with spatial structure: **phase transitions** between active and absorbing states are possible in infinite-size limit

Of interest in population dynamics, epidemiology, self-organized criticality, condensed-matter physics, social system modelling...

General references:
J Marro and R Dickman, *Nonequilibrium Phase Transitions in Lattice Models*, (Cambridge University Press, Cambridge, 1999).
H Hinrichsen, Adv. Phys. **49** 815 (2000).
G Odór, Rev. Mod. Phys. **76**, 663 (2004)

Principal universality classes of absorbing-state phase transitions:

Directed percolation (DP) (contact process)

Parity-conserving (branching-annihilating random walks)

Conserved DP* (conserved stochastic sandpile)

Pair contact process with diffusion (PDPC)

*Experiment: L Corté, P M Chaikin, J P Gollub and D J Pine, Nature Phys 2008
Transition between reversible and irreversible deformation in sheared colloidal suspension

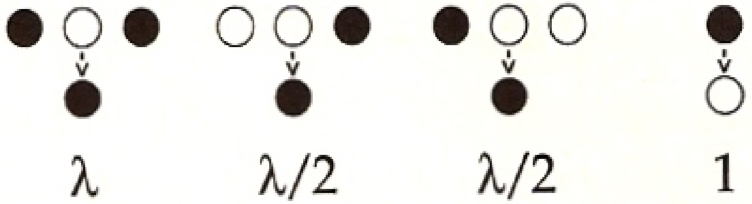
Contact Process (Harris 1972): a birth-and-death process with spatial structure

Lattice of L^d sites

Each site can be either active ($\sigma_i = 1$) or inactive ($\sigma_i = 0$)
An active site represents an organism

Active sites become inactive at a rate of unity, indep. of neighbors
An inactive site becomes active at a rate of λ times the fraction of active neighbors

The state with all sites inactive is absorbing



Rates for the one-dimensional CP.

Contact Process: order parameter ρ is fraction of active sites

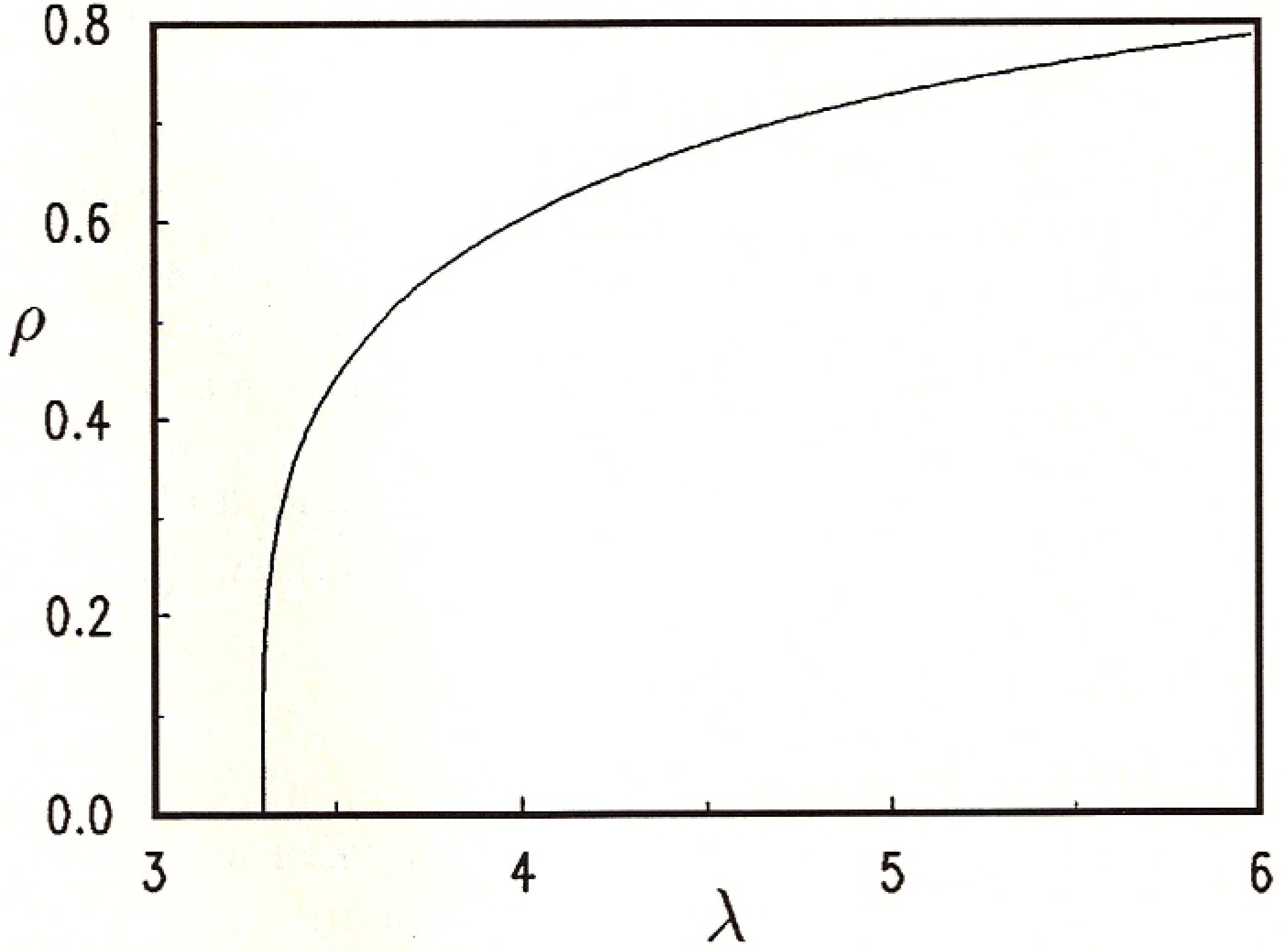
Rigorous results: continuous phase transition between active and absorbing state for d ≥ 1, at some λ_c (Harris, Grimmet...)

Order parameter: $\rho \sim (\lambda - \lambda_c)^{\beta}$

(Mean-field theory: $\lambda_c = 1$, $\beta = 1$)

Results for λ_c, critical exponents: series expansion, simulation, analysis of the master equation, ε-expansion

Types of critical behavior: static, dynamic, spread of activity

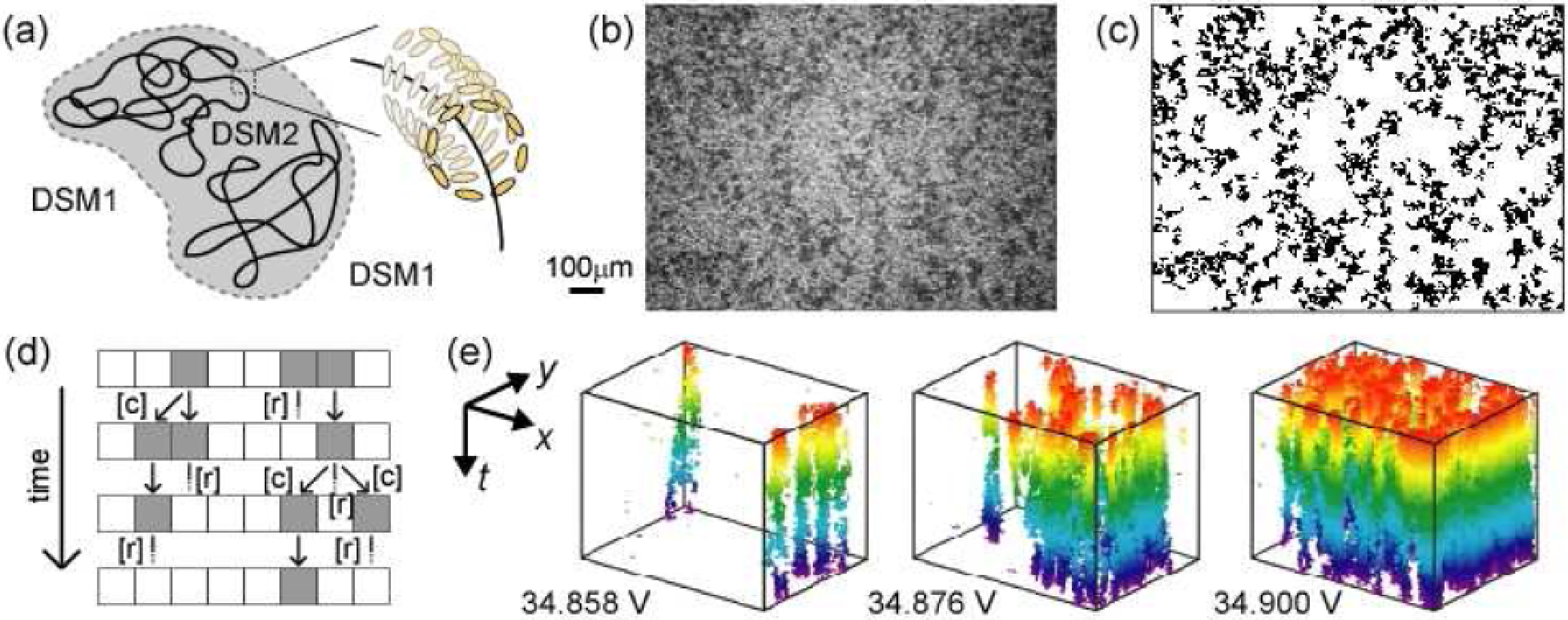


Order parameter in the one-dimensional contact process: series expansion analysis

