RESULTADOS TEÓRICOS E EXPERIMENTAIS RECENTES NO QUADRO DA *q*-ESTATÍSTICA

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Workshop INCT-SC, Maio 2012

ALGUNS RESULTADOS RECENTES

TEORIA DOS GRANDES DESVIOS

A ENTROPIA DO BURACO NEGRO

PHYSICAL REVIEW E **84**, 021121 (2011)

Group entropies, correlation laws, and zeta functions

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The notion of group entropy is proposed. It enables the unification and generaliztion of many different definitions of entropy known in the literature, such as those of Boltzmann-Gibbs, Tsallis, Abe, and Kaniadakis. Other entropic functionals are introduced, related to nontrivial correlation laws characterizing universality classes of systems out of equilibrium when the dynamics is weakly chaotic. The associated thermostatistics are discussed. The mathematical structure underlying our construction is that of formal group theory, which provides the general structure of the correlations among particles and dictates the associated entropic functionals. As an example of application, the role of group entropies in information theory is illustrated and generalizations of the Kullback-Leibler divergence are proposed. A new connection between statistical mechanics and zeta functions is established. In particular, Tsallis entropy is related to the classical Riemann zeta function.

$$S_q \leftrightarrow \varsigma(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

$$= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdot \frac{1}{1-11^{-s}} \mathrm{L}$$

Milan j. math. 76 (2008), 307–328

1424-9286/010307-22, *published online* 14.3.2008
DOI 10.1007/s00032-008-0087-y

Milan Journal of Mathematics

On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

JOURNAL OF MATHEMATICAL PHYSICS **51**, 033502 (2010)

Generalization of symmetric α-stable Lévy distributions for $q>1$

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See also:

H.J. Hilhorst, JSTAT P10023 (2010)

M. Jauregui and C. T., Phys Lett A **375**, 2085 (2011)

M. Jauregui, C. T. and E.M.F. Curado, JSTAT P10016 (2011)

A. Plastino and M.C. Rocca, 1112.1985 and 1112.1986 [math-ph]

CENTRAL LIMIT THEOREM

$N^{1/[\alpha(2-q)]}$ - *scaled attractor* $F(x)$ *when summing* $N \to \infty$ q - *independent identical random variables*

with symmetric distribution $f(x)$ *with* $\sigma_Q \equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q$ $\left(Q \equiv 2q-1,\ q_1 = \frac{1+q}{3-q} \right)$

<table>
<tr><td></td><td>$q=1$ [independent]</td><td>$q \neq 1$ (i.e., $Q \equiv 2q-1 \neq 1$) [globally correlated]</td></tr>
<tr><td>$\sigma_Q < \infty$
$(\alpha = 2)$</td><td>$F(x) =$ Gaussian $G(x)$,
with same σ_1 of $f(x)$

Classic CLT</td><td>$F(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, with same σ_Q of $f(x)$
$G_q(x) \sim \begin{cases} G(x) & \text{if } |x| \ll x_c(q,2) \\ f(x) \sim C_q / |x|^{2/(q-1)} & \text{if } |x| \gg x_c(q,2) \end{cases}$
with $\lim_{q \to 1} x_c(q,2) = \infty$
S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)</td></tr>
<tr><td>$\sigma_Q \to \infty$
$(0 < \alpha < 2)$</td><td>$F(x) =$ Levy distribution $L_\alpha(x)$,
with same $|x| \to \infty$ behavior
$L_\alpha(x) \sim \begin{cases} G(x) \\ \quad \text{if } |x| \ll x_c(1,\alpha) \\ f(x) \sim C_\alpha / |x|^{1+\alpha} \\ \quad \text{if } |x| \gg x_c(1,\alpha) \end{cases}$
with $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$
Levy-Gnedenko CLT</td><td>$F(x) = L_{q,\alpha}$, with same $|x| \to \infty$ asymptotic behavior
$L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\, \alpha}(x) \sim C^*_{q,\alpha} / |x|^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \quad (\textit{intermediate regime}) \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\, 2}(x) \sim C^L_{q,\alpha} / |x|^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \quad (\textit{distant regime}) \end{cases}$
S. Umarov, C. T., M. Gell-Mann and S. Steinberg
J Math Phys 51, 033502 (2010)</td></tr>
</table>

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg, Milan J Math **76,** 307 (2008)

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\, dx = \int_{-\infty}^{\infty} e_q^{ix\xi[f(x)]^{q-1}} f(x)\, dx$$

$(q \geq 1)$

(nonlinear!)

