Fokker-Planck description from empirical data

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OUTLINE

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- *Motivation*
- *Systems of interest*
- *Theoretical background*
  - *Kramers-Moyal description*
- *Evolution equation from data*
  - *Fokker-Planck equation*
    - *Linear-quadratic FPE*
- *Difficulties: finite-time effects*

MOTIVATION

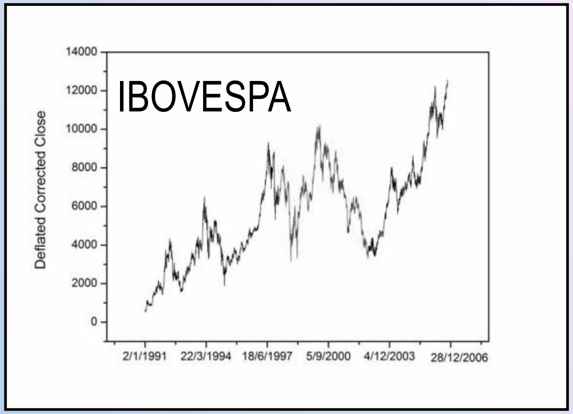
➢ *to model the dynamics of fluctuating observables*

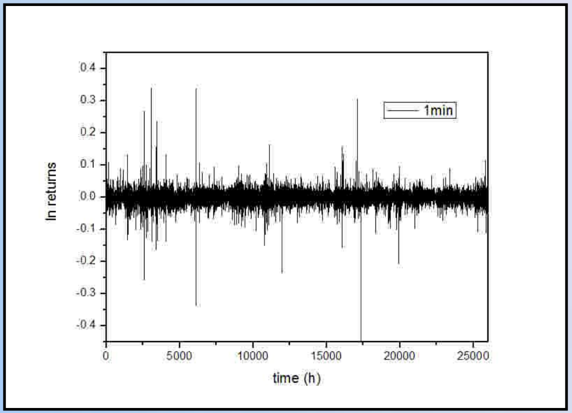
$x(t)$ = stock index at time t

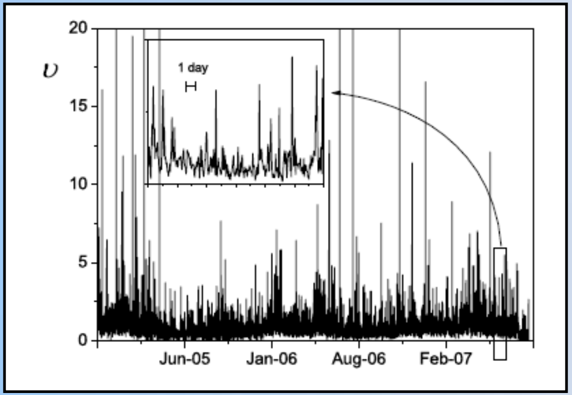
returns at different time scales Δt :

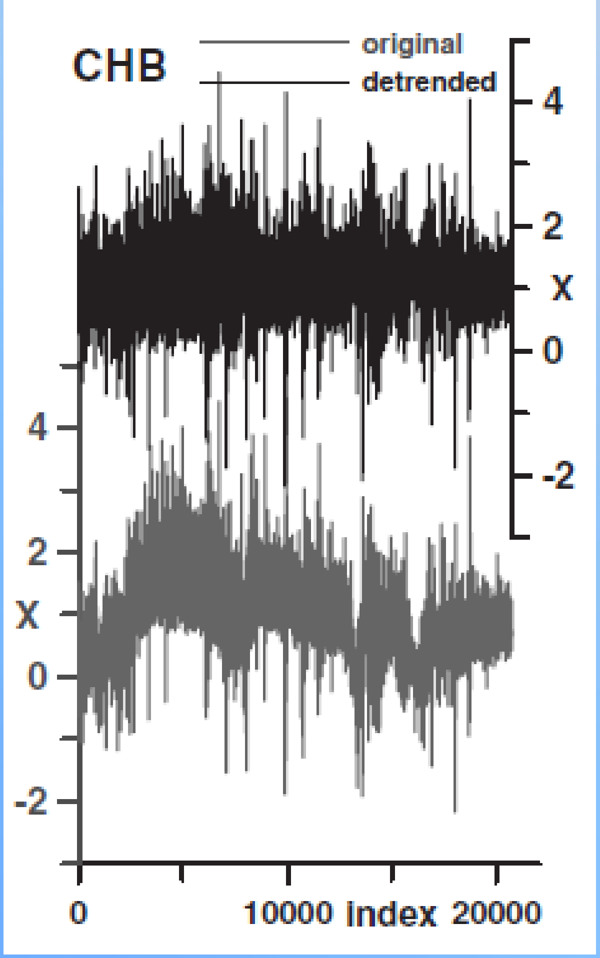
$$r(t) = \log x(t+\Delta t) - \log x(t)$$

$V(t)$ = volume of shares traded in Δt at time t









- heart rate variability
- EEG timeseries
- atmospheric data
- turbulence

Phenomenological models

- *generalized Fokker-Planck (nonlinear)*

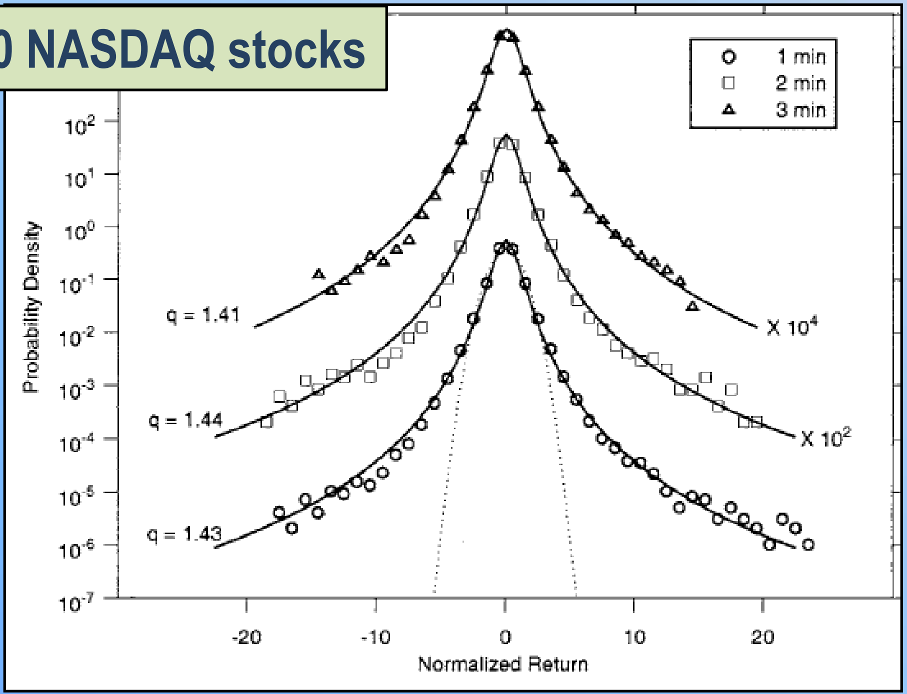
$$\partial_t P(z,t) = D\partial_{zz}\left[P(z,t)\right]^{\nu}$$

- *generalized Langevin*

$$\dot{z} = f(z,t) + g(z,t)\boldsymbol{\eta}(t)$$

- *returns*

10 NASDAQ stocks



PDFs of returns at timelags Δt

Tsallis, Anteneodo, Borland, Osorio; *Physica A* (2003)

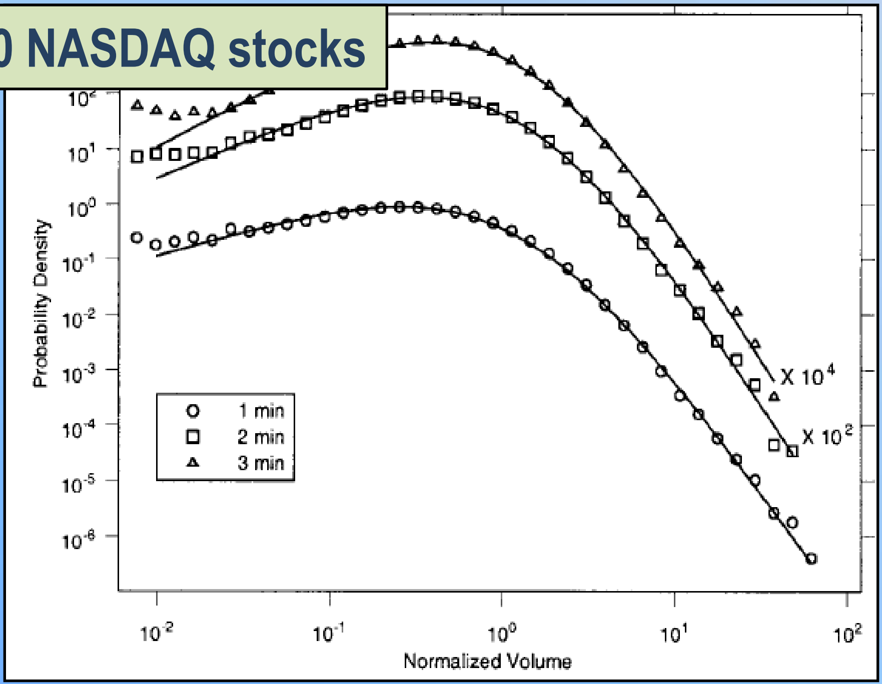
Borland; *PRE* (1998); *PRL* (2002)

$$\dot{x} = \mu + \sqrt{D}\,[P(x,t)]^{(\nu-1)/2}\,\eta(t)$$

$$\partial_t P(x,t) = -\partial_x[\mu P(x,t)] + D\partial_{xx}\left[P(x,t)\right]^{\nu}$$

- *volumes*

10 NASDAQ stocks



PDFs of volumes at aggregation times Δt

Tsallis, Anteneodo, Borland, Osorio; *Physica A* (2003)

Queiros; *EPL* (2005)
Queiros, Tsallis; *EPJB* (2005)
…

$$\dot{v} = -\gamma(v-\alpha) + \beta\sqrt{v}\,\eta(t)$$

with fluctuating α

→ *reproduce (1time) distributions*

Difficulties: *higher order statistics?*

correlations?

non unique

parameter meaning?

