2ª Reunião de Trabalho do Instituto Nacional de Ciência e Tecnologia de Sistemas Complexos

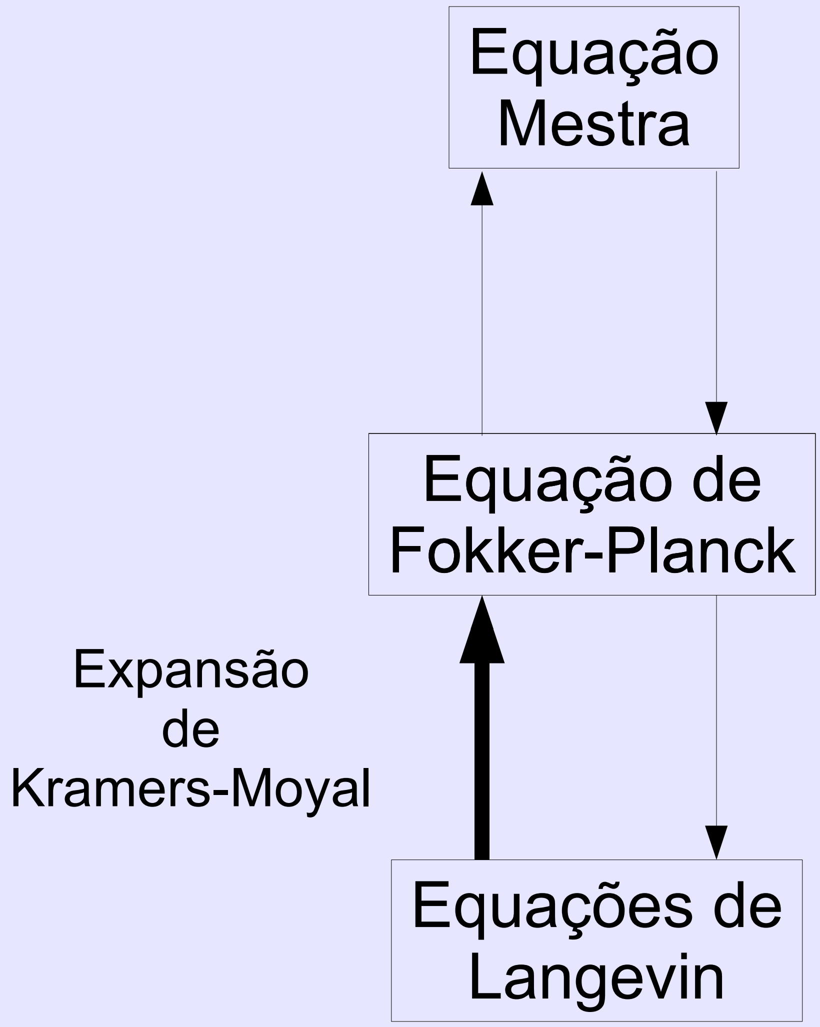
Unificação das equações estocásticas de Itô e Stratonovich
e
Evolução temporal para estados q-Gaussianos

Bernardo Coutinho e Constantino Tsallis
CBPF

RESUMO:

- Equações de Langevin
- Cálculo de Itô X Cálculo de Stratonovich
- Equações de Fokker-Planck
- q-momentos (q-curtose)
- Evolução temporal para estados estacionários q-Gaussianos

Processos estocásticos



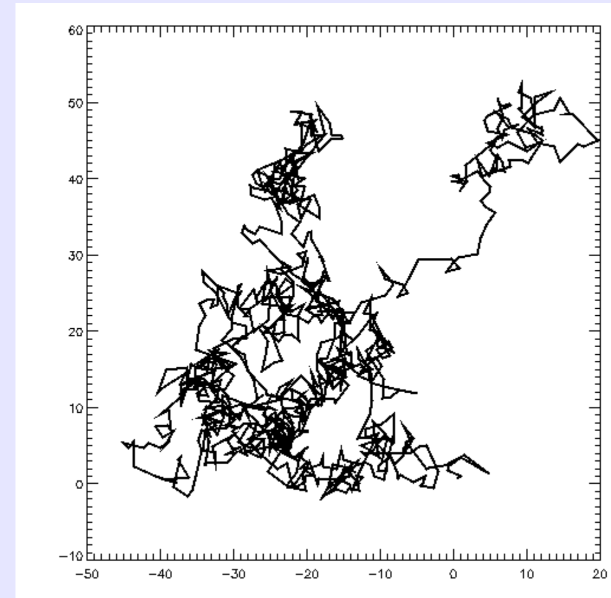
Equações de Langevin

$$\dot{x} = f(x) + \eta(t)$$

Força determinística
$f(x) = -\nabla U(x)$

Ruído gaussiano
$\langle \eta(t) \rangle = 0$ e $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$

