The problem of linear and non-linear thermal conduction for small systems

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1 - Abstract

We study the problem of thermal conductance for small systems in the context of general types of noise [1-7] and for linear and non-linear couplings between the particles [7].

We show that, for linear systems, thermal conductance is a purely mechanical property of the system [3]. However, non-linear couplings between the particles make the conductance a mechanical-thermodynamical property of the system.

Our treatment is essentially exact, dealing by making systematic expansions in powers of the non-linear interaction [7].

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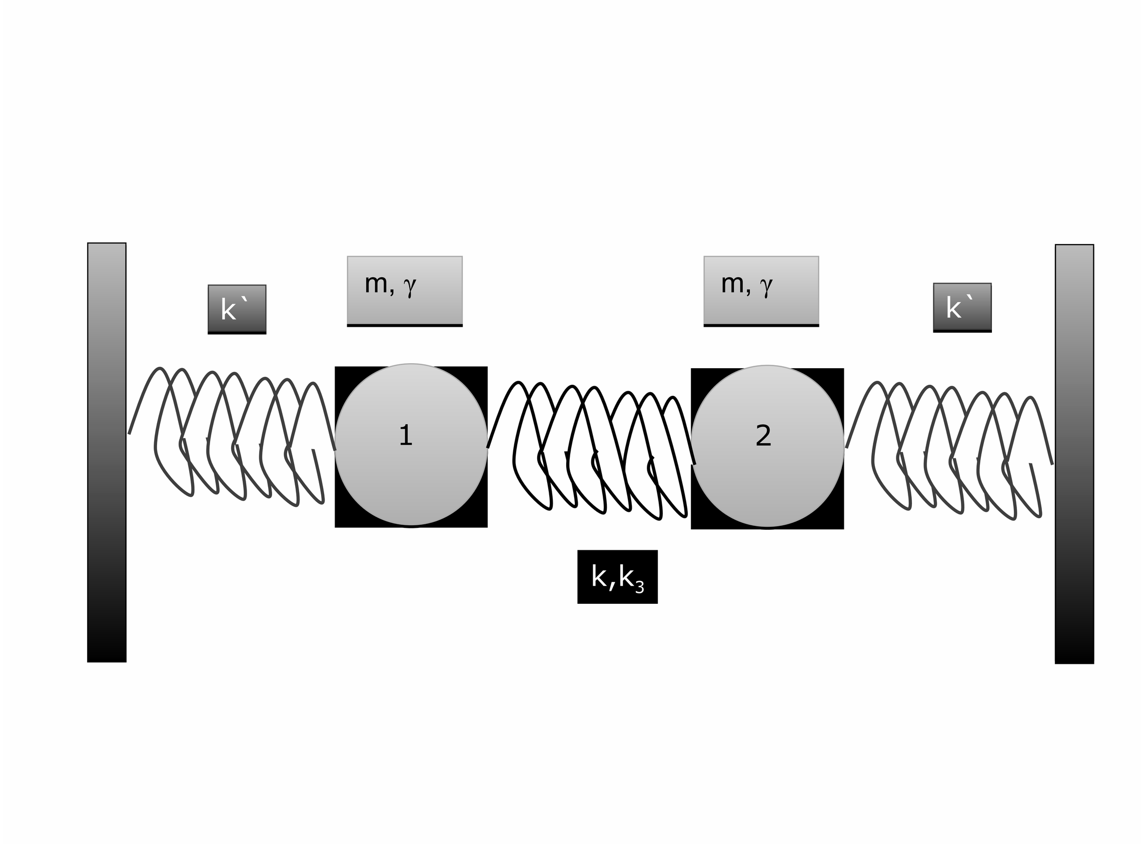
2 - Model