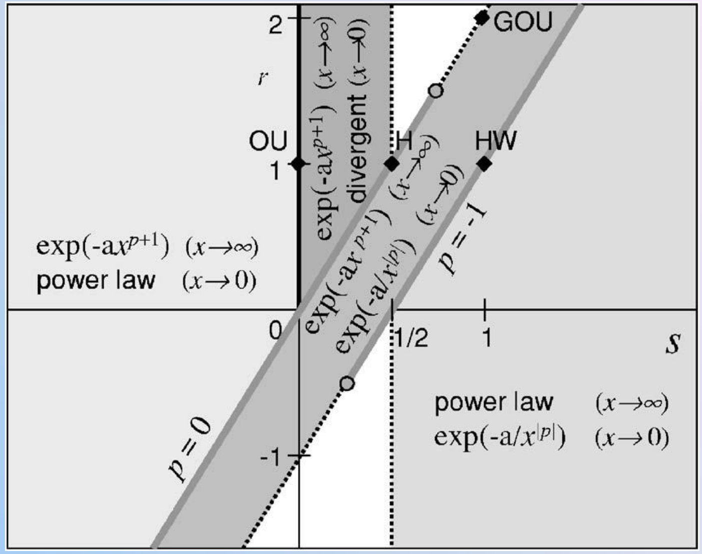
positivity, mean reversion

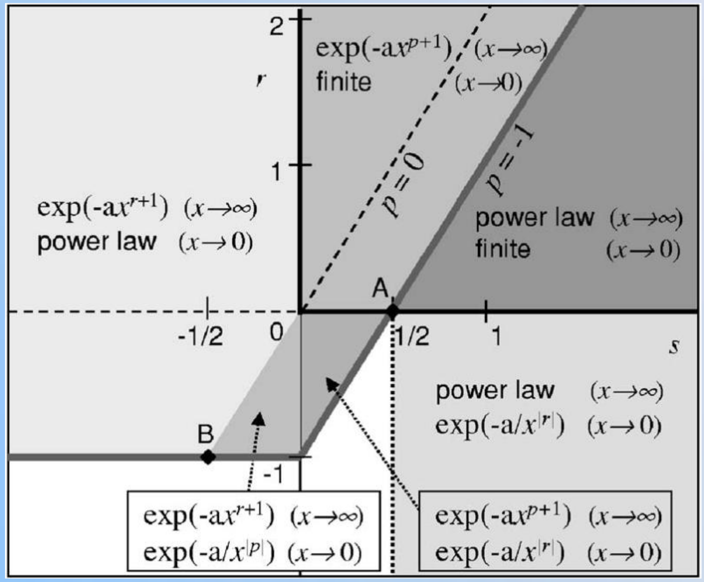
- *volumes or volatility*

$$\dot{x} = -\gamma(x-\theta)x^{r-1} + \mu x^{s}\,\eta(t)$$

OU	(1,0)
GOU	(2,1)
Hull-White	(1,1)
Heston	(1,1/2)

$$\dot{x} = -\gamma(x-\theta)x^{r-1} + \mu x^{s}\,\eta_1(t) + \alpha\,\eta_2(t)$$





Anteneodo, Riera; *PRE* (2005)

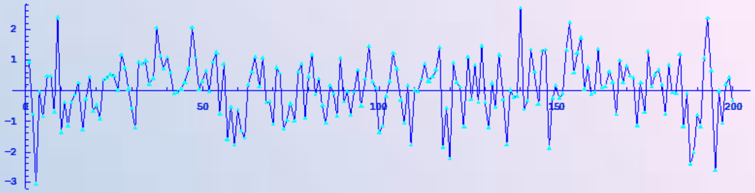
OBJECTIVE

- *determine the dynamics directly from the data series*

SYSTEMS/PROCESSES

- *Markovian*

Markov processes



$$P_n(x_1,t_1;x_2,t_2;\ldots;x_n,t_n) \equiv P_n\left(X_{t_1}=x_1;\ldots;X_{t_n}=x_n\right)$$

para n valores arbitrários de t, com $n = 1,2,\ldots$

Purely random

$(n>1)$

$$P_{1|n-1}(x_n,t_n \mid x_1,t_1;x_2,t_2;\ldots;x_{n-1},t_{n-1}) = P_1(x_n,t_n)$$

Markovian

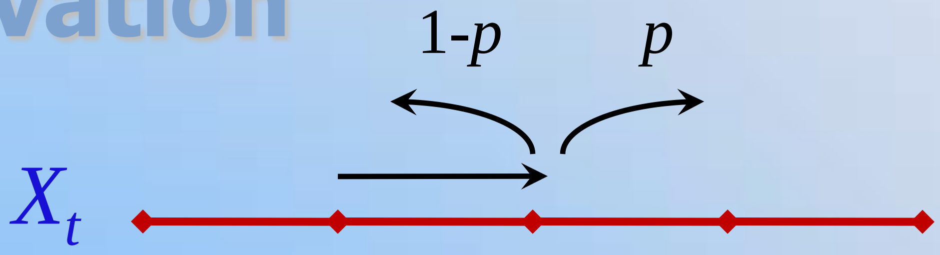
for any set $t_1<t_2<\ldots<t_n$, $n>1$

$$P_{1|n-1}(x_n,t_n \mid x_1,t_1;x_2,t_2;\ldots;x_{n-1},t_{n-1}) = P_{1|1}(x_n,t_n \mid x_{n-1},t_{n-1})$$

More general

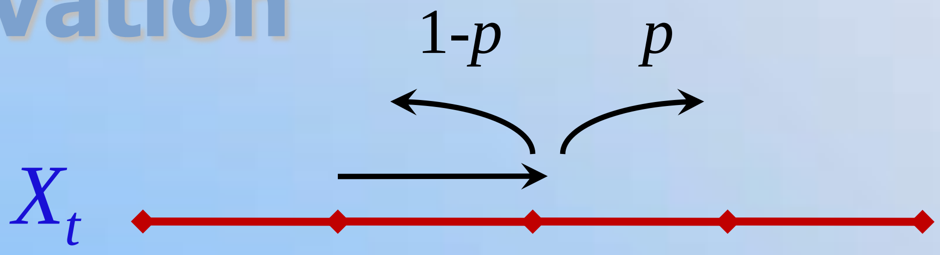
$$P_{1|n-1}(x_n,t_n \mid x_1,t_1;x_2,t_2;\ldots;x_{n-1},t_{n-1})$$

Observation



$$P(X_t \mid X_{t-1}, X_{t-2})$$

Observation



$$P(X_t \mid X_{t-1}, X_{t-2})$$

Cure: (X_t, X_{t-1})

$$P(s_1, s_2; t \mid r_1, r_2; t-1) = \delta_{s_2 r_1}[p\,\delta_{s_1-s_2, r_1-r_2} + (1-p)\,\delta_{s_1 r_2}]$$

Except if ALL steps (e.g., SAW)

Markov processes

$$P_{1|n-1}(x_n,t_n \mid x_1,t_1;x_2,t_2;\ldots;x_{n-1},t_{n-1}) = P_{1|1}(x_n,t_n \mid x_{n-1},t_{n-1})$$

Markov processes

$$P_{1|n-1}(x_n,t_n \mid x_1,t_1;x_2,t_2;\ldots;x_{n-1},t_{n-1}) = P_{1|1}(x_n,t_n \mid x_{n-1},t_{n-1})$$

$$P_3(x_1,t_1;x_2,t_2;x_3,t_3) = P_{1|2}(x_3,t_3 \mid x_1,t_1;x_2,t_2)P_2(x_1,t_1;x_2,t_2)$$

$$= P_{1|1}(x_3,t_3 \mid x_2,t_2)P_{1|1}(x_2,t_2 \mid x_1,t_1)P_1(x_1,t_1)$$

Markov processes

$$P_{1|n-1}(x_n,t_n \mid x_1,t_1;x_2,t_2;\ldots;x_{n-1},t_{n-1}) = P_{1|1}(x_n,t_n \mid x_{n-1},t_{n-1})$$

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$$= P_{1|1}(x_3,t_3 \mid x_2,t_2)P_{1|1}(x_2,t_2 \mid x_1,t_1)P_1(x_1,t_1)$$

Integration in x_2:

$$P_2(x_1,t_1;x_3,t_3) = P_1(x_1,t_1)\int dx_2 P_{1|1}(x_3,t_3 \mid x_2,t_2)P_{1|1}(x_2,t_2 \mid x_1,t_1)$$

Markov processes

$$P_{1|n-1}(x_n,t_n\,|\,x_1,t_1;x_2,t_2;\ldots;x_{n-1},t_{n-1}) = P_{1|1}(x_n,t_n\,|\,x_{n-1},t_{n-1})$$

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$$= P_{1|1}(x_3,t_3\,|\,x_2,t_2)P_{1|1}(x_2,t_2\,|\,x_1,t_1)P_1(x_1,t_1)$$

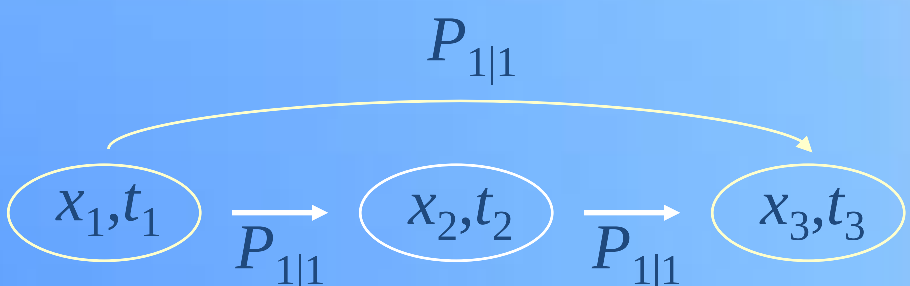
Integration in x_2:

$$P_2(x_1,t_1;x_3,t_3) = P_1(x_1,t_1)\int dx_2 P_{1|1}(x_3,t_3\,|\,x_2,t_2)P_{1|1}(x_2,t_2\,|\,x_1,t_1)$$

Division by $P_1(x_1,t_1)$:

$$P_{1|1}(x_3,t_3\,|\,x_1,t_1) = \int dx_2 P_{1|1}(x_3,t_3\,|\,x_2,t_2)P_{1|1}(x_2,t_2\,|\,x_1,t_1)$$

Chapman-Kolmogorov



Kramers-Moyal (KM) expansion

general evolution equation for the PDFs of Markovian processes

$$\frac{\partial}{\partial t} P(x,t) = \sum_{k \geq 1} \left[-\frac{\partial}{\partial x} \right]^k D^{(k)}(x,t) P(x,t)$$

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general evolution equation for the PDFs of Markovian processes

$$\frac{\partial}{\partial t}P(x,t)=\sum_{k\geq 1}\left[-\frac{\partial}{\partial x}\right]^k D^{(k)}(x,t)P(x,t)$$