Movimento Browniano
$f(x) = 0$



Caso geral: $$du = a(u,t)\ dt + b(u,t) M\ dW_1(t) + A\ dW_2(t)$$

Incrementos de Wiener

<u>Nosso interesse:</u> $$du = a(u)\ dt + b(u) M\ dW_1(t) + A\ dW_2(t)$$

Equação de Fokker-Planck

Expansão de Kramers-Moyal $$\partial_t P(u,t) = \sum_{n\geq 1} (-\partial_u)^n \left[D^{(n)}(u,t) P(u,t) \right]$$

$$D^{(n)}(u,t) = \frac{1}{n!} \lim_{\delta \to 0} \frac{\langle [u(t+\delta) - u(t)]^n \rangle}{\delta}$$

onde:

$$u(t+\delta) - x = \int_{u(t)=x}^{u(t+\delta)} du = \int_t^{t+\delta} dt' a(u(t')) + M \int_t^{t+\delta} dW_1(t')\, b(u(t')) + A \int_t^{t+\delta} dW_2(t')$$

Integrais estocásticas

- Cálculo de Itô
- Cálculo de Stratonovich

Itô vs. Stratonovich

A integral estocástica

$$I[G(t)] = \int_{t_0}^{t} dW(t')\, G(t')$$

Processos de Wiener

Cálculo de Itô:

$$I_I[G(t)] = \text{ms - }\lim_{n\to\infty} \sum_{i=1}^{n} G_{i-1}\, [W_i - W_{i-1}]$$

Cálculo de Stratonovich:

$$I_S[G(t)] = \text{ms - }\lim_{n\to\infty} \sum_{i=1}^{n} \frac{[G_i + G_{i-1}]}{2}\, [W_i - W_{i-1}]$$

Itô vs. Stratonovich

“Nossa” proposta unificadora:

$$I_\theta[G(t)] = \mathrm{ms}\text{ - }\lim_{n\to\infty} \sum_{i=1}^{n} [\theta G_i + (1-\theta) G_{i-1}] [W_i - W_{i-1}]$$

onde: $0 \leq \theta \leq 1$

$\theta = 0$ ⟶ Itô

$\theta = 1/2$ ⟶ Stratonovich

$\theta = 1$ ⟶ backward-Itô

Referência: *Germano, G., Politi, M., Scalas E. and Schilling, R.L., Stochastic calculus for uncoupled continuous-time random walks, Phys. Rev. E 79, 066102 (2009).*

Itô vs. Stratonovich

Caso: $G(t) = W(t)$

$$I_\theta[W(t)] \equiv \int_{t_0}^{t} dW(t')W(t')$$

$$= \text{ms -}\lim_{n\to\infty} \sum_{i=1}^{n} \left[\theta W_i + (1-\theta)W_{i-1}\right]\left[W_i - W_{i-1}\right]$$

$$= \frac{1}{2}\left\{[W(t)]^2 - [W(t_0)]^2 - (1-2\theta)(t-t_0)\right\} \; ,$$

Como: $\langle W^2(t)\rangle = t$

Então: $\langle I_\theta[W(t)]\rangle \equiv \left\langle \int_{t_0}^{t} dW(t')W(t') \right\rangle = \theta(t-t_0)$