Our problem focus on solving the set of equations,

$$m\frac{dv_i\,(t)}{dt} = -k\,x_i\,(t) - \gamma\,v_i\,(t) - \sum_{l=1}^{2} k_{2\,l-1}\left[x_i\,(t) - x_j\,(t)\right]^{2\,l-1} + \eta_i\,(t) \tag{1}$$

with,

$$\frac{dx_i\,(t)}{dt} = v_i\,(t) \tag{2}$$



The transfer flux between the two particles reads,

$$j_{12}\,(t) \equiv -\sum_{l=1}^{2}\frac{k_{2\,l-1}}{2}\left[x_1\,(t) - x_2\,(t)\right]^{2\,l-1}\left[v_1\,(t) + v_2\,(t)\right]. \tag{3}$$

The term, $\eta_i\,(t)$, represents a general uncorrelated Lévy class stochastic process the cumulants of which are defined as,

$$\left\langle \eta_{i_1}\,(t_1)\,\ldots\,\eta_{i_n}\,(t_n)\right\rangle_c = \mathcal{A}(t_1,n)\,\delta_{i_1\,i_2}\ldots\delta_{i_{n-1}\,i_n}\,\delta\,(t_1-t_2)\ldots\delta\,(t_{n-1}-t_n)\,. \tag{4}$$

3 - Laplace transformations

Laplace transforming Eqs. (1) and (2) we obtain,

$$\begin{cases} R\,(s)\ \ \tilde{x}_i(s) = k_1\,\tilde{x}_j\,(s) + \tilde{\eta}_i\,(s) \\ \tilde{x}_i(s) = \tilde{v}_i(s) \end{cases}, \tag{5}$$

(Re$(s) > 0$) with, $R(s) \equiv \left(m\,s^2 + \gamma\,s + k + k_1\right)$. The solutions to Eq. (5) are easily obtained when the relative position, $\tilde{r}_D\,(s) \equiv \tilde{x}_1\,(s) - \tilde{x}_2\,(s)$, and the mid-point position, $\tilde{r}_S\,(s) \equiv \left(\tilde{x}_1\,(s) + \tilde{x}_2\,(s)\right)/2$, as well as the respective noises $\tilde{\eta}_D\,(s) = \tilde{\eta}_1\,(s) - \tilde{\eta}_2\,(s)$ and $\tilde{\eta}_S\,(s) = \left(\tilde{\eta}_1\,(s) + \tilde{\eta}_2\,(s)\right)/2$. After some algebra it yields,

$$\begin{cases} \tilde{r}_D(s) = \frac{\tilde{\eta}_D(s)}{R'(s)} - \frac{2\,k_3}{R'(s)}\,\lim_{\alpha\to 0}\int_{-\infty}^{+\infty}\frac{\prod_{l=1}^{3}\frac{dq_l}{2\pi}\,\tilde{r}_D(\mathrm{i}\,q_l+\alpha)}{s-\sum_{l=1}^{3}(\mathrm{i}\,q_l+\alpha)} \\ \tilde{r}_S = \frac{\tilde{\eta}_S(s)}{R''(s)} \end{cases}, \tag{6}$$

with $R'(s) \equiv \left(m\,s^2 + \gamma\,s + k + 2\,k_1\right)$ and $R''(s) \equiv \left(m\,s^2 + \gamma\,s + k\right)$. Reverting the above equation, we get $\tilde{x}_1\,(s)$ and $\tilde{x}_2\,(s)$. To obtain the solutions, we must compute the cumulants of η_1 and η_2 in the Laplace space,

$$\begin{aligned}\left\langle \tilde{\eta}_{i_1}\,(z_1)\ \ldots\ \tilde{\eta}_{i_n}\,(z_n)\right\rangle_c &= \int_0^{\infty}\prod_{j=1}^{n}dt_{i_j}\,\exp\left[-\sum_{j=1}^{n}z_{i_j}\,t_{i_j}\right]\left\langle \eta_{i_1}\,(t_1)\ \ldots\ \eta_{i_n}\,(t_n)\right\rangle_c \\ &= \frac{\mathcal{A}(n)}{\sum_{j=1}^{n}z_{i_j}}\delta_{i_1\,i_2}\ldots\delta_{i_{n-1}\,i_n}. \end{aligned} \tag{7}$$

4 - Types of noise

We study the effect of distinct kinds of noise, namely Brownian (variance is the highest cumulant) and Poisson (all cumulants are nonzero) [5]. In fact, the method can be applied to any typo of noise with **convergent** cumulants.