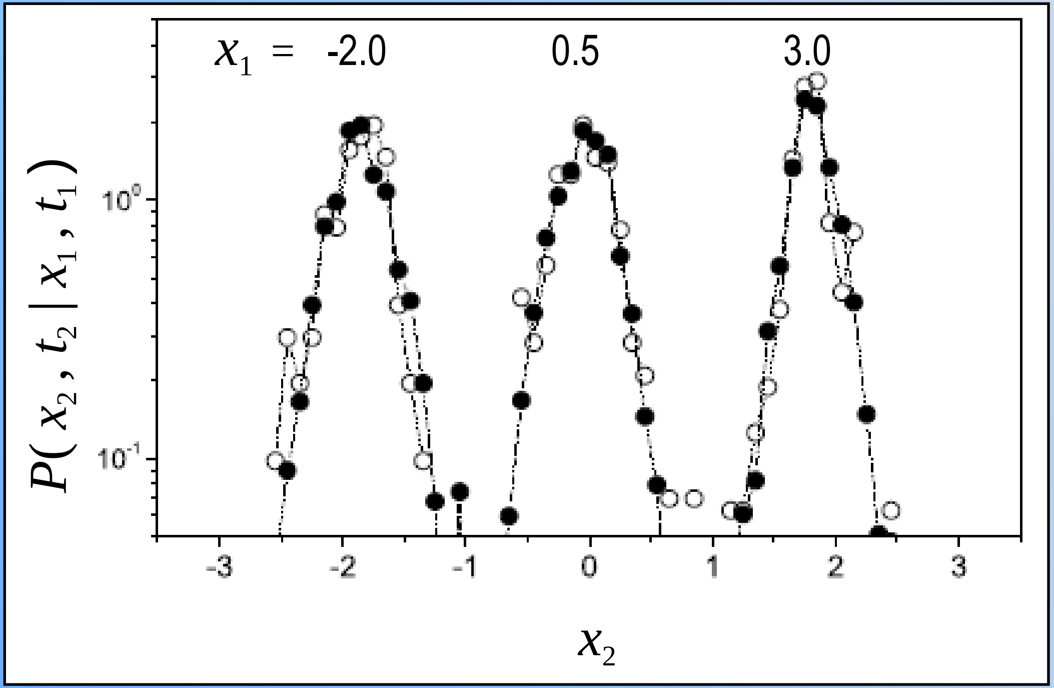
$$M^{(k)}=\int dx'\,(x'-x)^k\,P(x',t+\Delta t\,|\,x,t)$$

$$\tilde{D}^{(k)}(x,t,\Delta t)=\frac{1}{k!}\frac{M^{(k)}(x,t,\Delta t)}{\Delta t}\qquad D^{(k)}(x,t)=\lim_{\Delta t\to 0}\tilde{D}^{(k)}(x,t,\Delta t)$$

Markovianity check

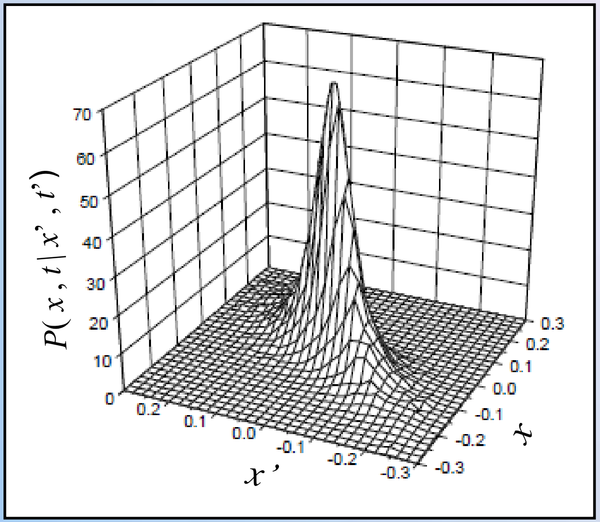
evaluation of the Chapman-Kolmogorov equation:

$$P(x_2,t_2\,|\,x_1,t_1)=\int dx' P(x_2,t_2\,|\,x',t')P(x',t'\,|\,x_1,t_1) \qquad t_1 < t' < t_2$$



Computation of KM coefficients

$$M^{(k)}(x,t,\Delta t) = \int dx' \, (x'-x)^k P(x',t+\Delta t \mid x,t)$$

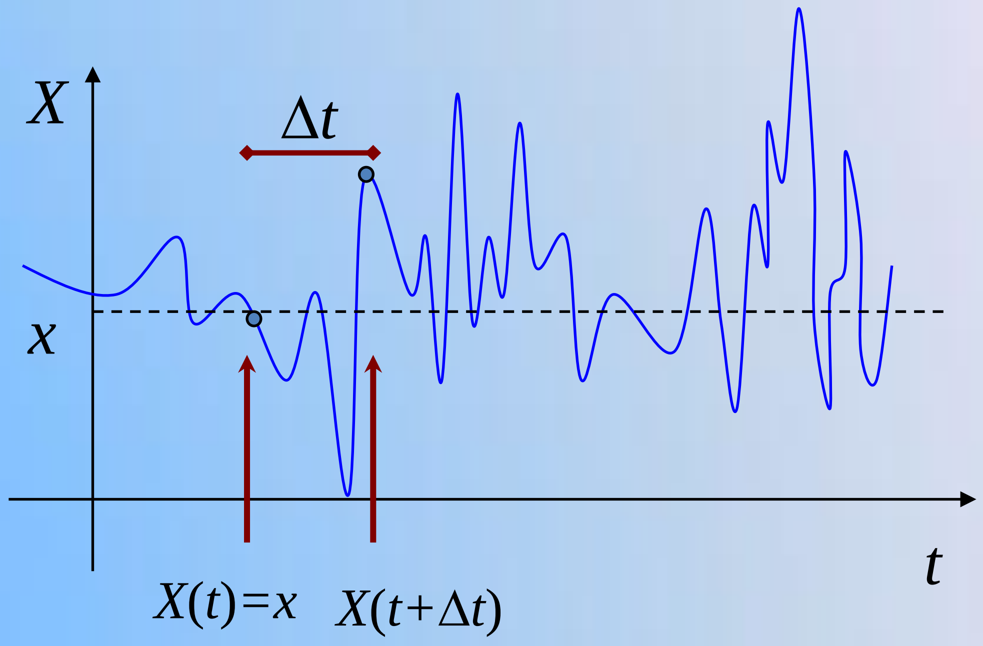


Conditional probabilities

$$P(x',t+\Delta t \mid x,t) = P(x',t+\Delta t; x,t) / P(x,t)$$

or equivalently

$$M^{(k)} = \left\langle \left(X(t+\Delta t) - X(t) \right)^k \right\rangle \Big|_{X(t)=x}$$



$$\frac{\partial}{\partial t}P(x,t)=\sum_{k\geq 1}\left[-\frac{\partial}{\partial x}\right]^k D^{(k)}(x,t)P(x,t)$$

$$\frac{\partial}{\partial t}P(x,t) = \sum_{k\geq 1}\left[-\frac{\partial}{\partial x}\right]^k D^{(k)}(x,t)P(x,t)$$

Important case: Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t) = \left[-\frac{\partial}{\partial x}D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2}D^{(2)}(x,t)\right]P(x,t)$$

drift coefficient
(deterministic)

diffusion coefficient
(random)

$$\frac{\partial}{\partial t}P(x,t)=\sum_{k\geq 1}\left[-\frac{\partial}{\partial x}\right]^k D^{(k)}(x,t)P(x,t)$$

Important case: Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t)=\left[-\frac{\partial}{\partial x}D^{(1)}(x,t)+\frac{\partial^2}{\partial x^2}D^{(2)}(x,t)\right]P(x,t)$$

drift coefficient (deterministic)

diffusion coefficient (random)

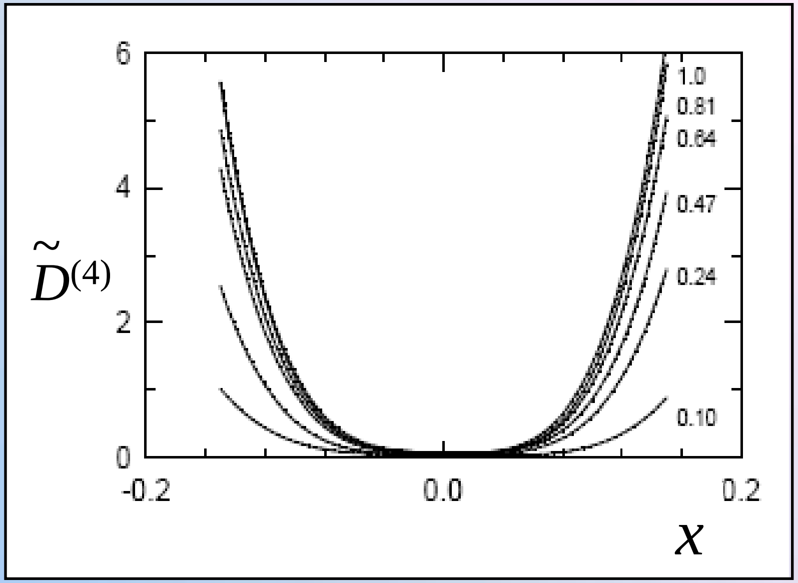
Itô-Langevin dynamics

$$\dot{x}=D^{(1)}(x,t)+\sqrt{D^{(2)}(x,t)}\,\eta(t)$$

$$\begin{cases}\langle \eta(t)\rangle = 0\\ \langle \eta(t)\eta(t')\rangle = 2\delta(t-t')\\ \text{Gaussian}\end{cases}$$

Pawula theorem

null 4th order coefficient $D^{(4)}$
$\rightarrow$ truncate KM expansion at 2nd order, reducing to a **Fokker-Planck equation**



consistent with $D^{(4)}(x,t) = 0$

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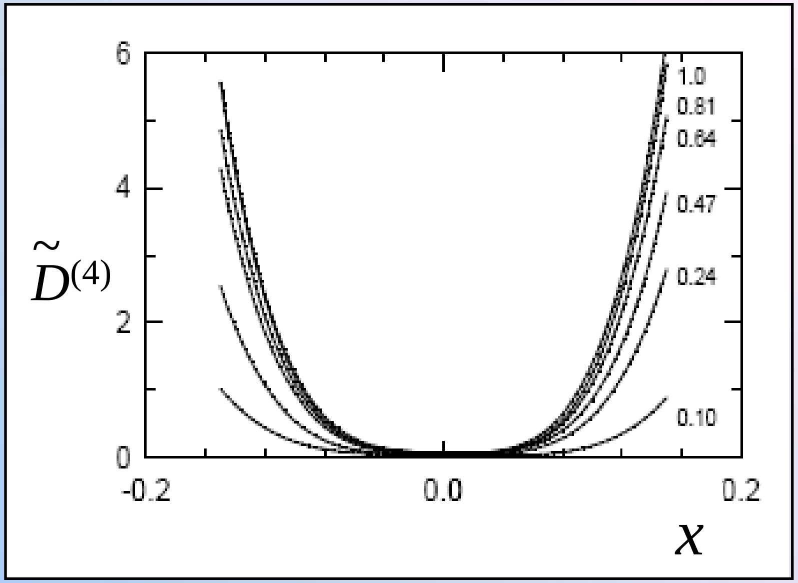

consistent with $D^{(4)}(x,t) = 0$

$$\left[\int f(x)g(x)P(x)dx\right]^2 \leq \int f^2(x)P(x)dx \int g^2(x)P(x)dx$$

Schwartz

arbitrary f,g
non-negative P

$$\int\int [f(x)g(y) - f(y)g(x)]^2 P(x)P(y)dxdy \geq 0$$

$$f(x) = (x - x')^n$$
$$g(x) = (x - x')^{n+m}$$
$$P(x) = P(x, t+\tau \mid x', t')$$