Equação de Fokker-Planck

Expansão de Kramers-Moyal $$\partial_t P(u,t) = \sum_{n\geq 1} (-\partial_u)^n [D^{(n)}(u,t) P(u,t)]$$

$$D^{(n)}(u,t) = \frac{1}{n!} \lim_{\delta\to 0} \frac{\langle [u(t+\delta) - u(t)]^n \rangle}{\delta}$$

onde:

$$u(t+\delta) - x = \int_{u(t)=x}^{u(t+\delta)} du = \int_t^{t+\delta} dt' a(u(t')) + M \int_t^{t+\delta} dW_1(t')\, b(u(t')) + A \int_t^{t+\delta} dW_2(t')$$

Se: $$\begin{cases} a(u(t+\delta), t+\delta) = a(x) + a'(x)\delta + ... \\ b(u(t+\delta), t+\delta) = b(x) + b'(x)\delta + ... \end{cases}$$

Iterando e mantendo termos proporcionais a δ:

$$u^{(2)}(t+\delta) - x = \delta a(x) + M b'(x) b(x) \int_t^{t+\delta} W(t') dW(t')$$

$$\langle u^{(2)}(t+\delta) - x \rangle = \delta \left[a(x) + \theta M b'(x) b(x) \right]$$

Equação de Fokker-Planck

$$D^{(1)}(u,t) = a(u) + \theta M b'(u) b(u) = J(u)$$

"noise induced drift"

$$D^{(2)}(u,t) = A + M b^2(u) = D(u)$$

Equação de Fokker-Planck

$$\partial_t P(u,t) = -\partial_u \left\{ \left\{ a(u) + \theta M b(u) b'(u) + \partial_u \left[A + M b^2(u) \right] \right\} P(u,t) \right\}$$

$$= -\partial_u j(u)$$

onde:

$$j(u) = J(u) P(u,t) - \partial_u \left[D(u) P(u,t) \right]$$

Estado estacionário: $j(\pm\infty) = j(u) = 0$

Equação de Fokker-Planck

Se: $a(u) = -\partial_u V(u) = (\tau/2)\partial_u g^2(u)$ e $g(u) \propto u$

$$P_{est}(u) \propto e_q^{-\beta u^2} \equiv [1 - \beta(1-q)u^2]_+^{1/(1-q)}$$

com: $[z]_+ = \begin{cases} z\ , & z > 0 \\ 0\ , & z < 0 \end{cases}$

onde: $\beta = \dfrac{\tau + M(1-\theta)}{A}$ e $q = \dfrac{\tau + M(2-\theta)}{\tau + M(1-\theta)}$

<u>Itô</u> ($\theta = 0$): $\beta = \dfrac{\tau + M}{A}$ e $q = \dfrac{\tau + 2M}{\tau + M}$

<u>Stratonovich</u> ($\theta = 1/2$): $\beta = \dfrac{\tau + M/2}{A}$ e $q = \dfrac{\tau + 3M/2}{\tau + M/2}$

<u>Referência:</u> Anteneodo, C. and Tsallis, C., Multiplicative noise: A mechanism leading to nonextensive Statistical mechanics., J. Math. Phys. 44, 5194, (2003).

q-momentos

Escort distributions:

$$\langle A(x) \rangle_{\tilde{q}} = \int_{-\infty}^{+\infty} A(x) f_{\tilde{q}}(x) dx \longrightarrow \langle A(x) \rangle_{\tilde{q}=1} = \int_{-\infty}^{+\infty} A(x) f(x) dx$$

onde: $$f_{\tilde{q}}(x) = \frac{[f(x)]^{\tilde{q}}}{\int_{-\infty}^{+\infty} [f(x)]^{\tilde{q}} dx} \longrightarrow f_{\tilde{q}=1}(x) = f(x)$$

q-transformada de Fourier:

$$\mathcal{F}_{\tilde{q}}[f](\xi) = \int_{-\infty}^{+\infty} f(x) e_{\tilde{q}}^{i\xi [f(x)]^{\tilde{q}-1}} dx \qquad (\tilde{q} \geq 1)$$

$$\longrightarrow \mathcal{F}_{\tilde{q}=1}[f](\xi) = \int_{-\infty}^{+\infty} f(x) e^{i\xi x} dx$$

q-momentos

curtose:

$$\kappa_1 = \frac{\langle x^4 \rangle}{3\langle x^2 \rangle^2} = \frac{\int_{-\infty}^{+\infty} x^4 f(x) dx}{3\left[\int_{-\infty}^{+\infty} x^2 f(x) dx\right]^2}$$

q-curtose:

$$\kappa_{\tilde{q}} = \frac{\langle x^4 \rangle_{\tilde{q}}}{3\langle x^2 \rangle_{\tilde{q}}^2} = \frac{\int_{-\infty}^{+\infty} x^4 \left[f(x)\right]^{4\tilde{q}-3} dx}{3\left\{\int_{-\infty}^{+\infty} x^2 \left[f(x)\right]^{2\tilde{q}-1} dx\right\}^2}$$