subcritical critical supercritical

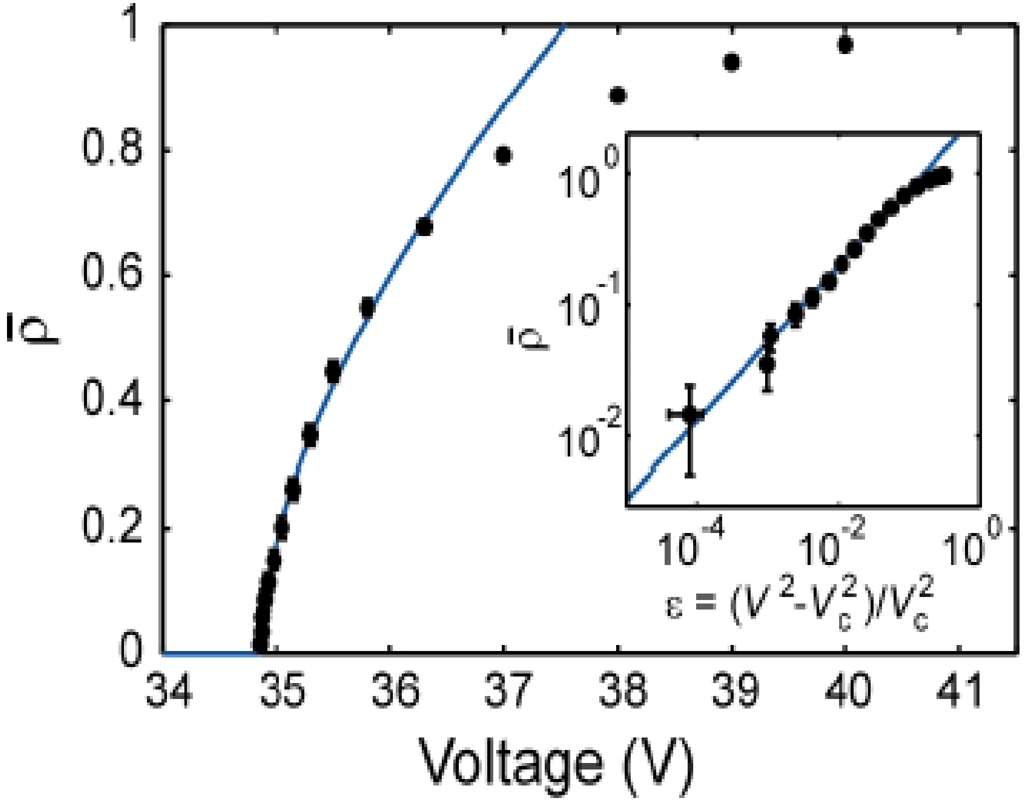
Spread of activity in contact process (avalanches)

Experimental realization of the contact process/directed percolation (Takeuchi et al, PRL **99** 234503 (2007))



Absorbing-state phase transiton between two turbulent regimes in electrohydrodynamic convection of liquid crystals in a thin layer

Takeuchi et al: order parameter vs control parameter



Experiments confirm critical exponents of DP in 2 space dimensions, for example: β = 0.59(4) (expt), β = 0.583(3) (sim)

Physics

Physics **2**, 96 (2009)

Viewpoint

Observation of directed percolation—a class of nonequilibrium phase transitions

Haye Hinrichsen
Fakultät für Physik und Astronomie, Universität Würzburg, 97074 Würzburg, Germany

Published November 16, 2009

Directed percolation, a class of nonequilibrium phase transitions as prominent as the Ising model in equilibrium statistical mechanics, is realized experimentally for the first time, after more than fifty years of research.

Subject Areas: **Statistical Mechanics, Soft Matter**

A Viewpoint on:
Experimental realization of directed percolation criticality in turbulent liquid crystals
Kazumasa A. Takeuchi, Masafumi Kuroda, Hugues Chaté and Masaki Sano
Phys. Rev. E **80**, 051116 (2009) – Published November 16, 2009

Effect of disorder on the contact process

Harris criterion ($d\nu < 2$): quenched disorder relevant for contact process (CP) and directed percolation (DP)
(For recent perspective: T Vojta and M Dickison, PRE **72**)

Harris criterion for CP

Local fluctuation in λ is ~v
In a block of length b, summed fluctuation is $\sim b^{d/2} v$, by central limit theorem

Treat this as equivalent to a uniform variation over block,
$\sim v b^{-d/2}$

Under a block transformation (Kadanoff) $v \rightarrow v' = v b^{y} b^{-d/2}$

Note: $y = 1/\nu_{\perp}$
Then disorder is relevant if $d\nu_{\perp} < 2$

Effect of disorder on the contact process

Harris criterion ($d\nu < 2$): quenched disorder relevant for contact process/directed percolation
(For recent perspective: T Vojta and M Dickison, PRE **72**)

What about **mobile disorder**? Is it irrelevant?
Does it cause Fisher renormalization of critical exponents?
Or something more dramatic?

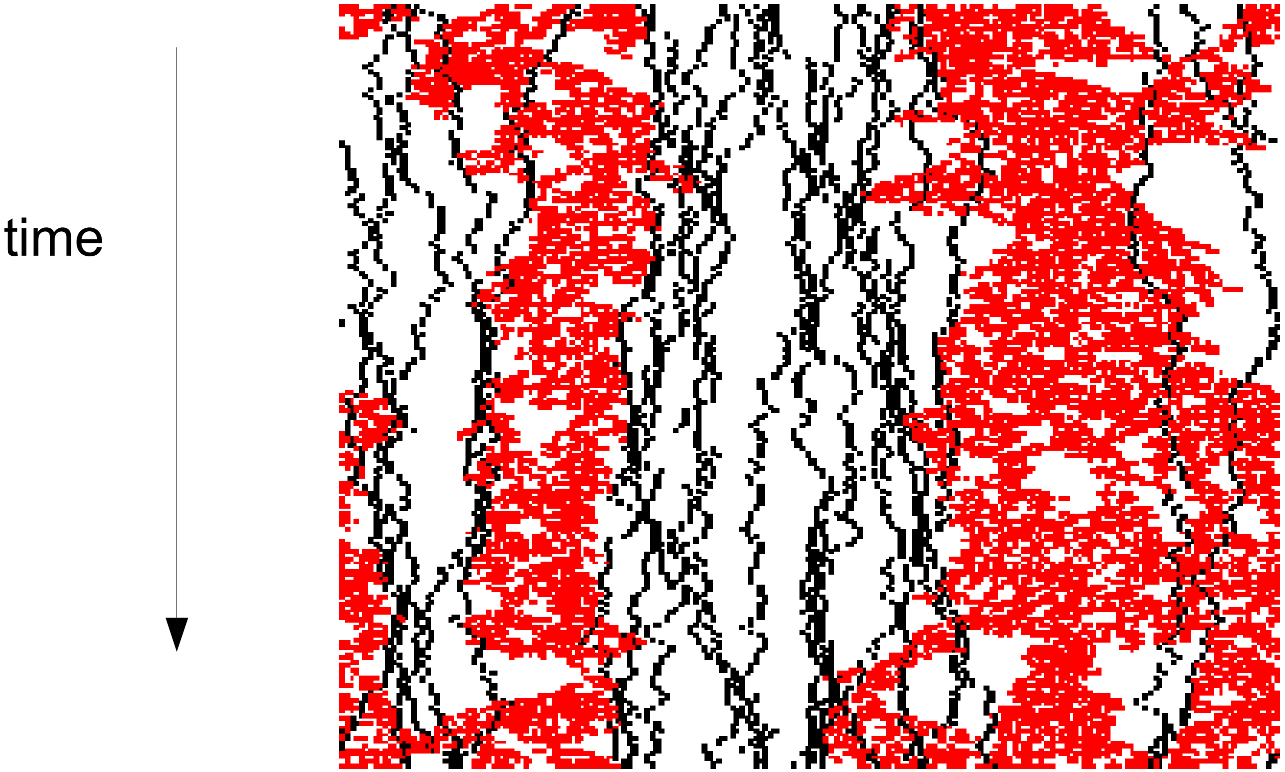
Model: Contact process with mobile vacancies (CPMV)

Vacancies are permanently inactive but diffuse at rate D, exchanging positions with the other sites, which host a basic contact process (Individuals with permanent immunity)

A fraction **v** of sites are vacancies

Nondiluted sites may be active or inactive

CP with mobile vacancies: simulation in one dimension



Typical evolution near critical point. Red: active; black: vacancies
$v=0.1$, $D=1$, $\lambda = 4.1$

Related model: CP with diffusive background (Evron et al., arXiv:0808-0592)
“good” (large λ) and “bad” (small λ) sites instead of vacancies

In principle both models should have the same continuum description:

$$\partial_t \boldsymbol{\rho} = D_a \boldsymbol{\nabla}^{\mathbf{2}} \rho + (a + \boldsymbol{\gamma\phi})\boldsymbol{\rho} - b\boldsymbol{\rho}^{\mathbf{2}} + \boldsymbol{\eta}(x,t)$$

$$\partial_t \boldsymbol{\phi} = \boldsymbol{\nabla}^{\mathbf{2}} \boldsymbol{\phi} + \boldsymbol{\nabla} \bullet \boldsymbol{\xi}(x,t)$$

$\boldsymbol{\rho}$: order parameter density; $\boldsymbol{\phi}$: density of nondiluted (or “good”) sites

η and ξ are suitable noise terms.