$\Rightarrow$ $M_{2n+m}^2 \leq M_{2n} M_{2n+2m}$ $\quad n,m \geq 0$

Pawula theorem

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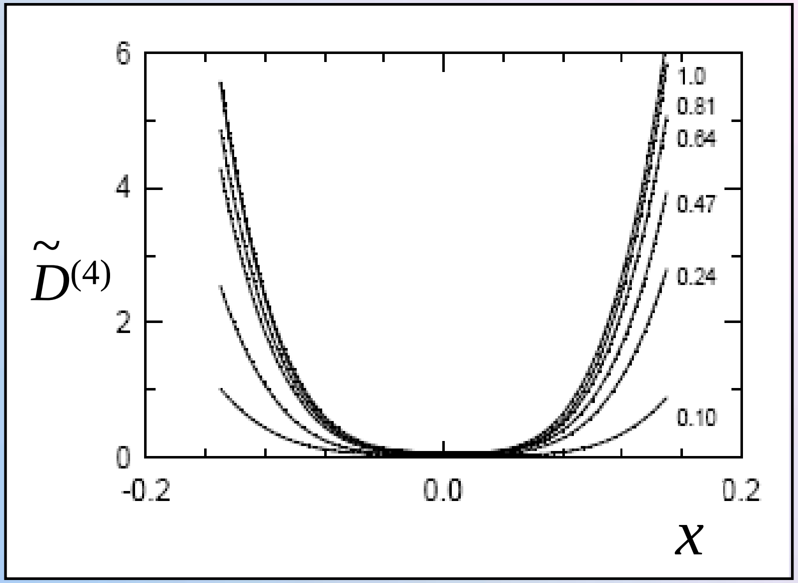

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Schwartz

- arbitrary f,g
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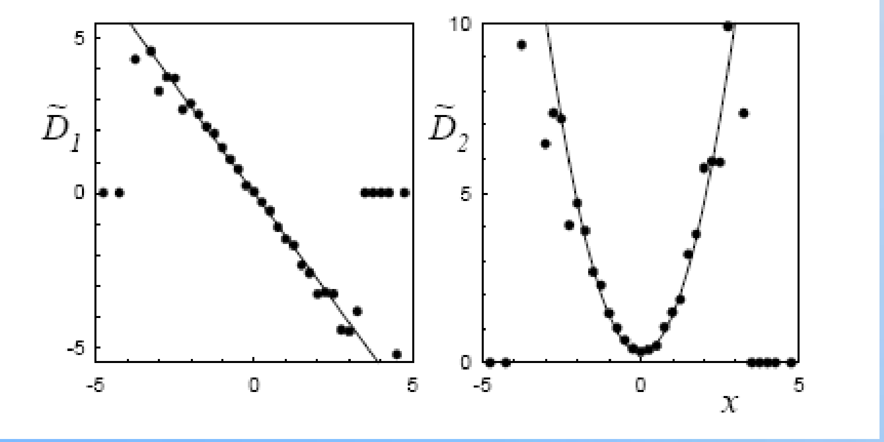
$$P(x) = P(x, t+\tau \mid x', t')$$

$\Rightarrow$ $M^2_{2n+m} \le M_{2n} M_{2n+2m}$ $\quad n,m \ge 0$

$\Rightarrow$ $D^2_{2n+m} \le cte\, D_{2n} D_{2n+2m}$ $\quad n,m \ge 1$

Exemplo 1



$\widetilde{D}^{(1)} \to D^{(1)}$

$\widetilde{D}^{(2)} \to D^{(2)}$

$\widetilde{D}^{(4)} \to 0$

$\Delta t \to 0$

$$\partial_t P(x,t) = \left[-\partial_x D^{(1)}(x,t) + \partial_{xx} D^{(2)}(x,t) \right] P(x,t)$$

$$\dot{x} = -D^{(1)}(x) + \sqrt{D^{(2)}(x)}\,\eta(t)$$

$\langle \eta(t) \rangle = 0$

$\langle \eta(t)\eta(t') \rangle = 2\delta(t-t')$

Gaussian

Exemplo 2

Poisson process

$$\dot{P}(m,t) = \lambda P(m-1,t) - \lambda P(m,t)$$

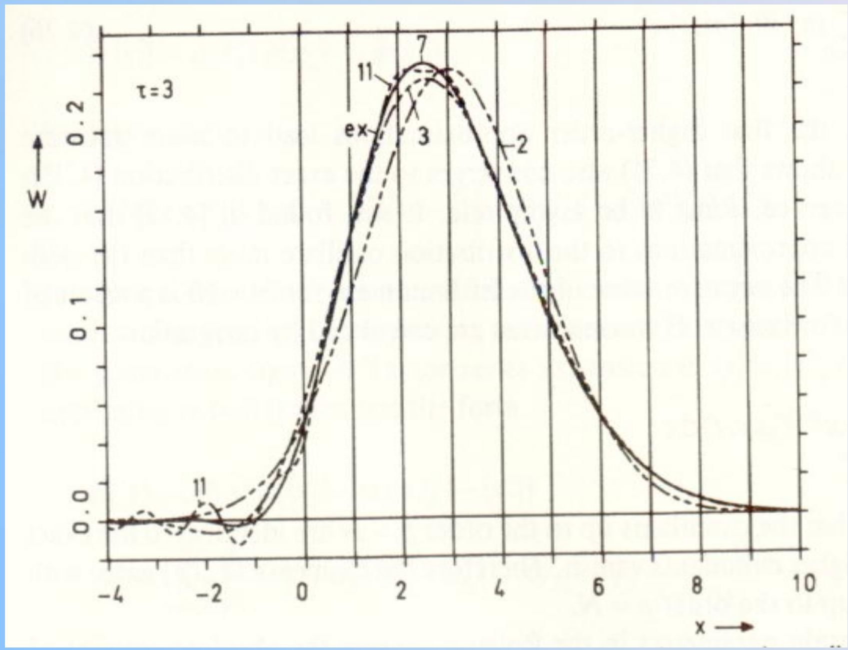
$$m \to x$$

$$\partial_t P(x,t) = \sum_{n\geq 1} \lambda \left[-\partial_x\right]^n P(x,t)/n!$$

$$P(x,t) = \lambda^x \, \mathrm{e}^{-\lambda} / \Gamma(x+1)$$

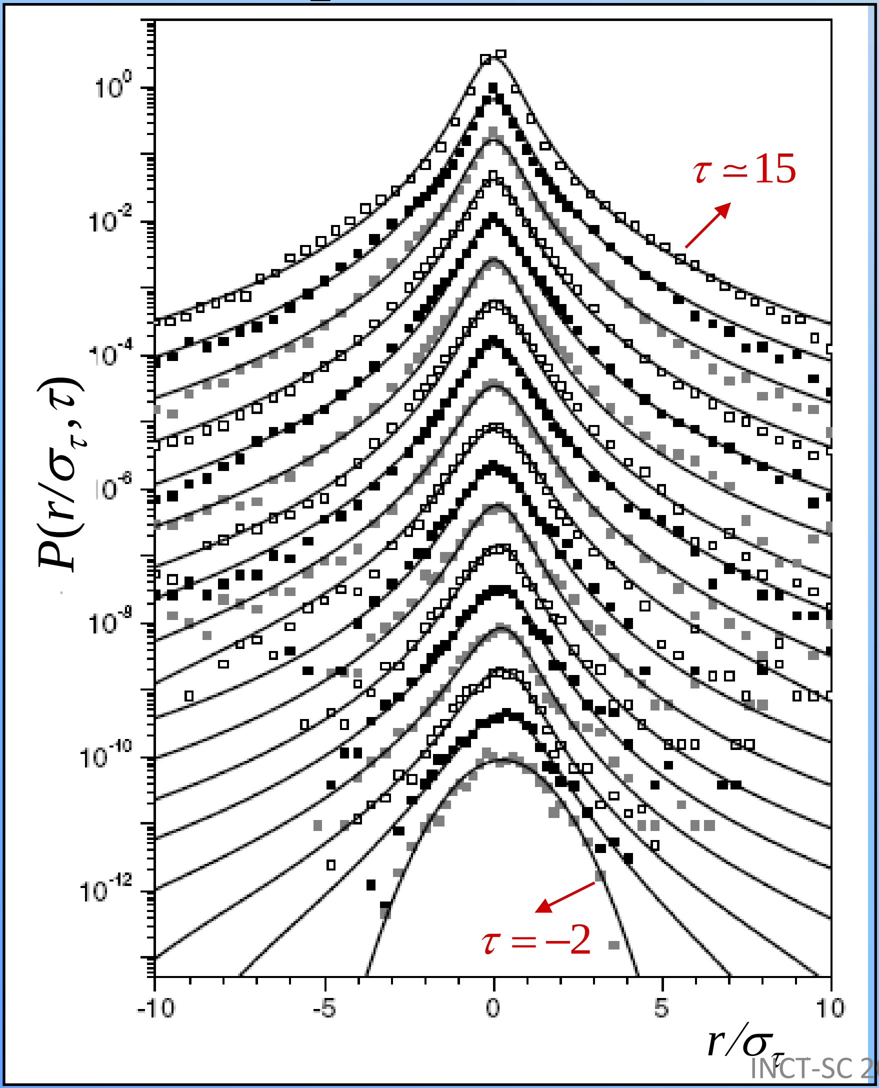
$$\partial_t P_N(x,t) = \sum_{n=1}^{N} \lambda \left[-\partial_x\right]^n P_N(x,t)/n!$$

$$P_N(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ikx + \sum_{n=1}^{N} [-ik]^n \lambda t/n!} \, dk$$



Applications: IBOVESPA

$$\partial_\tau P(r,\tau) = \left[-\partial_r D^{(1)}(r,\tau) + \partial^2_{rr} D^{(2)}(r,\tau) \right] P(r,\tau)$$