<u>Referência:</u> Tsallis, C., Plastino, A.R. and Alvarez-Estrada, R.F., Escort mean values and the characterization of power-lawdecaying probability densities, J. Math. Phys. 50, 043303.

q-momentos

Qual o valor correto de $\tilde{q}$?

$$\lim_{x\to\infty} f(x) \sim |x|^{-\gamma} \xrightarrow[(\gamma > 0)]{} \gamma = \frac{1}{\tilde{q}-1}$$

Se: $f(x) = e_q^{-\beta x^2}$

$$\lim_{x\to\infty} e_q^{-\beta x^2} \sim |x|^{2/(q-1)} \xrightarrow[(q > 1)]{} \gamma = 2/(q-1)$$

$$\Rightarrow \quad \tilde{q} - 1 = \frac{q-1}{2}$$

Referência: *Tsallis, C., Plastino, A.R. and Alvarez-Estrada, R.F., Escort mean values and the characterization of power-lawdecaying probability densities, J. Math. Phys. 50, 043303.*

q-momentos

$$\left\langle x^2 \right\rangle_q = \begin{cases} \dfrac{\beta^{-1}\Gamma\left(\frac{q}{q-1}-\frac{3}{2}\right)}{2(q-1)\Gamma\left(\frac{q}{q-1}-\frac{1}{2}\right)} & , \quad \text{para} \quad 1<q<3 \\ \dfrac{\beta^{-1}}{2} & , \quad \text{para} \quad q=1 \\ \dfrac{\beta^{-1}\Gamma\left(\frac{q}{1-q}+\frac{3}{2}\right)}{2(1-q)\Gamma\left(\frac{q}{1-q}+\frac{5}{2}\right)} & , \quad \text{para} \quad 0<q<1 \end{cases}$$

$$\left\langle x^4 \right\rangle_{2q-1} = \begin{cases} \dfrac{3\beta^{-2}\Gamma\left(\frac{q}{q-1}-\frac{3}{2}\right)}{4(q-1)^2\Gamma\left(\frac{q}{q-1}+\frac{1}{2}\right)} & , \quad \text{para} \quad 1<q<3 \\ \dfrac{3\beta^{-1}}{4} & , \quad \text{para} \quad q=1 \\ \dfrac{3\beta^{-2}\Gamma\left(\frac{q}{1-q}+\frac{1}{2}\right)}{4(1-q)^2\Gamma\left(\frac{q}{1-q}+\frac{5}{2}\right)} & , \quad \text{para} \quad 0<q<1 \end{cases}$$

q-momentos

q-curtose:

$$\kappa_q(\infty) = \lim_{t\to\infty} \kappa_q(t) = \kappa_q(\infty) = \begin{cases} \dfrac{\Gamma^2\left(\frac{q}{q-1} - \frac{1}{2}\right)}{\Gamma\left(\frac{q}{q-1} + \frac{1}{2}\right)\Gamma\left(\frac{q}{q-1} - \frac{3}{2}\right)} & , \quad \text{para } 1 < q < 3 \\ 1 & , \quad \text{para} \quad q = 1 \\ \dfrac{\Gamma^2\left(\frac{q}{1-q} + \frac{1}{2}\right)\Gamma\left(\frac{q}{1-q} + \frac{5}{2}\right)}{\Gamma^2\left(\frac{q}{1-q} + \frac{3}{2}\right)} & , \quad \text{para } 0 < q < 1 \end{cases}$$