Mobile disorder is relevant for finite D

Consider a correlated region in the CP, with characteristic size ξ and duration τ

If fluctuations in the vacancy density on this spatial scale relax on a time scale $\tau_\phi \ll \tau$, then the CP will be subject, effectively, to a disorder that is uncorrelated in time, which is **irrelevant**

But fluctuations in ϕ relax via diffusion, so $\tau_\phi \sim \xi^2$

In the neighborhood of the critical point, $\xi \sim |\lambda - \lambda_c|^{-\nu_\perp}$

and $\tau \sim \xi^z$, so that $\tau_\phi \sim \tau^{2/z}$

This suggests that diffusing disorder is **relevant** for z < 2, provided that quenched disorder is relevant

In directed percolation these conditions are satisfied in d < 4 space dimensions

CP with mobile vacancies: limiting situations

D = 0: In ***one-dimension,*** this corresponds to a CP on finite strips, which must always fall in the absorbing state.
Thus for any v > 0, $\lambda_c \rightarrow \infty$ as $D \rightarrow 0$.

In ***two or more dimensions,*** the CP with fixed vacancies is active (for suff. large λ) if nondiluted sites percolate (v < 1-pc).
Thus $\lambda_c \rightarrow \infty$ as $D \rightarrow 0$ for v > 1-pc

D$\rightarrow$ **∞**: In ***one dimension***, diffusing vacancies do not change order of active and inactive (nondiluted) sites
Thus $D \rightarrow \infty$ is not a mean-field limit
Instead it represents a regular CP with $\lambda_{eff} = (1-v)\lambda$, so one expects $\lambda_c \rightarrow \lambda_{c,pure} / (1-v)$, with DP scaling, in this limit

In two or more dimensions $D \rightarrow \infty$ should correspond to a mean-field limit

Studies of CPMV in one dimension

(RD, J Stat Mech (2009) P08016)

Determine λ_c and scaling properties as functions of vacancy fraction v and diffusion rate D

Three kinds of simulation:

- conventional (stationary regime)
- quasistationary
- spreading

A “first look”: moderate dilution (v=0.1), vary D

Monte Carlo simulations (conventional)

Rings of L = 100, 200,...,1600 sites - all nondiluted sites initially active

Determine (1) fraction **ρ**(t) of active sites

(2) moment ratio m(t) = $\langle \rho^2 \rangle / \rho^2$ in averages over surviving realizations

(3) mean lifetime τ from the decay of the survival probability, $P_s(t) \sim \exp[-t/\tau]$

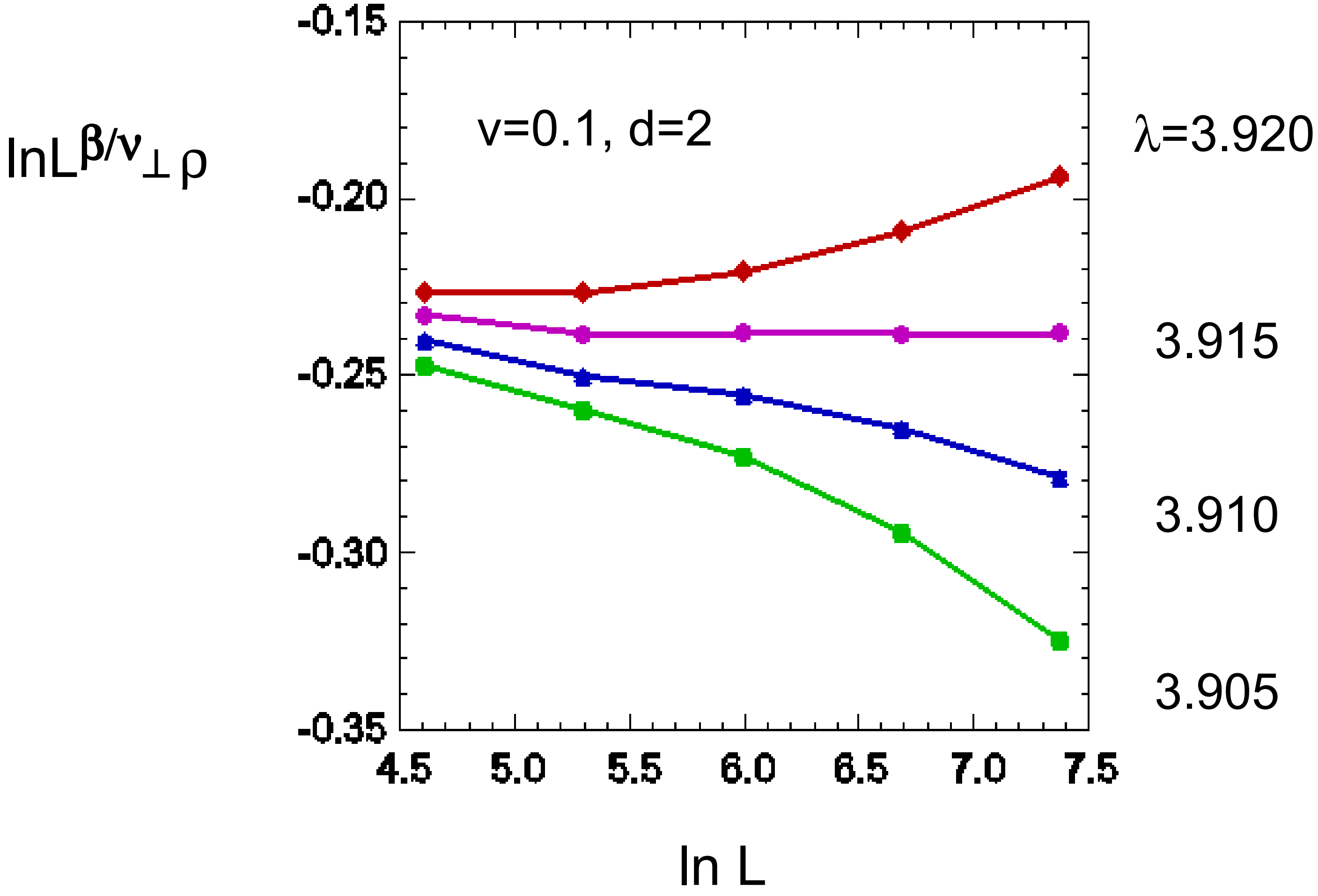
In large (pure) systems at critical point, ρ and m approach their quasistationary (QS) values via

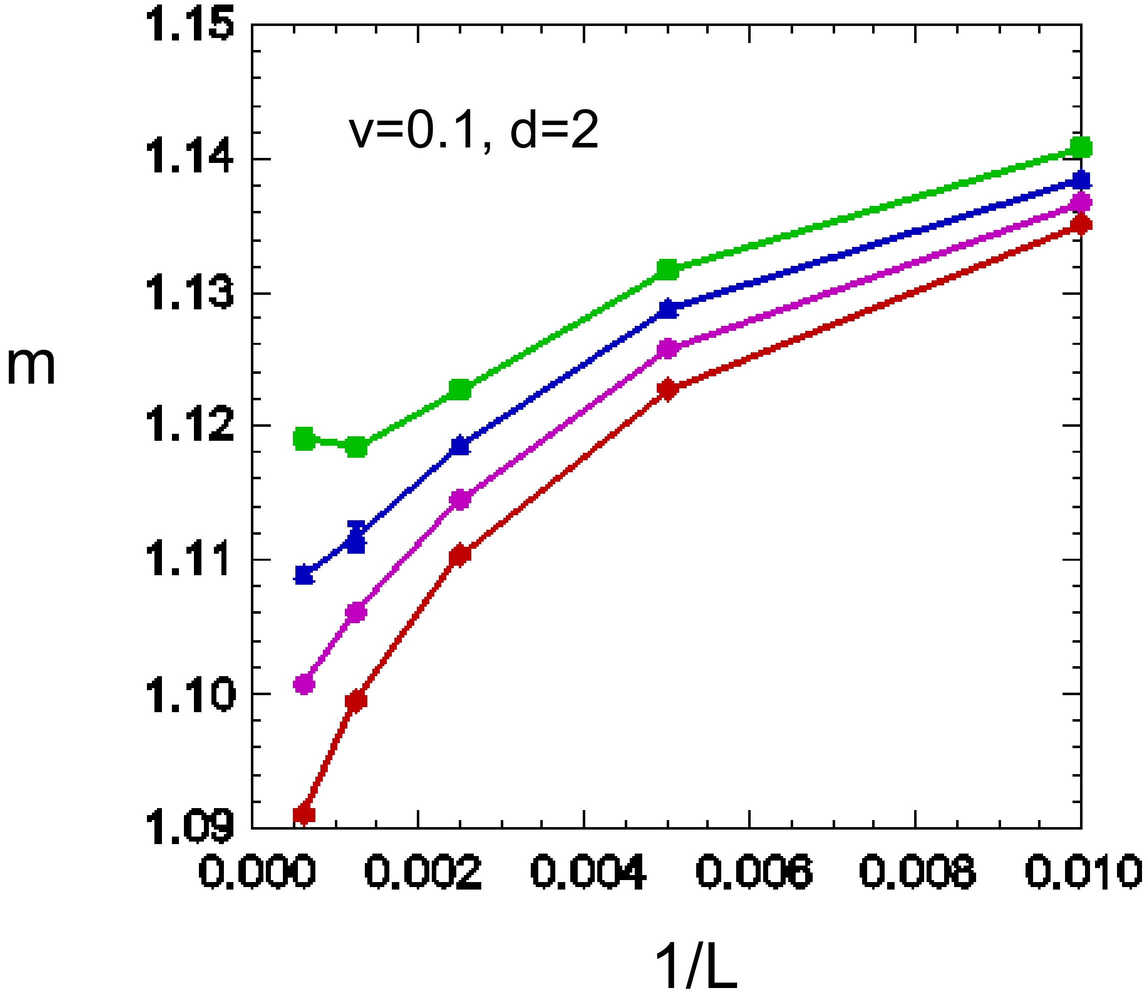
$$\rho(t) \sim t^{-\delta} \quad \text{and} \quad m(t) - 1 \sim t^{1/z}$$