For q<1 see K.P. Nelson and S. Umarov, Physica A **389**, 2157 (2010)

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Note on a q-modified central limit theorem

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Abstract. A q-modified version of the central limit theorem due to Umarov *et al* affirms that q-Gaussians are attractors under addition and rescaling of certain classes of strongly correlated random variables. The proof of this theorem rests on a nonlinear q-modified Fourier transform. By exhibiting an invariance property we show that this Fourier transform does not have an inverse. As a consequence, the theorem falls short of achieving its stated goal.

J. Stat. Mech. (2010) P10023

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q-Generalization of the inverse Fourier transform

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ON THE INVERSE q-FOURIER TRANSFORM:

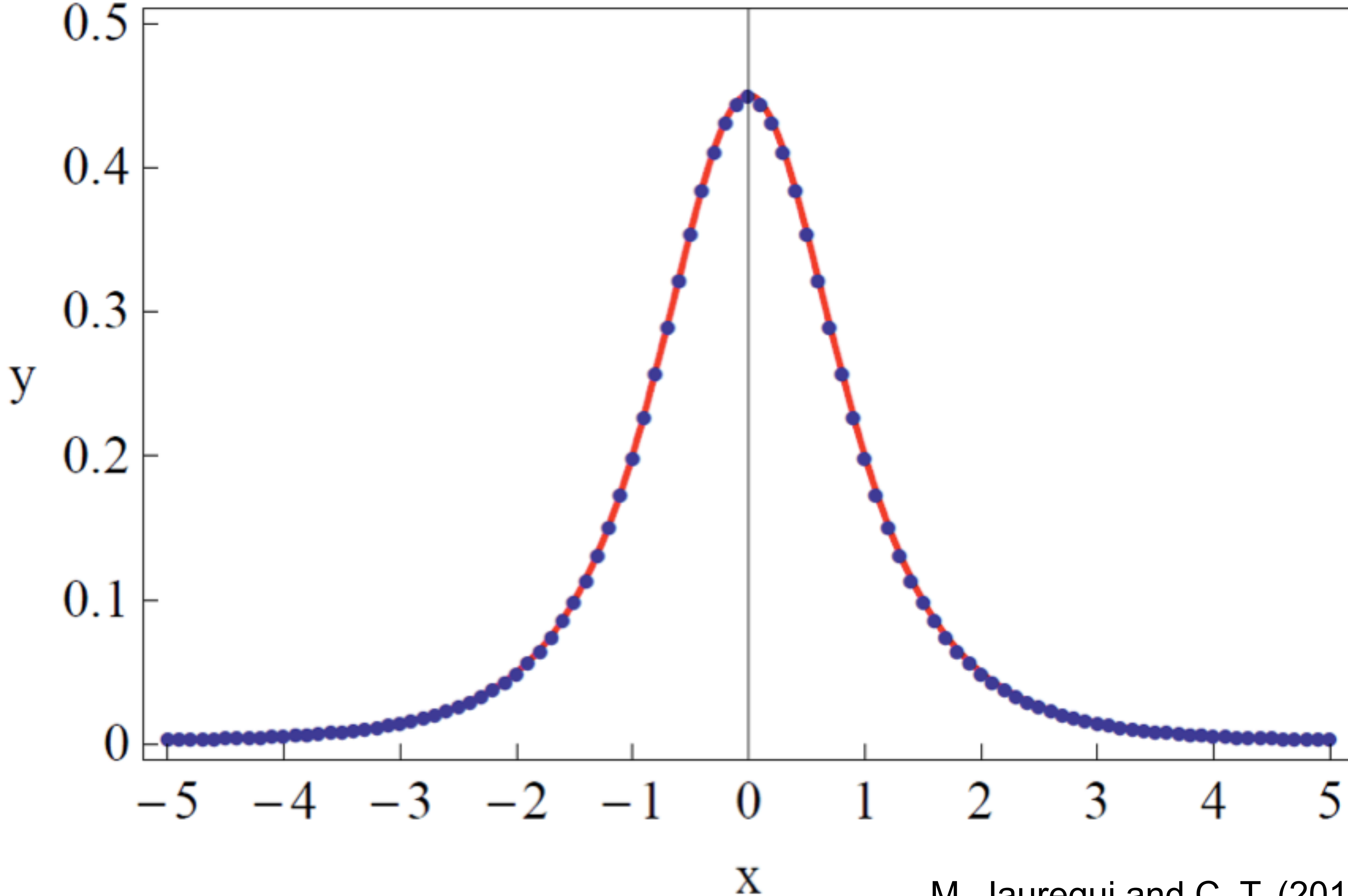
$$f(y) = \left[\frac{2-q}{2\pi} \int_{-\infty}^{+\infty} F_q[f(x+y)](\xi, y)\, d\xi \right]^{1/(2-q)} \qquad (1 \le q < 2)$$