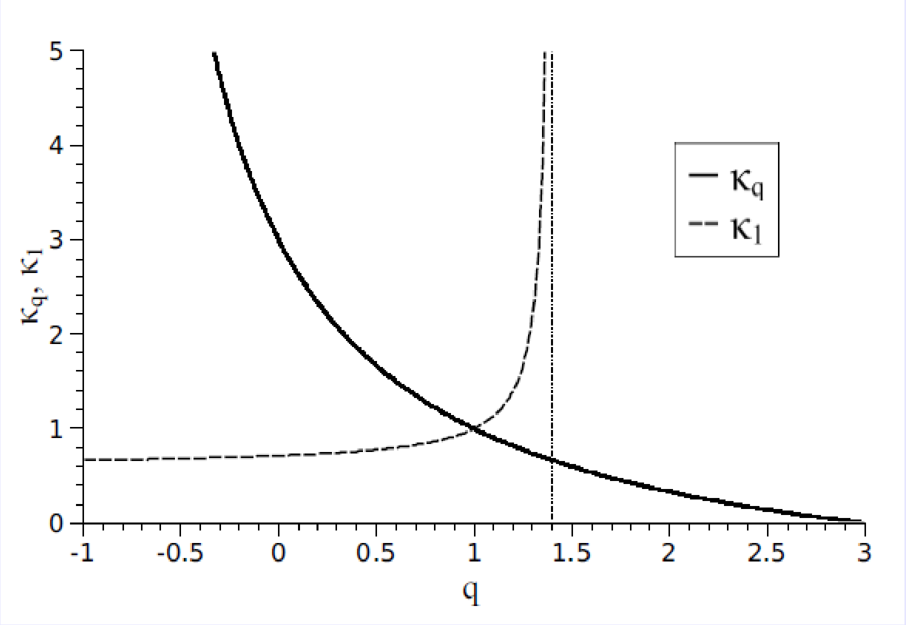
Vantagens da q-curtose sobre a curtose:

$$\langle x^2 \rangle_1 \to \infty\,, \text{ para } 5/3 < q < 3$$

$$\langle x^4 \rangle_1 \to \infty\,, \text{ para } 7/5 < q < 3$$

$$\langle x^2 \rangle_q \text{ é finito, para } 0 < q < 3$$

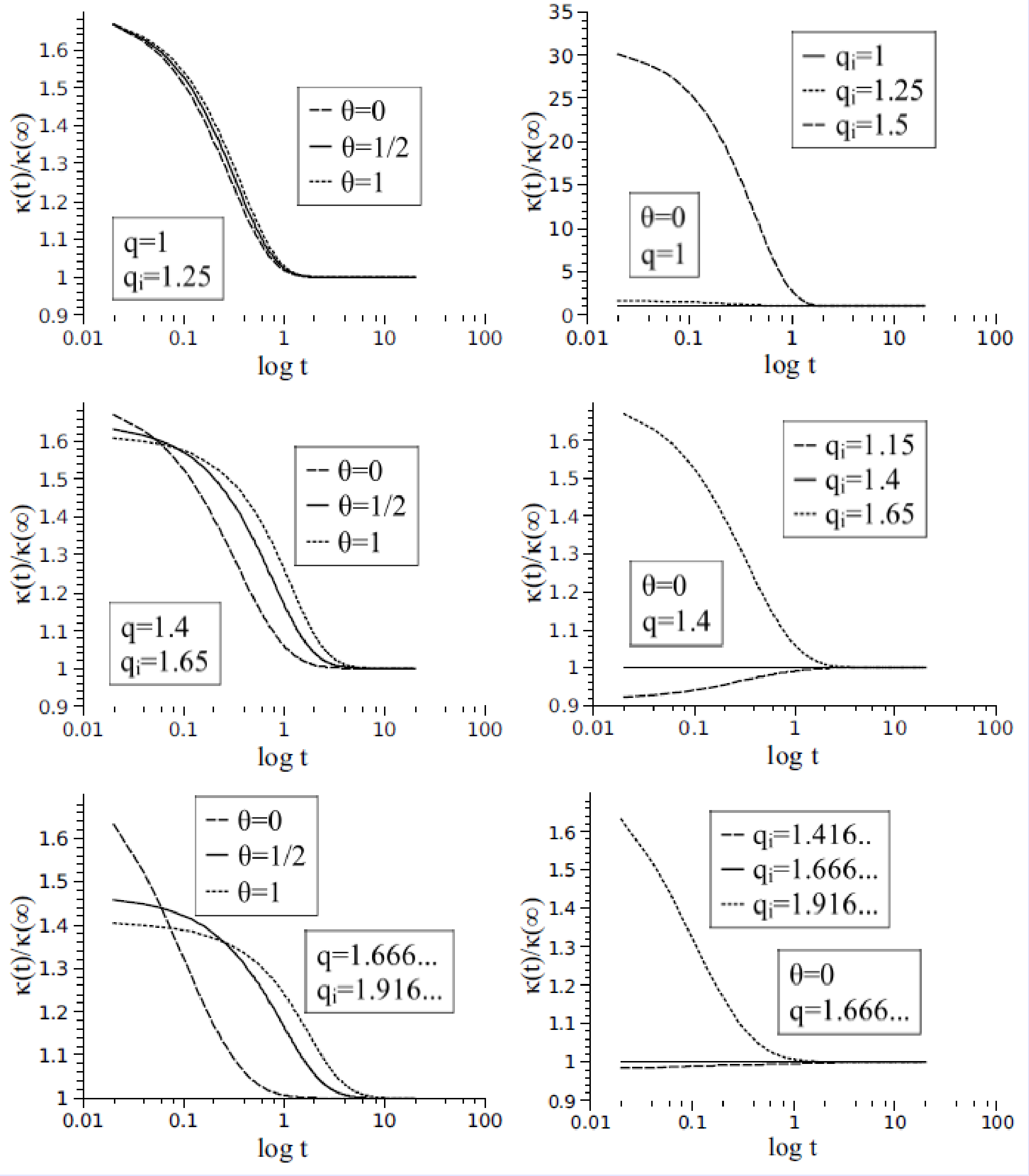
$$\langle x^4 \rangle_q \text{ é finito}\,, \text{ para } 0 < q < 3$$