In first approximation in powers of k_3, the conductance reads

$$\overline{\left\langle j_{12}^{(B)}\right\rangle} = \overline{\left\langle j_{12}^{(0)}\right\rangle} - \frac{3}{8}\gamma\,k_1\,k_3\frac{(2\,k+k_1)\left[\mathcal{A}_1(2)^2 - \mathcal{A}_2(2)^2\right]}{(k+2\,k_1)\left[\gamma^2\,(k+k_1) + m\,k_1^2\right]^2} + \mathcal{O}\left(k_3^3\right), \tag{8}$$

for Brownian particles and,

$$\overline{\left\langle j_{12}^{(P)}\right\rangle} = \overline{\left\langle j_{12}^{(B)}\right\rangle} - \frac{27}{2}\gamma^2\,k_1\,k_3\,\lambda\frac{\mathcal{N}}{\mathcal{D}}\left[\mathcal{A}_1(2)^2 - \mathcal{A}_2(2)^2\right] + \mathcal{O}\left(k_3^3\right), \tag{9}$$

for Poissonian particles, with,

$$\overline{\left\langle j_{12}^{(0)}\right\rangle} = -\frac{k_1^2}{4}\frac{\left(\mathcal{A}_1(2) - \mathcal{A}_2(2)\right)}{m\,k_1^2 + \gamma^2\,(k+k_1)}, \tag{10}$$

and

$$\mathcal{N} \equiv \gamma^2\,(5\ k + 3\,k_1) + m\left(3\,k_1^2 + 4\,k^2 + 11\,k\,k_1\right), \tag{11}$$

$$\mathcal{D} \equiv \left[\gamma^2\,(k+k_1)\right]\left[m\,(4\,k+9\,k_1)^2 + 6\,\gamma^2\,(2\,k+3\,k_1)\right]\left[3\,\gamma^4 + m^2\,k_1^2 + 4\,m\,\gamma^2\,(k+k_1)\right] \tag{12}$$

Thence, we are finally in the position to compute the thermal conductance,

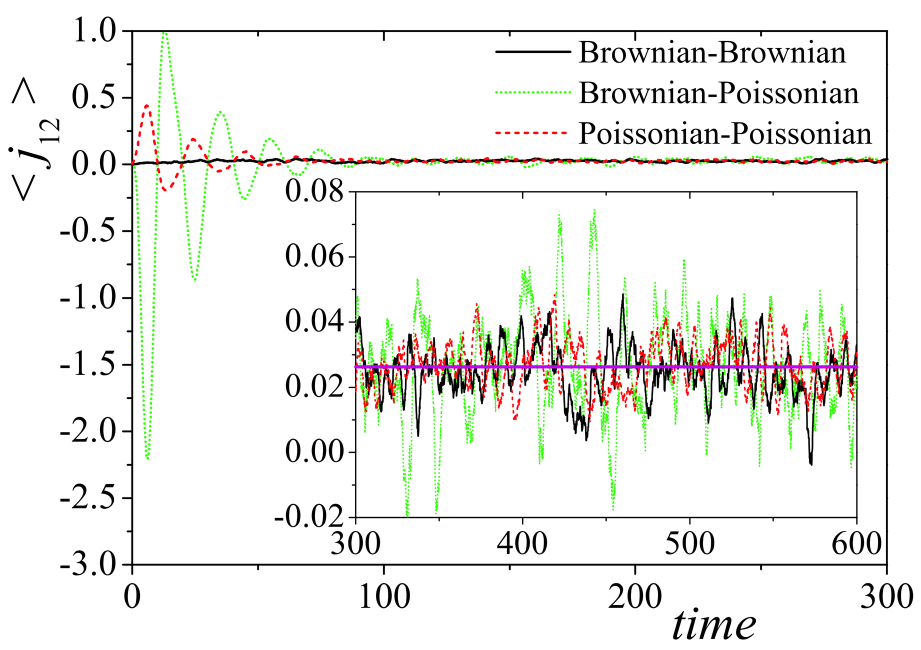
$$\kappa \equiv -\frac{\partial}{\partial\Delta T}\left\langle j_{12}\right\rangle_{\Delta T} = -\frac{\overline{\left\langle j_{12}\right\rangle}}{T_1 - T_2}, \tag{13}$$