Finite-size scaling: at the critical point, $\rho_{QS} \sim L^{-\beta/\nu_\perp}$,

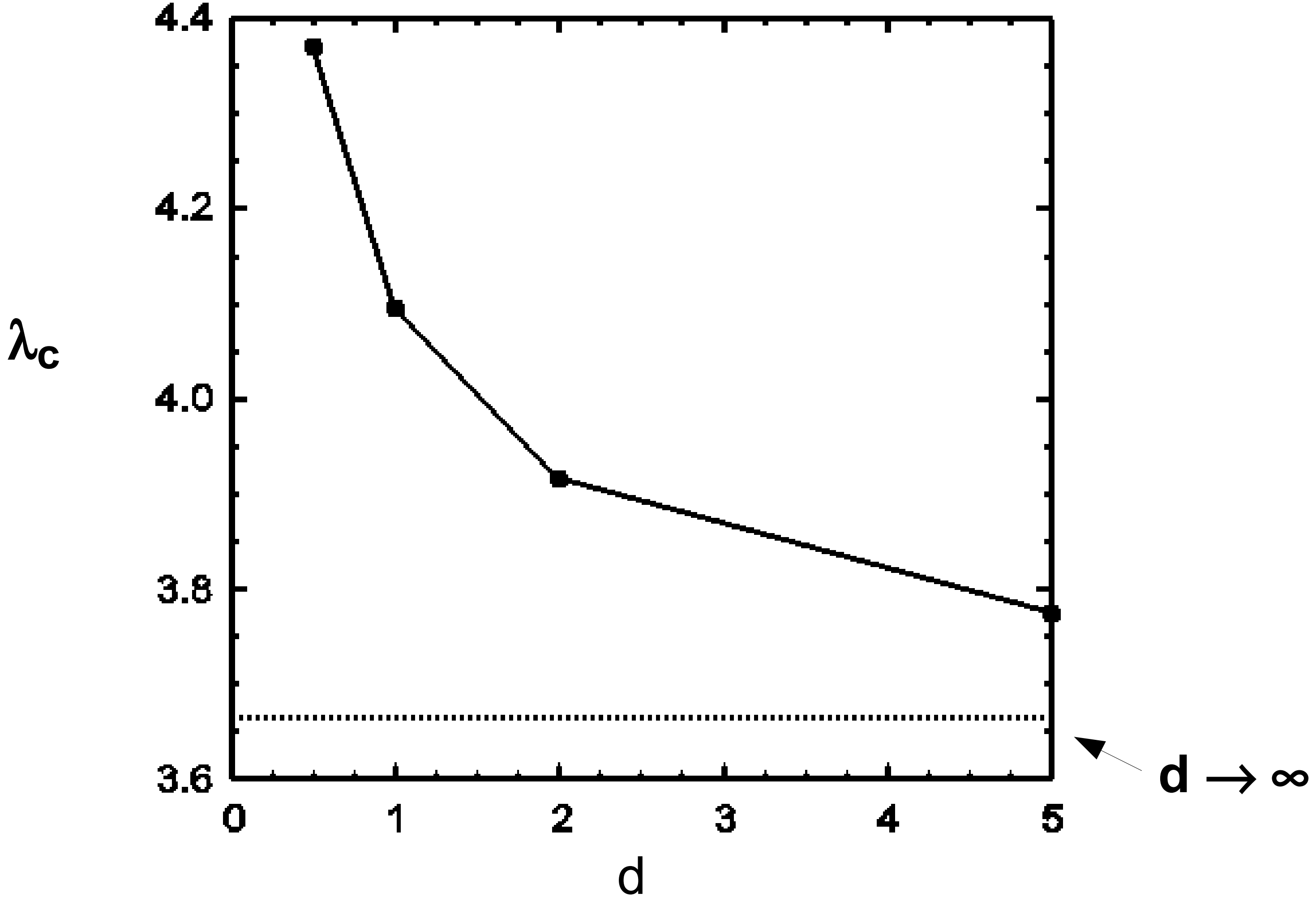
$$\tau \sim L^{z} \quad \text{and} \quad m \to m_c \quad \text{(a universal quantity)}$$

Criteria for determining λ_c: power-law scaling of ρ with L, convergence of moment ratio m to a finite limiting value




v=0.1, d=2
m
1/L
1.15
1.14
1.13
1.12
1.11
1.10
1.09
0.000
0.002
0.004
0.006
0.008
0.010

Phase boundary, v=0.1



Order parameter: data collapse

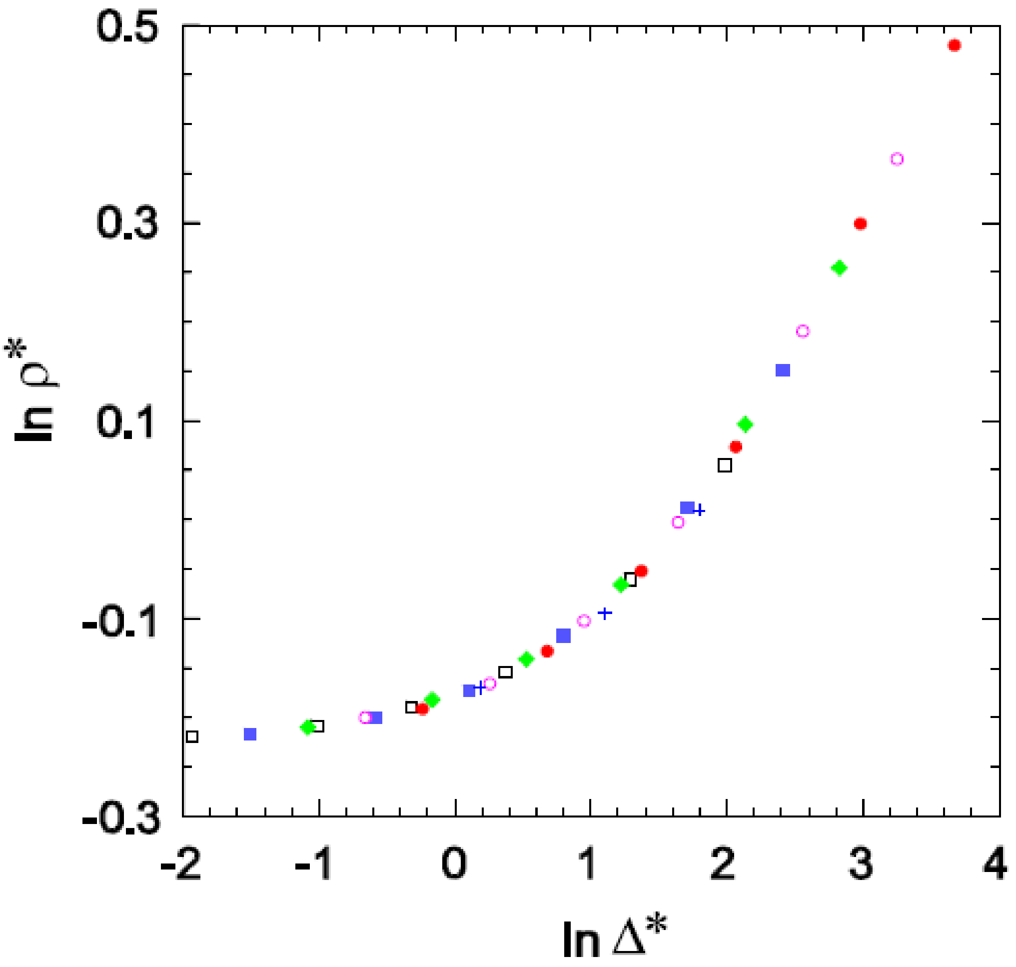


Figure 8. Order parameter scaling plot, $\rho^* \equiv L^{\beta/\nu_\perp}\rho$ versus $\Delta^* = L^{1/\nu_\perp}[(\lambda - \lambda_c)/\lambda_c]$, for $v = 0.1$ and $D = 0.5$. System sizes $L = 100$ (open squares); 200 (filled squares); 400 (diamonds); 800 (open circles); 1600 (filled circles); 3200 (+).

Anomalous behavior: lifetime grows more slowly than power-law at critical point! (Crossover to smaller z? Apparent exponent for small sizes is 2.4, might expect z=2.)