(τ = -log $\Delta t/\Delta t_0$)

- ***Unified description***

Cortines, Riera, Anteneodo; *EPJB* (2007)

Applications: Worldwide markets

- $D^{(1)} = -a_1 r \qquad D^{(2)} = b_2 r^2 + b_0$

universal r-dependence for all analyzed markets at the monthly/weekly timescales

- $D^{(1)}$ *and* $D^{(2)}$ *reveal market inner mechanisms*
  - a_1 *(restoring force) without significant differences across markets, meaning similar deterministic rules constrain the dynamics of markets*
  - b_0 *(additive noise) measure of market efficiency*
  - b_2 *(multiplicative noise) main determining factor of the power-law exponents of the tails of the PDFs, has a wide relative variation for different markets*

Cortines, Anteneodo, Riera; *EPJB* (2008)

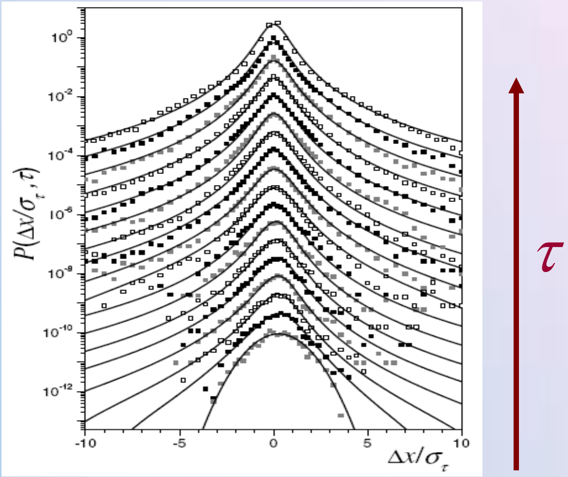
Observation

returns at different lags Δt :

$$r(t) = \log x(t+\Delta t) - \log x(t)$$

$$\partial_\tau P(r,\tau) = \left[-\partial_r D^{(1)}(r,\tau) + \partial^2_{rr} D^{(2)}(r,\tau) \right] P(r,\tau)$$

$\tau = -\log \Delta t / \Delta t_0$



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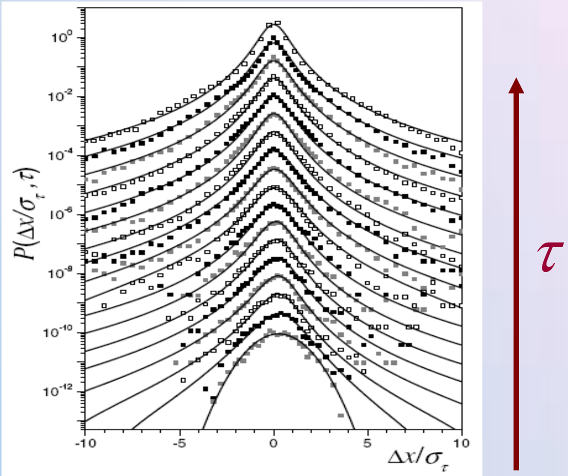
$$\partial_\tau P(r,\tau) = \left[-\partial_r D^{(1)}(r,\tau) + \partial_{rr}^2 D^{(2)}(r,\tau) \right] P(r,\tau)$$

$$\Delta\tau \rightarrow 0$$

$$\tilde{D}^{(k)}(r,\tau,\Delta\tau)$$

τ = -log $\Delta t/\Delta t_0$



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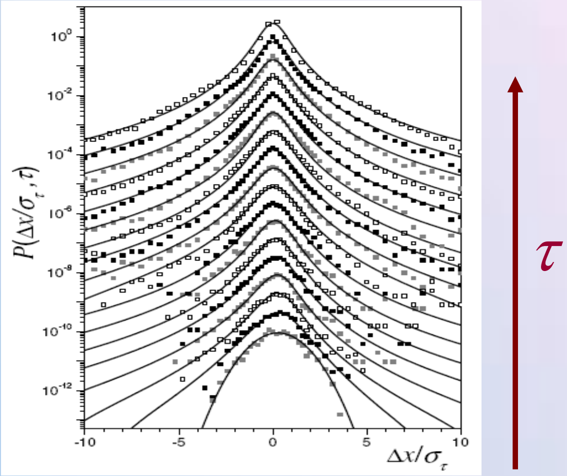

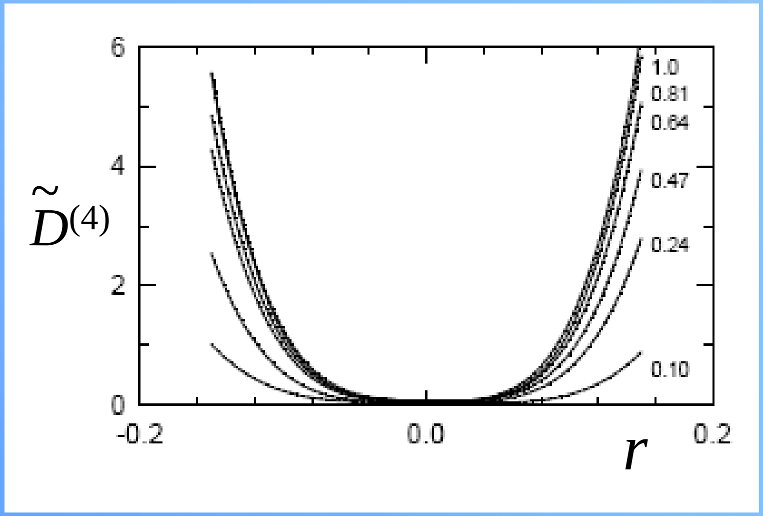

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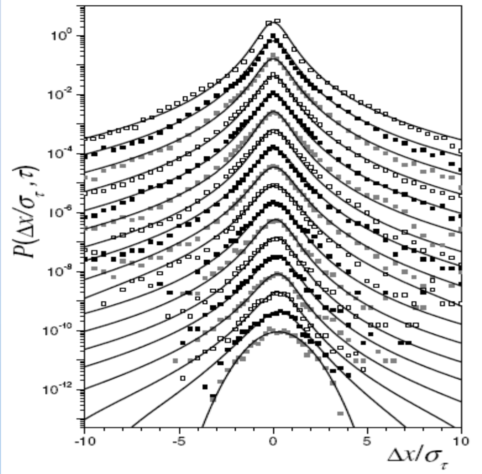
$$\partial_\tau P(r,\tau) = \left[-\partial_r D^{(1)}(r,\tau) + \partial^2_{rr} D^{(2)}(r,\tau) \right] P(r,\tau)$$

$$\Delta\tau \rightarrow 0$$

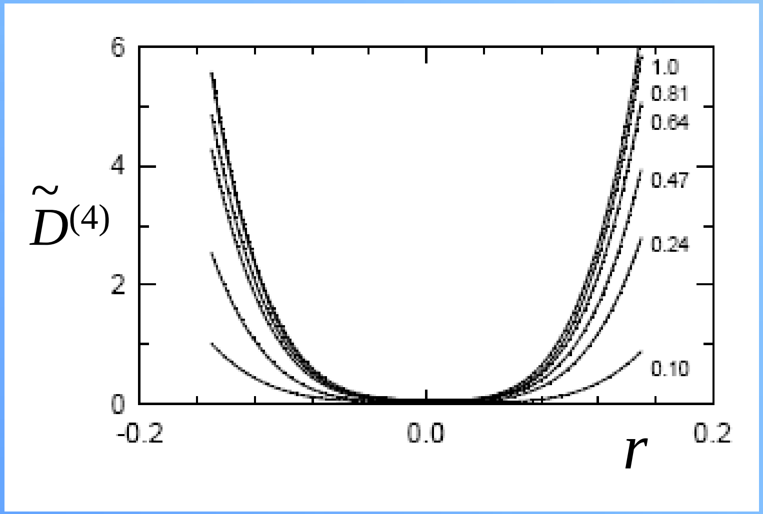
$$\tilde{D}^{(k)}(r,\tau,\Delta\tau)$$

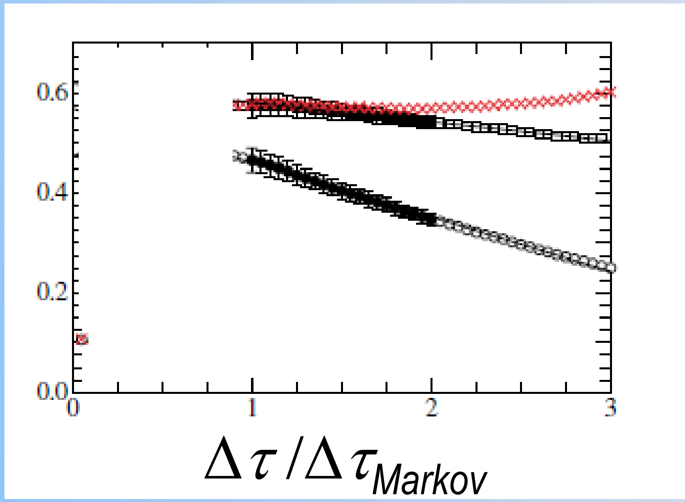
$$\tau = -\log \Delta t / \Delta t_0$$



τ

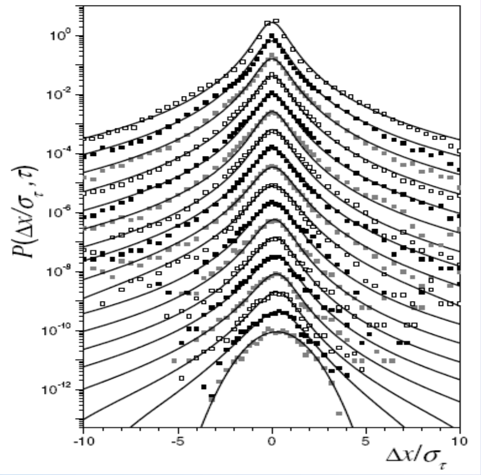
$$\tilde{D}^{(4)}(r,\tau,\Delta\tau) = a_{\tau,\Delta\tau} r^4 + b_{\tau,\Delta\tau} r^2 + c_{\tau,\Delta\tau}$$