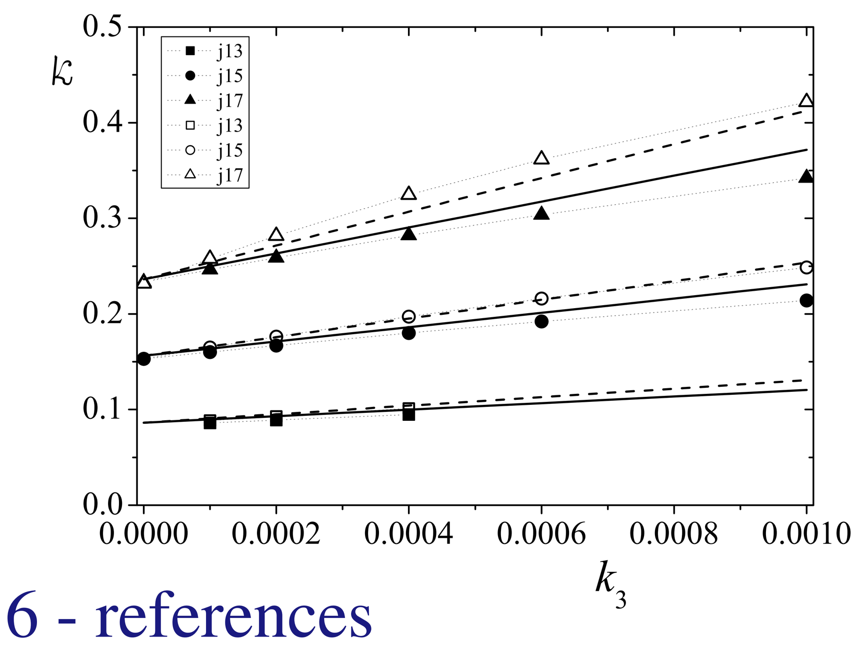
yielding

$$\begin{aligned}\kappa^{(0)} &= \frac{1}{2}\frac{k_1^2\gamma}{m\,k_1^2 + \gamma^2\,(k+k_1)} + \frac{3}{8}\gamma\,k_1\,k_3\frac{(2\,k+k_1)\left[\mathcal{A}_1(2) + \mathcal{A}_2(2)\right]}{(k+2\,k_1)\left[\gamma^2\,(k+k_1) + m\,k_1^2\right]^2} + \\ &+ \frac{27}{2}\gamma^2\,k_1\,k_3\,\lambda\frac{\mathcal{N}}{\mathcal{D}}\left[\mathcal{A}_1(2) + \mathcal{A}_2(2)\right] + \mathcal{O}\left(k_3^2\right), \end{aligned} \tag{14}$$

depending on the type of noise.

5 - Numerical Results





6 - references

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Conclusions

The results above show that the conductance is consistent with interpreting the distinct cumulants of the noise as distinct sources of heat [6]. We observe that for Poisson-Poisson sources the thermal conductance is highest due to the effect of kurtosis, which is not present on the usual Brownian Gaussian noise.

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