Particular case q = 1:

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F[f(x+y)](\xi, y)\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F[f(x)](\xi)\, e^{-i\xi y}\, d\xi$$

M. Jauregui and C. T. (2011)

3/2-Gaussian



M. Jauregui and C. T. (2011)

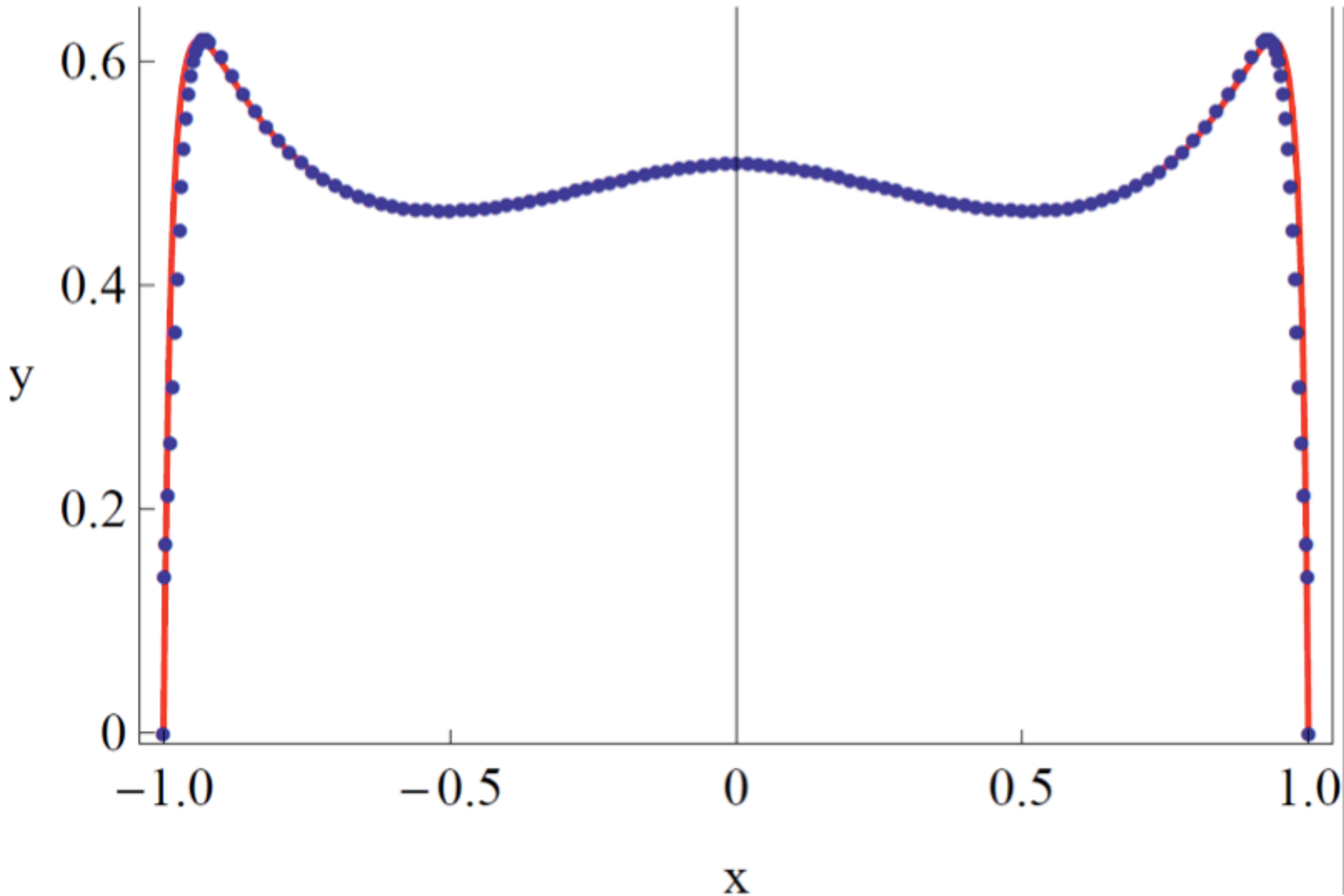
Hilhorst function

$$f_A(x) = \begin{cases} \dfrac{\left[\ |x|^{(q-2)/(q-1)} - A\right]^{1/(q-2)}}{C_q\ |x|^{1/(q-1)}\ \left\{1+(q-1)\left[\ |x|^{(q-2)/(q-1)} - A\right]^{2(q-1)/(q-2)}\right\}^{1/(q-1)}} & \text{if } 0 \le A < |x|^{(q-2)/(q-1)} \\ 0 & \text{if } 0 \le |x|^{(q-2)/(q-1)} \le A \\ \text{with} \quad \int_{-\infty}^{\infty} dx\, f_A(x) = 1 & \end{cases}$$

Particular case: $A = 0$

$$f_0(x) = \frac{1}{C_q\ [1+(q-1)x^2]^{1/(q-1)}}$$

Hilhorst function (*q*=5/4; *A*=1)



M. Jauregui and C. T. (2011)

Journal of Statistical Mechanics: Theory and Experiment
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q-moments remove the degeneracy associated with the inversion of the q-Fourier transform