Observation

$\tau = -\log \Delta t/\Delta t_0$



returns at different lags Δt :

$$r(t) = \log x(t+\Delta t) \;-\; \log x(t)$$

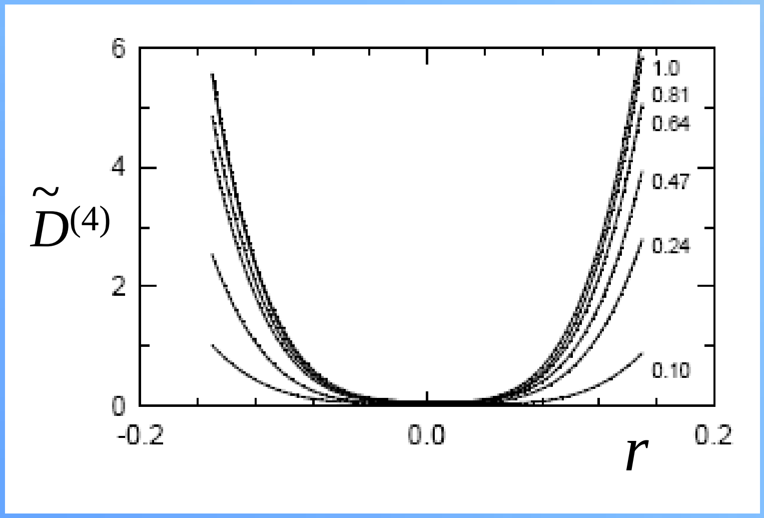
$$\partial_\tau P(r,\tau) = \left[-\partial_r D^{(1)}(r,\tau) + \partial_{rr}^2 D^{(2)}(r,\tau) \right] P(r,\tau)$$

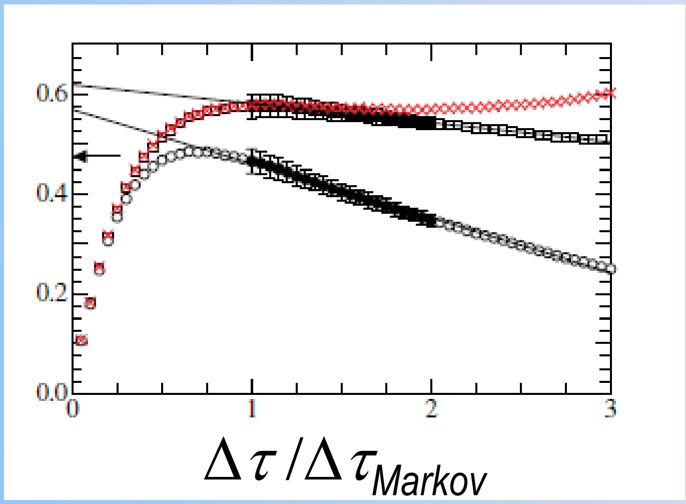
τ

$$\Delta\tau \to 0$$

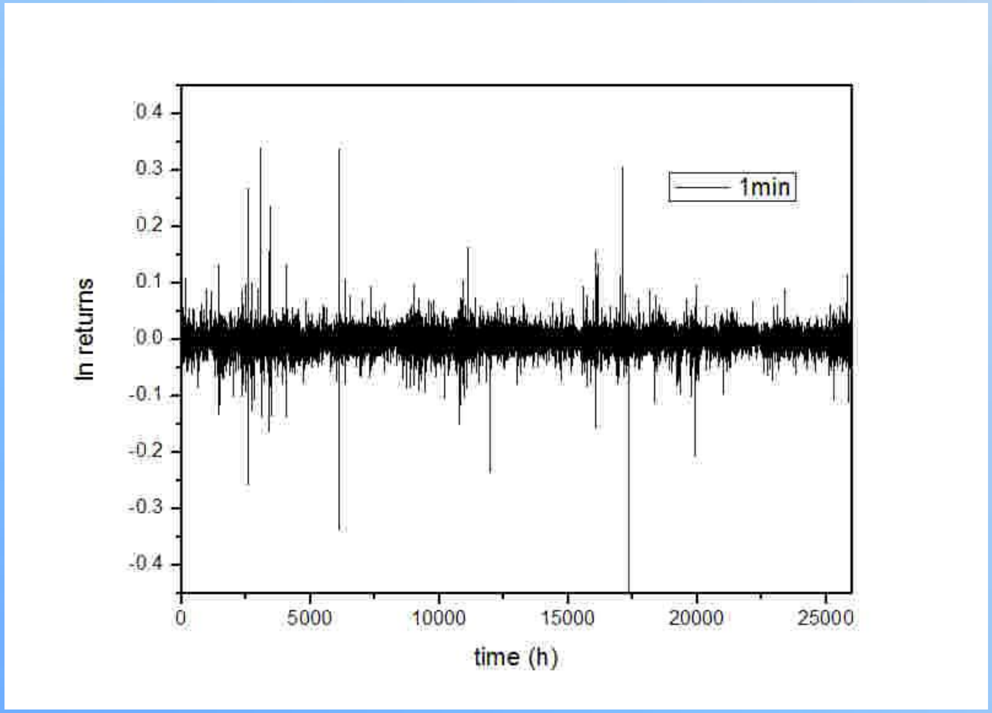
$$\tilde{D}^{(k)}(r,\tau,\Delta\tau)$$

$$\tilde{D}^{(4)}(r,\tau,\Delta\tau) = a_{\tau,\Delta\tau} r^4 + b_{\tau,\Delta\tau} r^2 + c_{\tau,\Delta\tau}$$

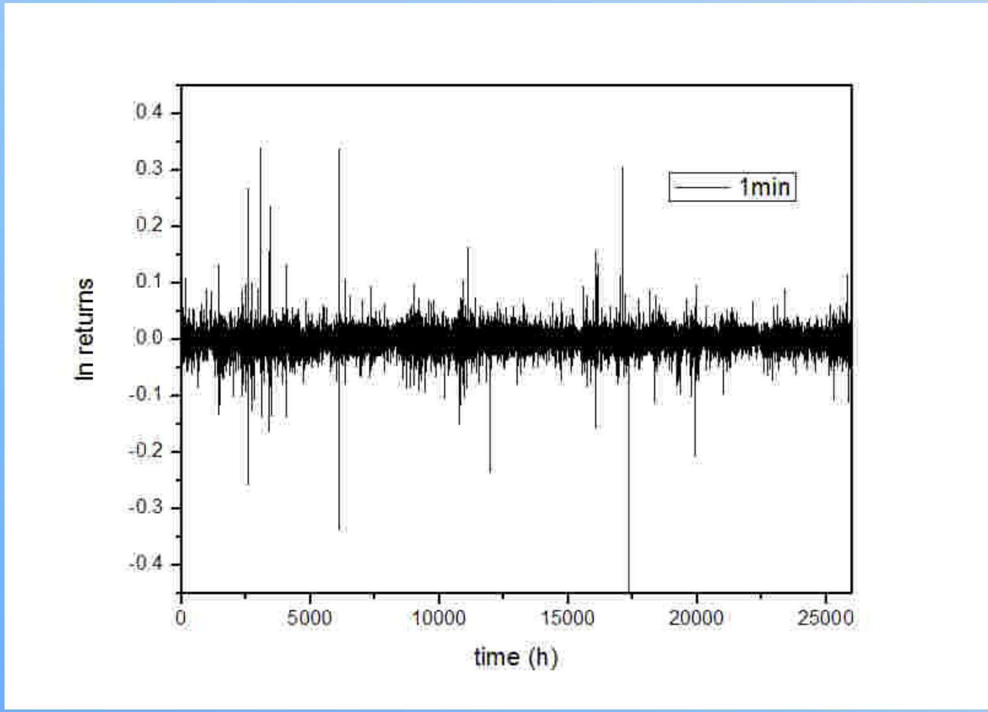




Now: timescale t (given fixed lag "Δt")



Now: timescale t (given fixed lag "Δt")



$$D^{(k)}(x,t) = \lim_{\Delta t \to 0} \tilde{D}^{(k)}(x,t,\Delta t)$$

$$\Delta t \to 0 \; ?$$

$$\Delta t_{min} \equiv 1/\text{data rate}$$

$$\Delta t_{max} << t_{char}$$

Applications

I

VOLUME 78, NUMBER 5 PHYSICAL REVIEW LETTERS 3 FEBRUARY 1997

Description of a Turbulent Cascade by a Fokker-Planck Equation

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J. Peinke
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(Received 1 July 1996)

VOLUME 83, NUMBER 26 PHYSICAL REVIEW LETTERS 27 DECEMBER 1999

Uniform Statistical Description of the Transition between Near and Far Field Turbulence in a Wake Flow

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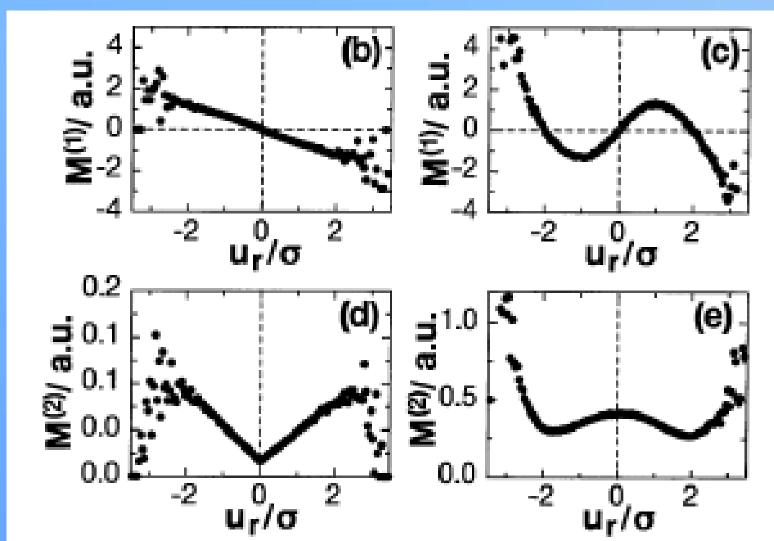
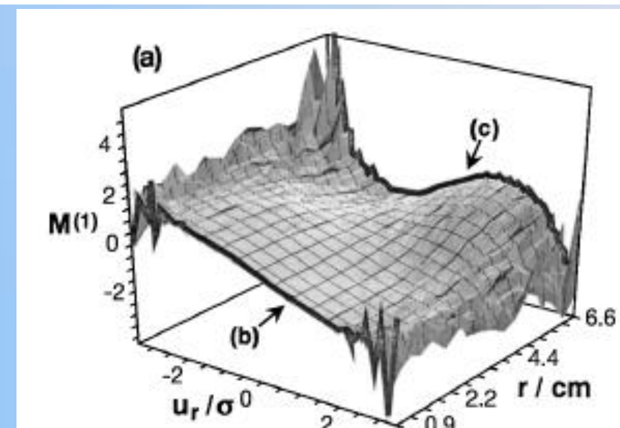
(Received 4 February 1999)





II

RAPID COMMUNICATIONS

PHYSICAL REVIEW E **75**, 060102(R) (2007)

Markov analysis and Kramers-Moyal expansion of nonstationary stochastic processes with application to the fluctuations in the oil price