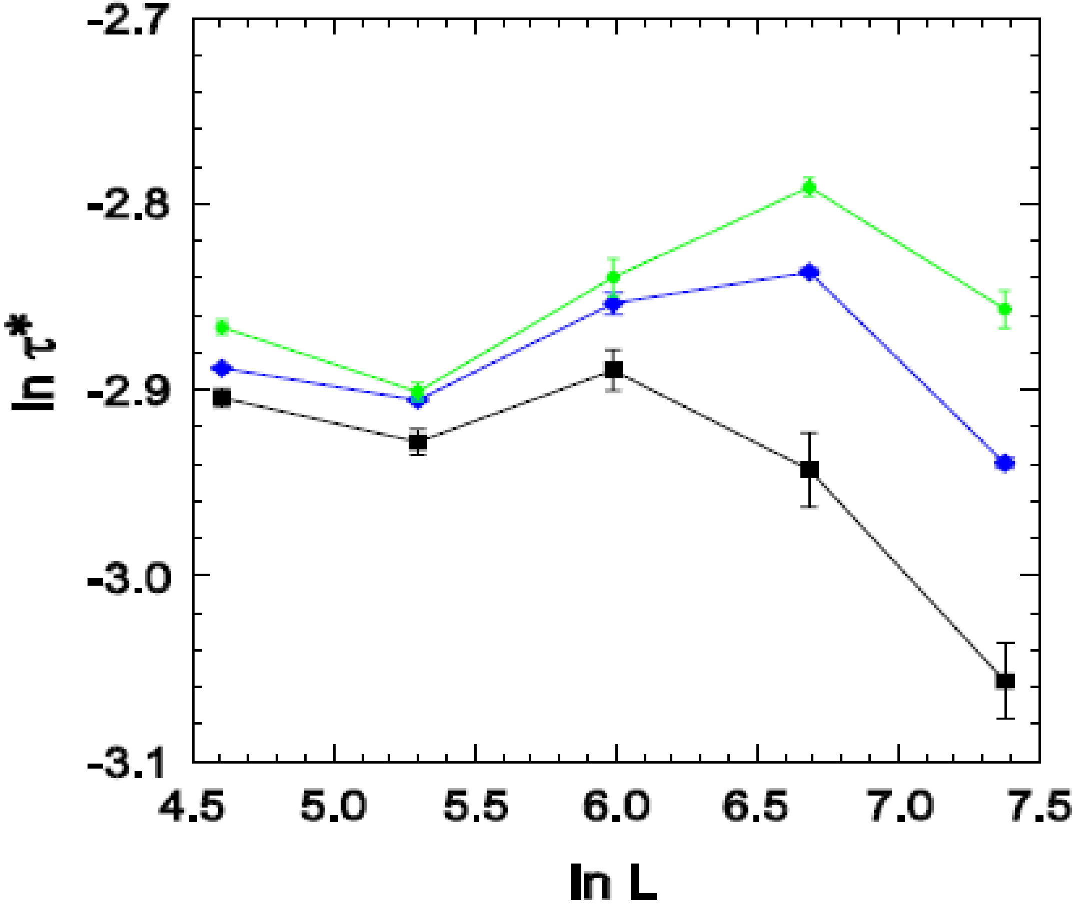
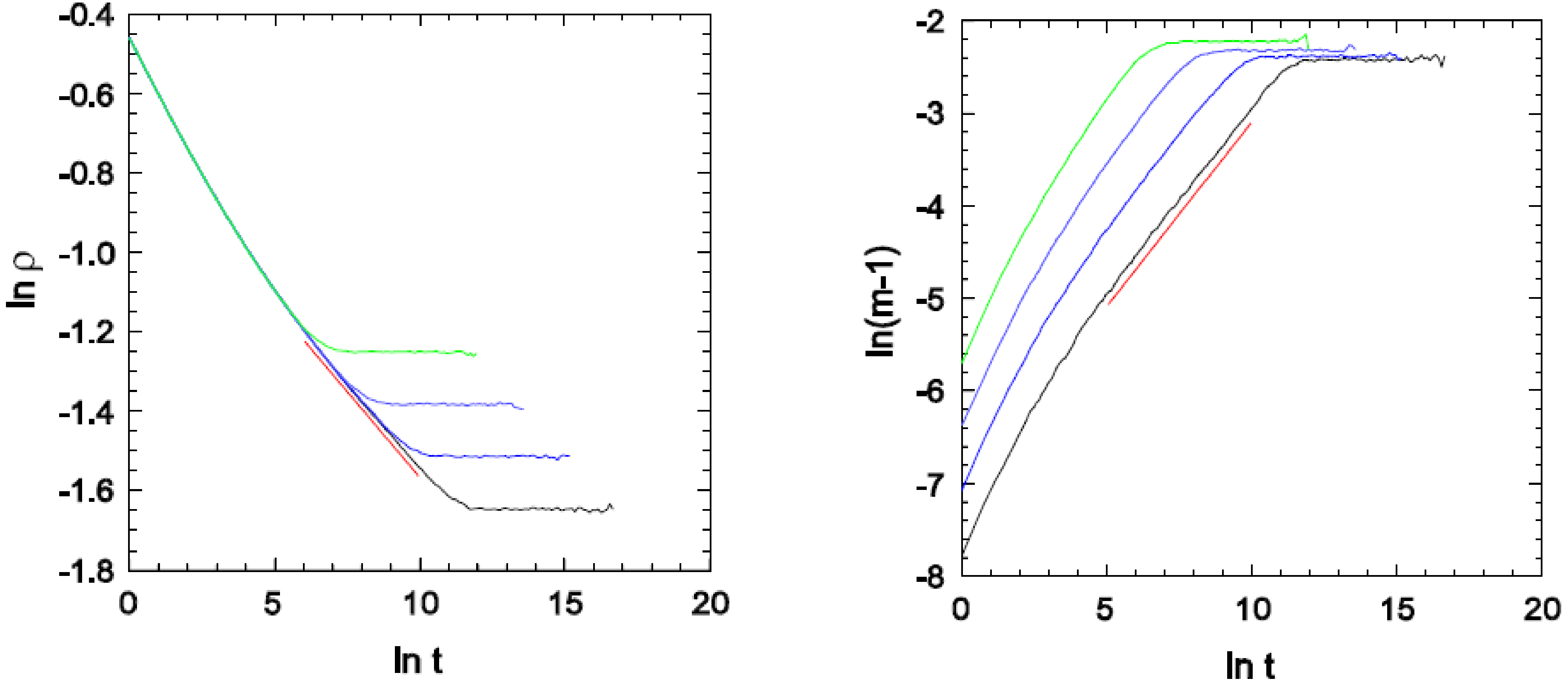


Figure 7. Scaled lifetime $\tau^* = L^{-z}\tau$ versus system size for $v = 0.1$, $D = 1$, and (lower to upper) $\lambda = 4.097$, 4.099, and 4.101.

Anomalous behavior: m(t) and ρ (t) cannot be collapsed



Spreading simulations: one active site initially

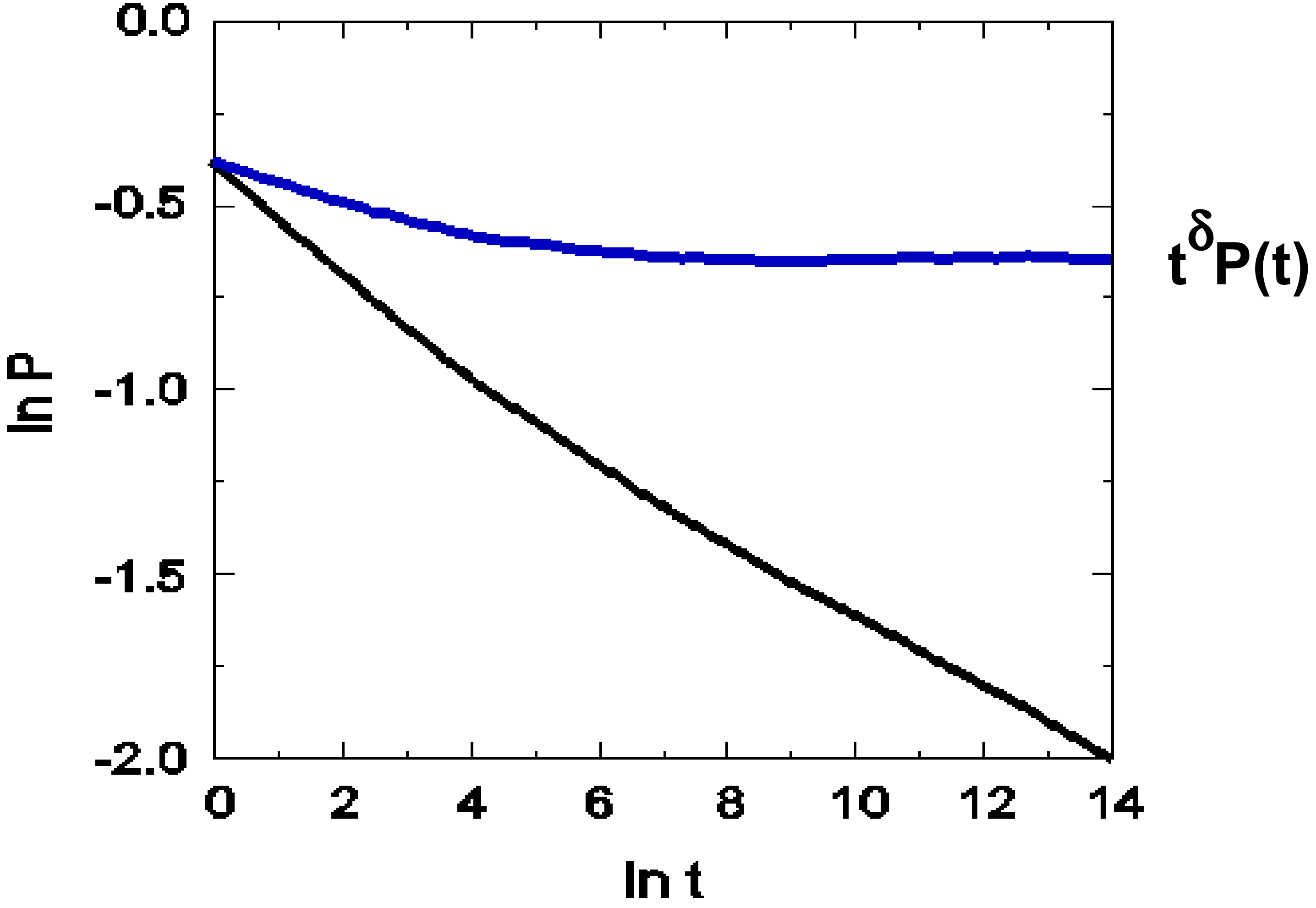
Determine survival probability $P(t)$, mean number of active sites $n(t)$, and mean-square spread, $R^2(t) = \langle \Sigma_j x_j(t)^2 \rangle / n(t)$