Soluções estacionárias da Fokker-Planck

Condições iniciais: $q_i \neq q_f$

$$q_f = \frac{\tau + 2M(2-\theta)}{\tau + 2M(1-\theta)}$$



Conclusões finais

- Unificação dos procedimentos de Itô e Stratonovich
- Construimos uma equação de Fokker-Planck unificada
- Soluções estacionárias são q-Gaussianas
  (verificado numericamente)
- Conjectura de que a q-curtose pode ser usada para determinar o valor de q a partir de dados experimentais ou computacionais.

A integral $I_\theta[W(t)] \equiv \int_{t_0}^{t} dW(t')W(t')$

$$I_\theta[W(t)] \equiv \int_{t_0}^{t} dW(t')W(t') = \text{ms - }\lim_{n\to\infty} S_n$$

onde: $S_n = \sum_{i=1}^{n} [\theta W_i \Delta W_i + (1-\theta) W_{i-1}\Delta W_i]$

Note que:

$$\sum_{i=1}^{n} W_i \Delta W_i = \frac{1}{2}\sum_{i=1}^{n}\left[(W_i)^2 + (\Delta W_i)^2 - (\overbrace{W_i - \Delta W_i}^{W_{i-1}})^2\right] = \frac{W^2(t) - W^2(t_0) + \sum_{i=1}^{n}(\Delta W_i)^2}{2}$$

$$\sum_{i=1}^{n} W_{i-1} \Delta W_i = \frac{1}{2}\sum_{i=1}^{n}\left[(\underbrace{W_{i-1} + \Delta W_i}_{W_i})^2 - (W_{i-1})^2 - (\Delta W_i)^2\right] = \frac{W^2(t) - W^2(t_0) - \sum_{i=1}^{n}(\Delta W_i)^2}{2}$$

$$\text{ms - }\lim_{n\to\infty} \sum_{i=1}^{n} (\Delta W_i)^2 = t - t_0$$

Assim sendo:

$$I_\theta[W(t)] \equiv \int_{t_0}^{t} dW(t')W(t') = \text{ms - }\lim_{n\to\infty} S_n = \frac{1}{2}\left\{W^2(t) - W^2(t_0) - (1-2\theta)(t-t_0)\right\}$$