Fatemeh Ghasemi,[1] Muhammad Sahimi,[2,*] J. Peinke,[3] R. Friedrich,[4] G. Reza Jafari,[5] and M. Reza Rahimi Tabar[3,6,7,†]

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[4]*Institute for Theoretical Physics, University of Münster, D-48149 Münster, Germany*
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(Received 10 January 2007; revised manuscript received 9 May 2007; published 18 June 2007)

$$D^{(1)}(y) = -1.09y,$$

$$D^{(2)}(y) = 0.0033 - 0.003y + 0.716y^2$$

III

PRL **98,** 138103 (2007)

PHYSICAL REVIEW LETTERS

week ending
30 MARCH 2007

Stochastic Qualifiers of Epileptic Brain Dynamics

Jens Prusseit[1,2,*] and Klaus Lehnertz[1,2,3,†]

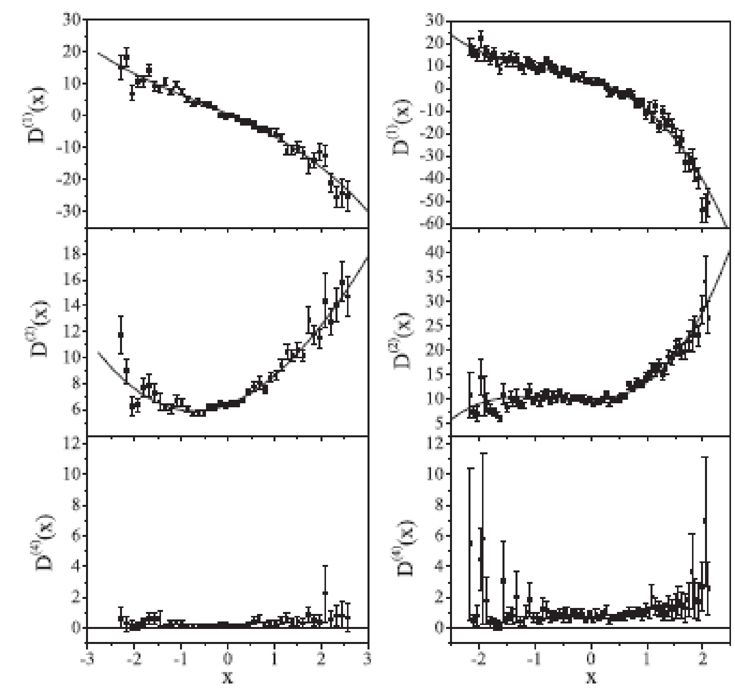

FIG. 1. Estimated coefficients $D^{(1)}$, $D^{(2)}$, and $D^{(4)}$ for exemplary EEG time series (left: from a distant brain region; right: from within the epileptic focus). Shown are estimates for time

IV

PHYSICAL REVIEW E **80**, 031127 (2009)

Kramers-Moyal coefficients in the analysis and modeling of heart rate variability

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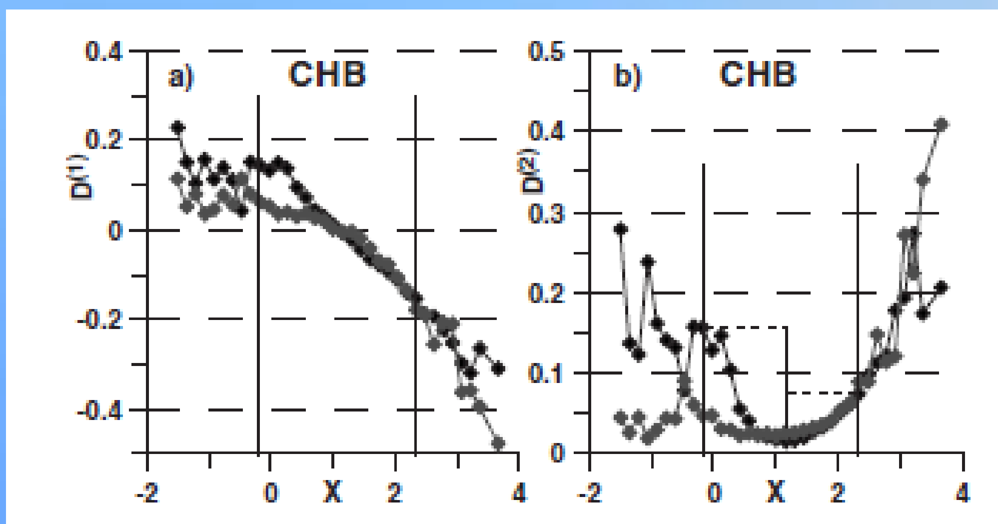
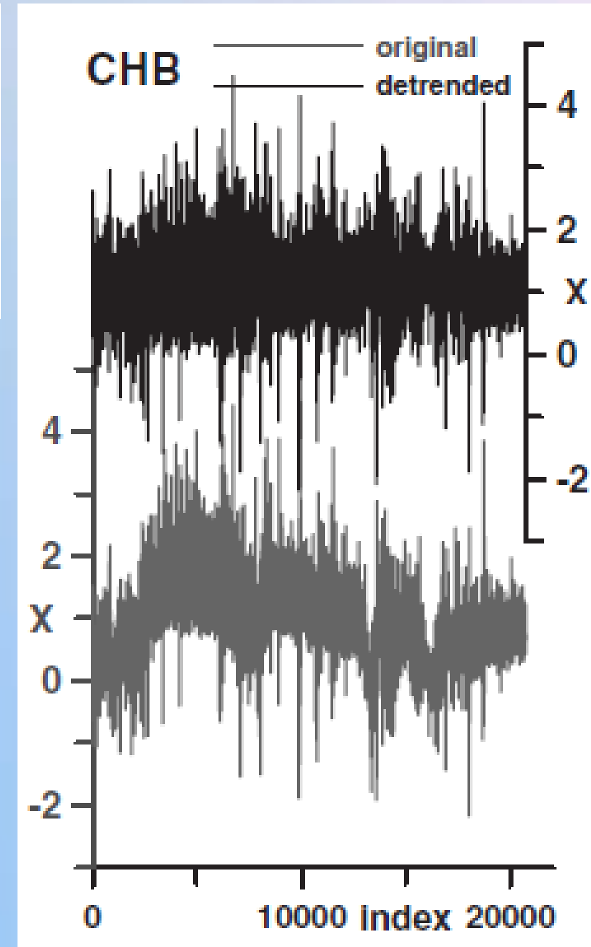
(Received 8 February 2009; revised manuscript received 25 May 2009; published 18 September 2009)





HOWEVER …

$$D^{(k)}(x,t) = \lim_{\Delta t \to 0} \tilde{D}^{(k)}(x,t,\Delta t)$$

$\Delta t \;\to 0$?

* Δt_{min}

* Finite time corrections (first order)

Observation

V

Available online at www.sciencedirect.com

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Physica D 193 (2004) 278–291

PHYSICA D

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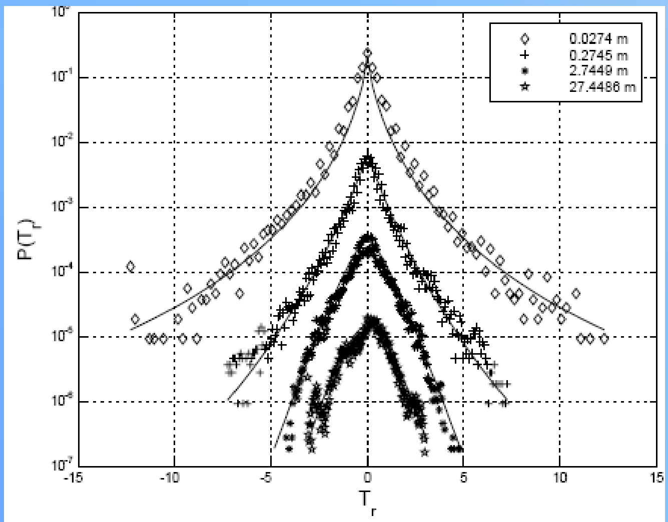
ELSEVIER

Atmospheric turbulence within and above an Amazon forest

Fernando Manuel Ramos [a,*], Maurício José Alves Bolzan [b], Leonardo Deane Abreu Sá [a,c], Reinaldo Roberto Rosa [a]

[a] *Instituto Nacional de Pesquisas Espaciais, INPE, São José dos Campos, SP, Brazil*
[b] *Universidade do Vale do Paraiba, UNIVAP, São José dos Campos, SP, Brazil*
[c] *Museu Paraense Emílio Goeldi (Campus de Pesquisa), Coordenação de Ciências da Terra e Ecologia (CCTE), Escritório do CPTEC/INPE, Belém, PA, Brazil*



Observation

V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Physica D 193 (2004) 278–291

PHYSICA D

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Atmospheric turbulence within and above an Amazon forest

Fernando Manuel Ramos [a,*], Maurício José Alves Bolzan [b], Leonardo Deane Abreu Sá [a,c], Reinaldo Roberto Rosa [a]