Expected scaling behaviors at the critical point (pure CP):

$$P(t) \sim t^{-\delta}, \qquad n(t) \sim t^{\eta} \qquad \text{and} \qquad R^2(t) \sim t^{2/z}$$

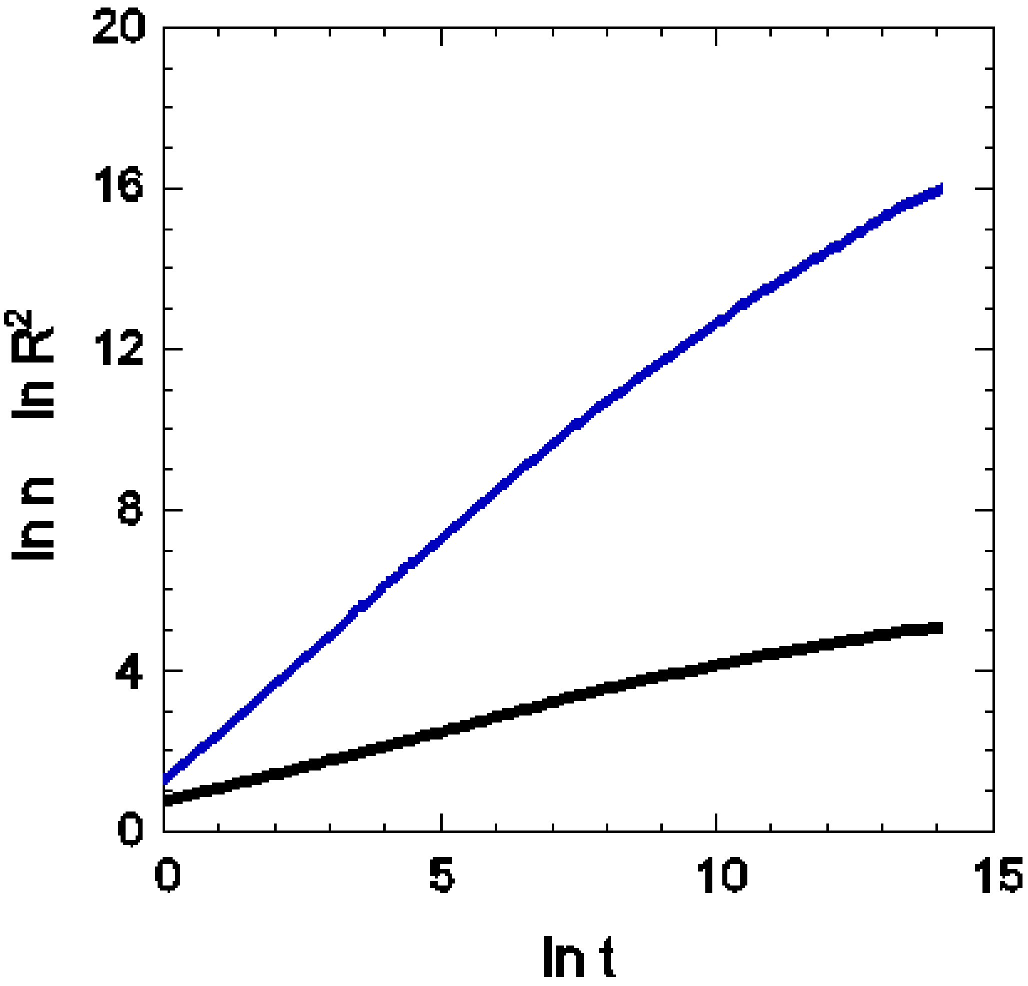
Spreading studies of CPMV confirm power-law scaling of survival probability and value of exponent δ
For $v=0.1$, $D=1$, spreading simulations yield $\delta=0.084(1)$,
$\delta=0.129(1)$ for $D=5$

Surprisingly n and R^2 grow *more slowly* than power laws



Spreading simulation: survival probability, v=0.1, D=2

Mean number of active sites and mean-square spread, v=0.1, D=2



Short-time behavior similar to DP. Possible crossover to much smaller η (and larger z) at long times.

Summary of Results for v=0.1

Critical exponents z, δ, $\beta/\nu_\perp$, and moment ratio m_c appear to vary continuously with vacancy diffusion rate d, and approach DP-class values as d increases

Spreading simulations confirm scaling of survival probability, $P \sim t^{-\delta}$ but other quantities show anomalous scaling

The lifetime τ grows more slowly than a power law at the critical point, for small D

Summing up, static scaling is observed, but certain aspects of time-dependent behavior are anomalous.

SIMULATION RESULTS: v=0.1

D	λ_c	$\beta/\nu_\perp$	m	**z**	δ
0.5	**4.375(2)**	**0.175(3)**	**1.076(2)**	**2.65(4)**	**0.076(2)**
1.0	**4.099(1)**	**0.191(3)**	**1.085(2)**	**2.49(1)**	**0.085(2)**
2.0	**3.915(1)**	**0.205(3)**	**1.096(3)**	**2.36(5)**	**0.101(4)**
5.0	**3.7746(10)**	**0.235(4)**	**1.123(4)**	**1.92(2)**	**0.135(3)**
CP	**3.2979**	**0.2521**	**1.1736**	**1.5808**	**0.1598**

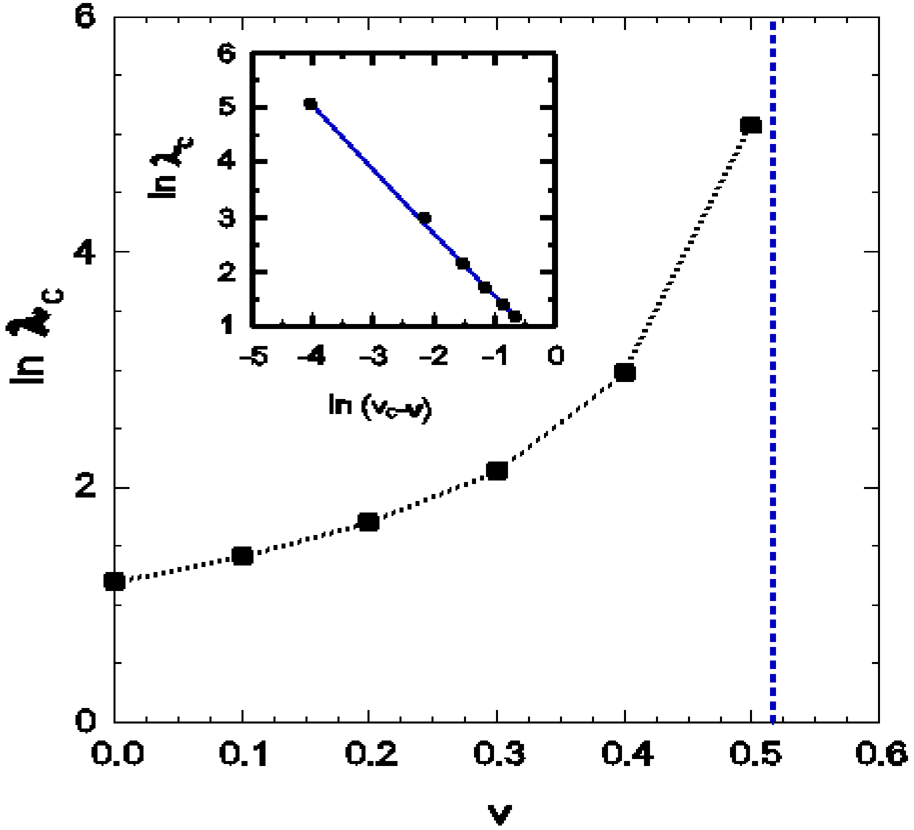
The critical exponents violate the scaling relation

$$\delta = \beta/\nu_{||} = \beta/(\nu_\perp z)$$

- stronger violation for larger D; seem to approach DP values as D grows

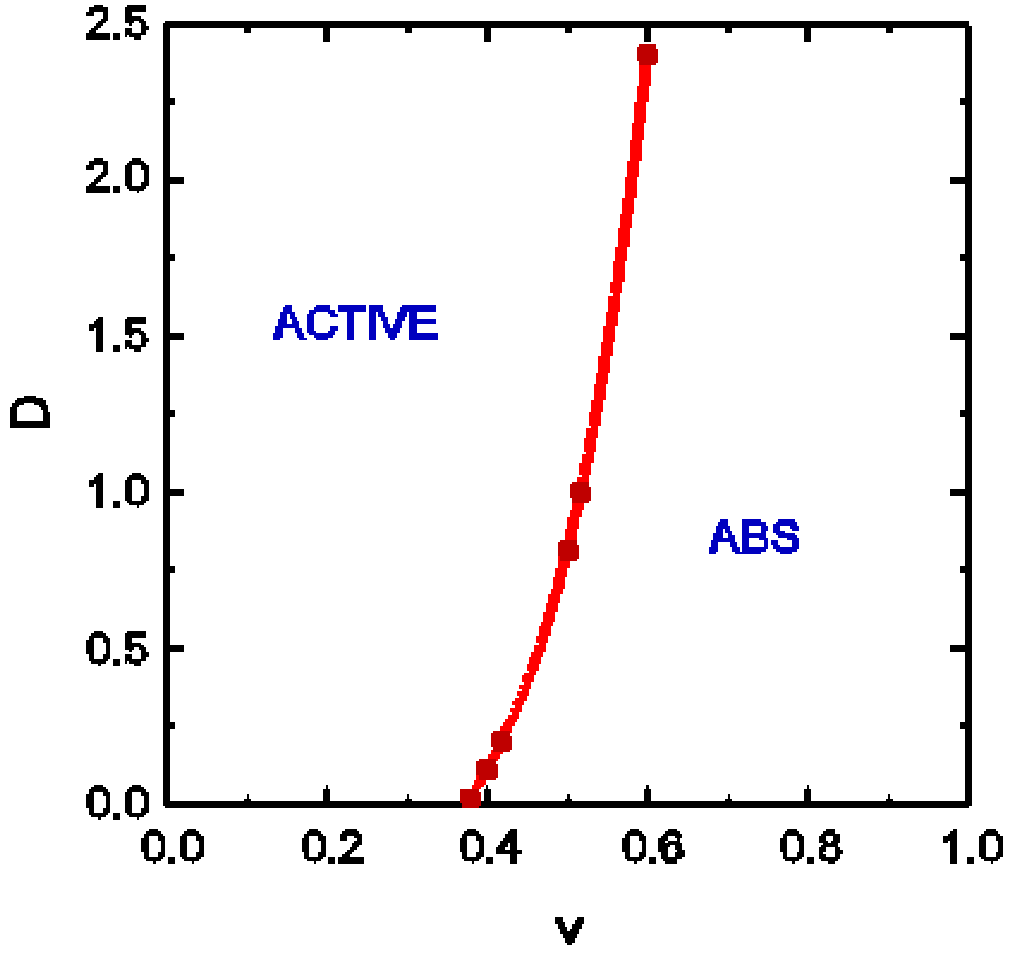
*These exponents are also quite different from those of the DEP with equal diffusion rates

A second look: CPMV at the Critical Vacancy Density



For fixed diffusion rate D, critical reproduction rate λ_c grows with vacancy density v and *diverges* at v_c(D)

Critical vacancy density line in the v-D plane (simulation)

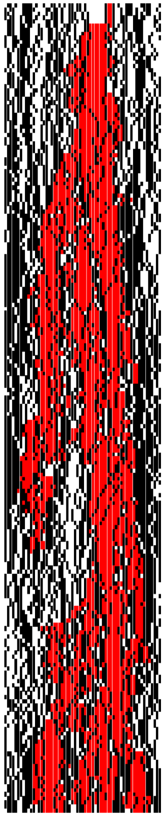


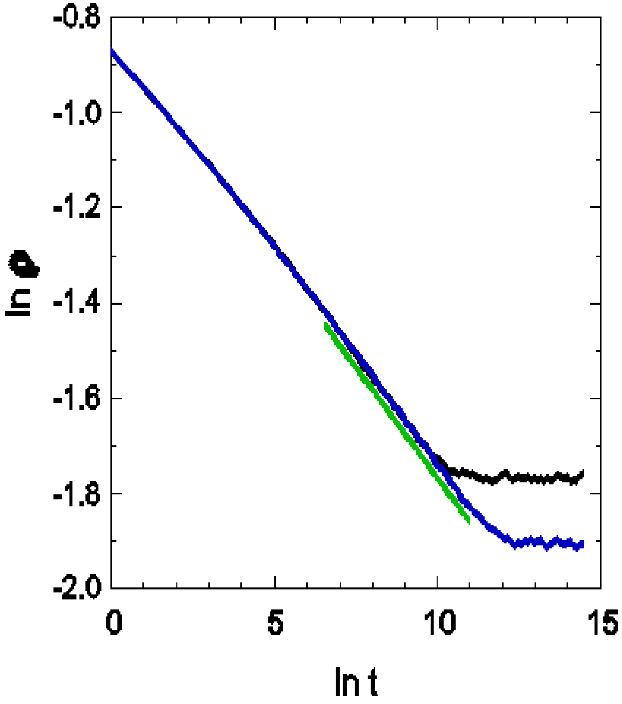
For $v < 0.38$, λ_c diverges only when $D \longrightarrow 0$

Simulation with $\lambda = \infty$: allow only *isolated* active sites to become inactive (at a rate of unity), and activate any nondiluted site the instant it gains an active neighbor

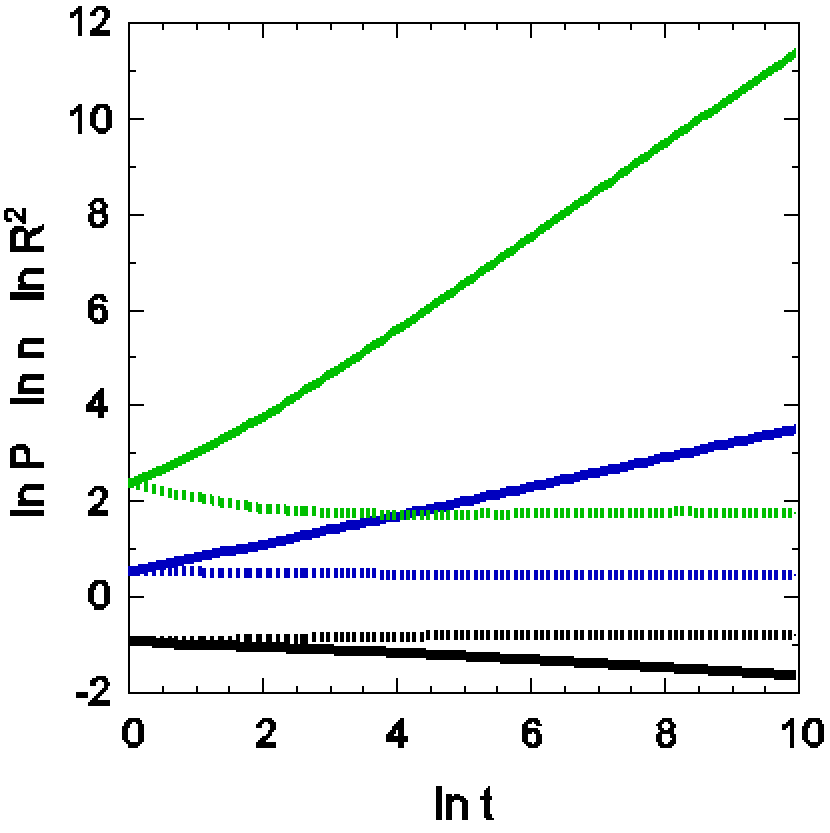
Typical evolution starting from a single active site