[a] Instituto Nacional de Pesquisas Espaciais, INPE, São José dos Campos, SP, Brazil
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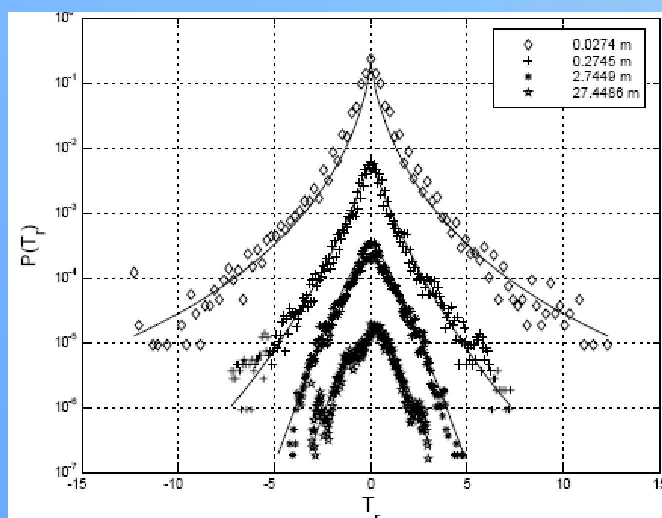
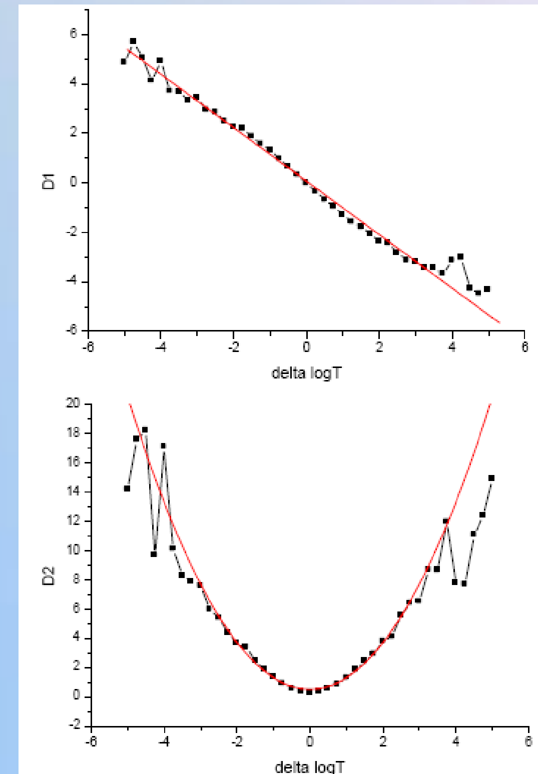
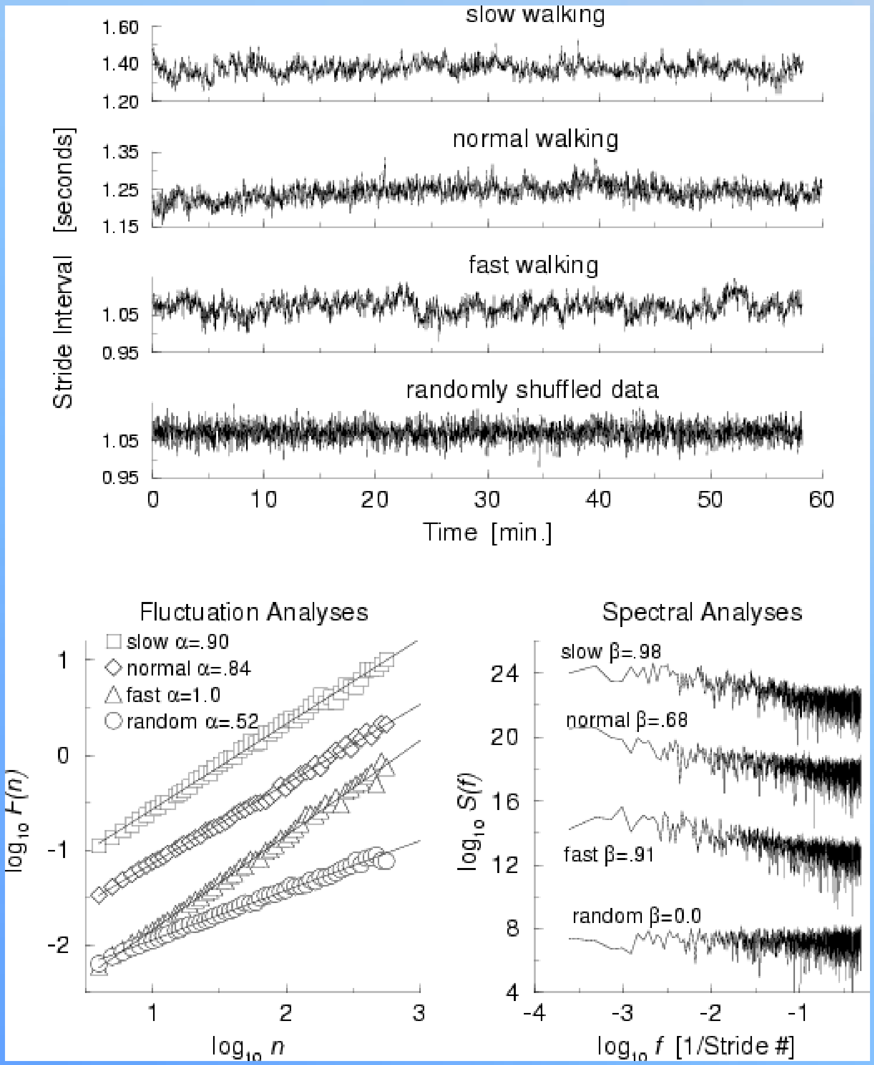
[c] Museu Paraense Emílio Goeldi (Campus de Pesquisa), Coordenação de Ciências da Terra e Ecologia (CCTE), Escritório do CPTEC/INPE, Belém, PA, Brazil





Long-term Recordings of Gait Dynamics: Unconstrained and Metronomic Walking

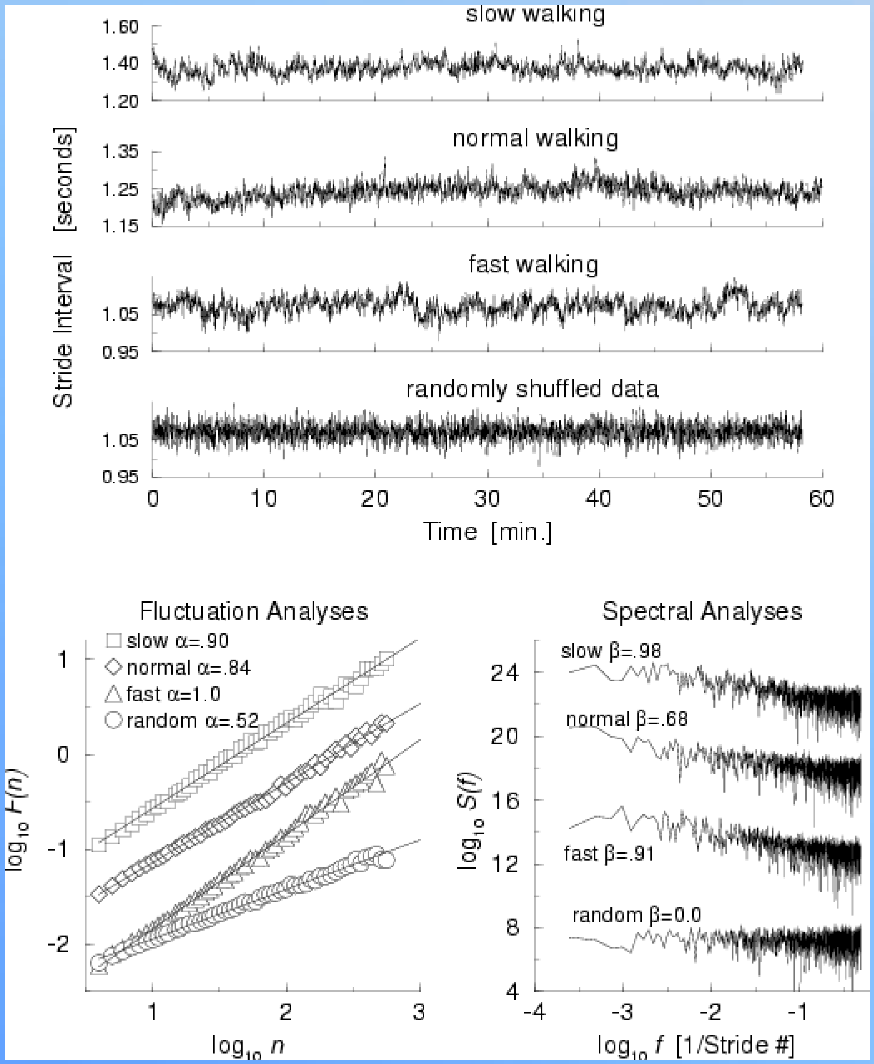
Hausdorff et al.; *J Appl Physiol* 8(1996)

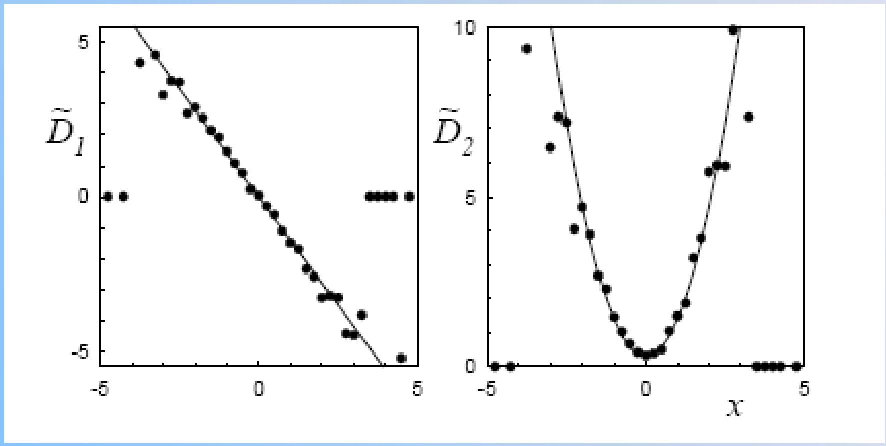


VI

Long-term Recordings of Gait Dynamics: Unconstrained and Metronomic Walking

Hausdorff et al.; *J Appl Physiol* 8(1996)





$$\partial_t P(z,t) = \Big[-\partial_z \underbrace{D^{(1)}(z,t)}_{-a_1(t)z} + \partial_{zz} \underbrace{D^{(2)}(z,t)}_{[b_0(t)+b_2(t)z^2]} \Big] P(z,t)$$

$$\dot{z} = D^{(1)}(z,t) + \sqrt{D^{(2)}(z,t)}\,\boldsymbol{\eta}(t)$$

Itô-Langevin dynamics

$$\begin{cases} \langle \boldsymbol{\eta}(t) \rangle = 0 \\ \langle \boldsymbol{\eta}(t)\boldsymbol{\eta}(t') \rangle = 2\boldsymbol{\delta}(t-t') \\ \text{Gaussian} \end{cases}$$

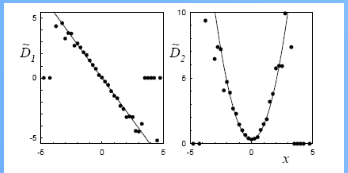
$$\dot{z} = -a_1(t)z + \sqrt{b_0(t) + b_2(t)z^2}\,\boldsymbol{\eta}(t)$$

$$\dot{z} = -a_1(t)z + \sqrt{b_0(t)}\,\boldsymbol{\eta}_1(t) + \sqrt{b_2(t)}\, z\,\boldsymbol{\eta}_2(t)$$

η_1, η_2: independent

Anteneodo, Tsallis, *JMP* (2003)
Anteneodo, *Physica A* (2005)

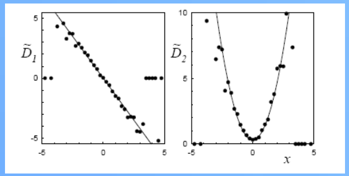
- *Finite-time effects*
- *Linear-quadratic*



⟹ *Exact corrections*

- *Finite-time effects*

- *Linear-quadratic*



Exact corrections

- *See next talk*