D=1, v=0.515



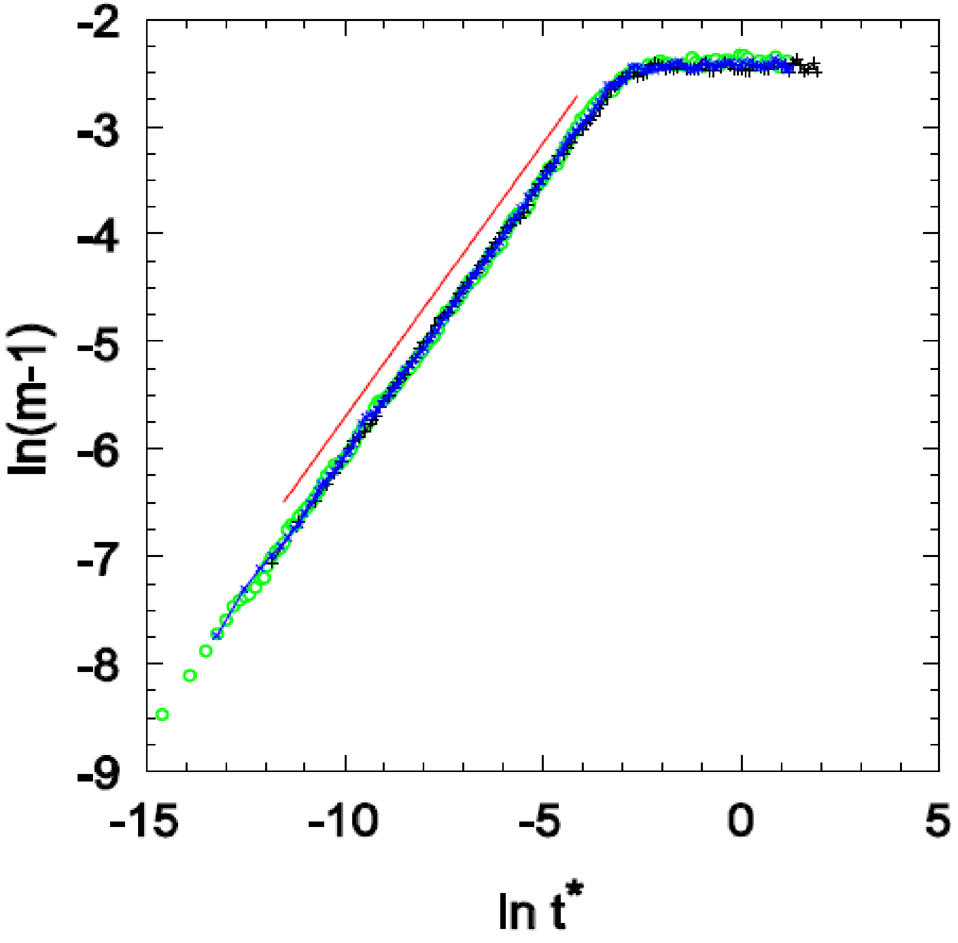


Simpler scaling behavior at v_c than for smaller v



At critical vacancy density, P, n and R^2 ***all*** follow power laws

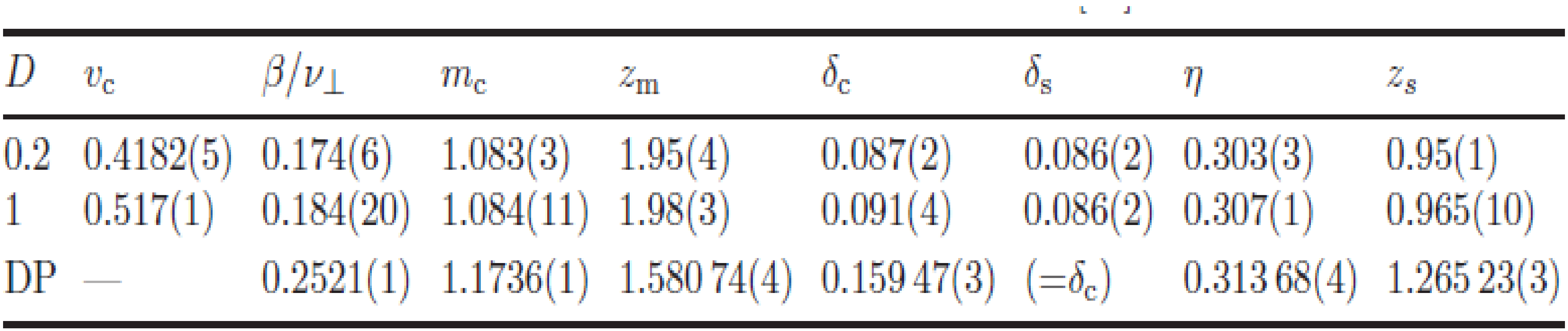
Collapse of m(t)



Scaling plot of $m-1$ versus $t^* = t/L^z$ using $z = 1.98$. Parameters $v = 0.5176$, $\lambda = \infty$, and $D = 1$.

System sizes $L = 398$ $(+)$, $L = 796$ $(\times)$, and 1592 (circles). The slope of the straight line is 0.51.

Critical properties along the critical vacancy density line



D	v_c	$\beta/\nu_\perp$	m_c	z_m	δ_c	δ_s	η	z_s
0.2	0.4182(5)	0.174(6)	1.083(3)	1.95(4)	0.087(2)	0.086(2)	0.303(3)	0.95(1)
1	0.517(1)	0.184(20)	1.084(11)	1.98(3)	0.091(4)	0.086(2)	0.307(1)	0.965(10)
DP	—	0.2521(1)	1.1736(1)	1.580 74(4)	0.159 47(3)	(=δ_c)	0.313 68(4)	1.265 23(3)

Similar results are found for v=0.4, 0.5, and 0.6

The hyperscaling relation $4\delta + 2\eta = dz$ is satisfied to within uncertainty

These results suggest that critical exponents are ***independent*** of D along the critical line v**$_c$**

Does the CP with mobile vacancies belong to the diffusive epidemic process (DEP) class?

The continuum description proposed for CPMV corresponds to that suggested for DEP by Kree, Schaub and Schmittmann. There is reasonable agreement for values of some critical exponents, but more precise results are needed.

The conclusions of this study differ from those of Evron et al., who find $\delta = \delta_{DP}$, with anomalous scaling away from critical Point. These authors study a weaker form of disorder

Ongoing studies:

Characterize more precisely the critical behavior along the line v_C, and the critical exponents of the DEP continuum theory

Diffusive epidemic process (DEP) [Kree et al, 1989]:

A and B particles diffuse on a lattice at rates D_A and D_B
Reactions: $B \rightarrow A$ (rate r)
$A + B \rightarrow 2B$ (rate AB at each site)
- No limit on the number of particles at a given site
- Total number of particles is conserved.

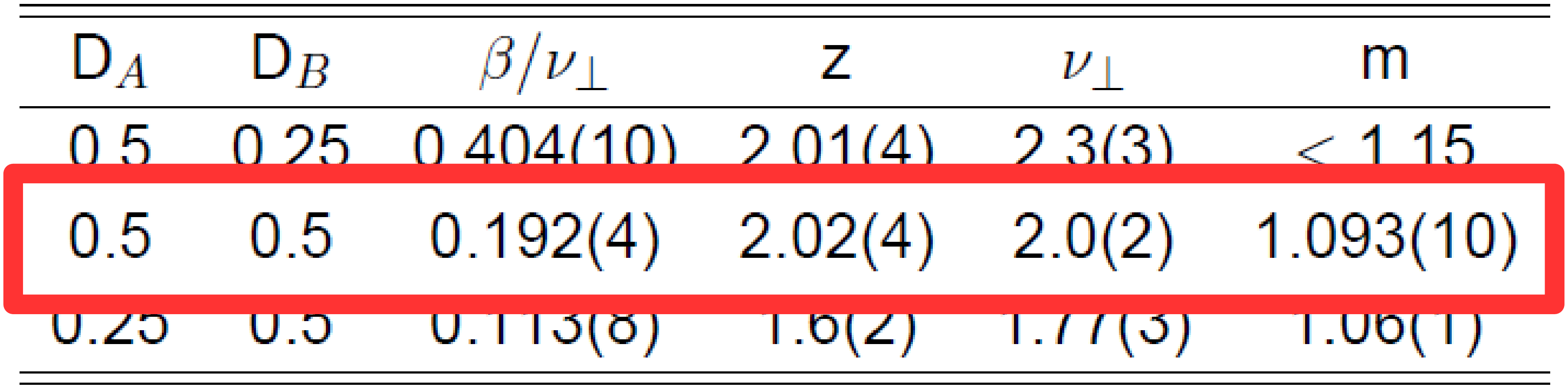
Epidemic interpretation: A represents a healthy organism, B an infected one
Reactions correspond to spontaneous recovery and transmission of disease on contact

B-free state is absorbing

For equal diffusion rates, the A and B particles in DEP correspond to nondiluted sites in CPMBV $(\phi \leftrightarrow \rho_A + \rho_B)$

Critical Parameters of Diffusive Epidemic Process in 1d

Compare values for $D_A = D_B$ with CPMBV at critical vacancy density



D_A	D_B	$\beta/\nu_\perp$	z	$\nu_\perp$	m
0.5	0.25	0.404(10)	2.01(4)	2.3(3)	< 1.15
0.5	0.5	0.192(4)	2.02(4)	2.0(2)	1.093(10)
0.25	0.5	0.113(8)	1.6(2)	1.77(3)	1.06(1)

CPMBV: 0.18(2) 1.97(4) 1.084(10)

The conclusions of this study differ from those of Evron et al., who find $\delta = \delta_{DP}$, with anomalous scaling away from critical point. These authors study a weaker form of disorder

Ongoing studies:

Characterize more precisely the critical behavior along the line v_C, and the critical exponents of the DEP continuum theory

CPMBV in two dimensions (Rajesh Ravindran)

Contact process with mobile vacancies - Summary

Simple scaling behavior at critical vacancy density, with clearly non-DP critical exponents, possible connection to DEP

For smaller v, apparently variable exponents: Is this a crossover between DP and a new fixed point?

Future work:

Map out $v_C(D)$ and associated exponents with higher precision, verify universality along this line of critical points

Apply exact QSD analysis, series expansions

Two and three dimensions

Investigate other forms of slowly evolving disorder, and effect of mobile vacancies on other classes of absorbing-state phase transitions

Thanks to: Thomas Vojta, Jose Hoyos, Rajesh Ravindran, and Miguel Muñoz