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J. Stat. Mech. (2

STILL ANOTHER PATH TO INVERT THE *q*-FOURIER TRANSFORM

A. Plastino and M.C. Rocca, 1112.1985 [math-ph] and 1112.1986 [math-ph]

$$F(k,q) \equiv \left[H(q-1) - H(q-2) \right] \int_{-\infty}^{\infty} dx \; e_q^{ikx[f(x)]^{q-1}} f(x)$$

$$= \left[H(q-1) - H(q-2) \right] \int_{-\infty}^{\infty} dx \; \left\{ 1 + i(1-q)kx[f(x)]^{q-1} \right\}^{\frac{1}{1-q}} f(x)$$

where $H(t) \equiv$ Heaviside step function

It can be proved that the inverse is uniquely given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\lim_{\varepsilon \to +0} \int_1^2 F(k,q) \; \delta(q-1-\varepsilon) \, dq \right] e^{-ikx} dk$$

CONSERVATIVE MC MILLAN MAP:

$$x_{n+1} = y_n$$

$$y_{n+1} = -x_n + 2\mu \frac{y_n}{1 + y_n^2} + \varepsilon y_n$$

$\mu \neq 0 \Leftrightarrow$ nonlinear dynamics

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2012), in press

$(\mu, \varepsilon) = (1.6, 1.2)$ $(\lambda_{max} \approx 0.05)$

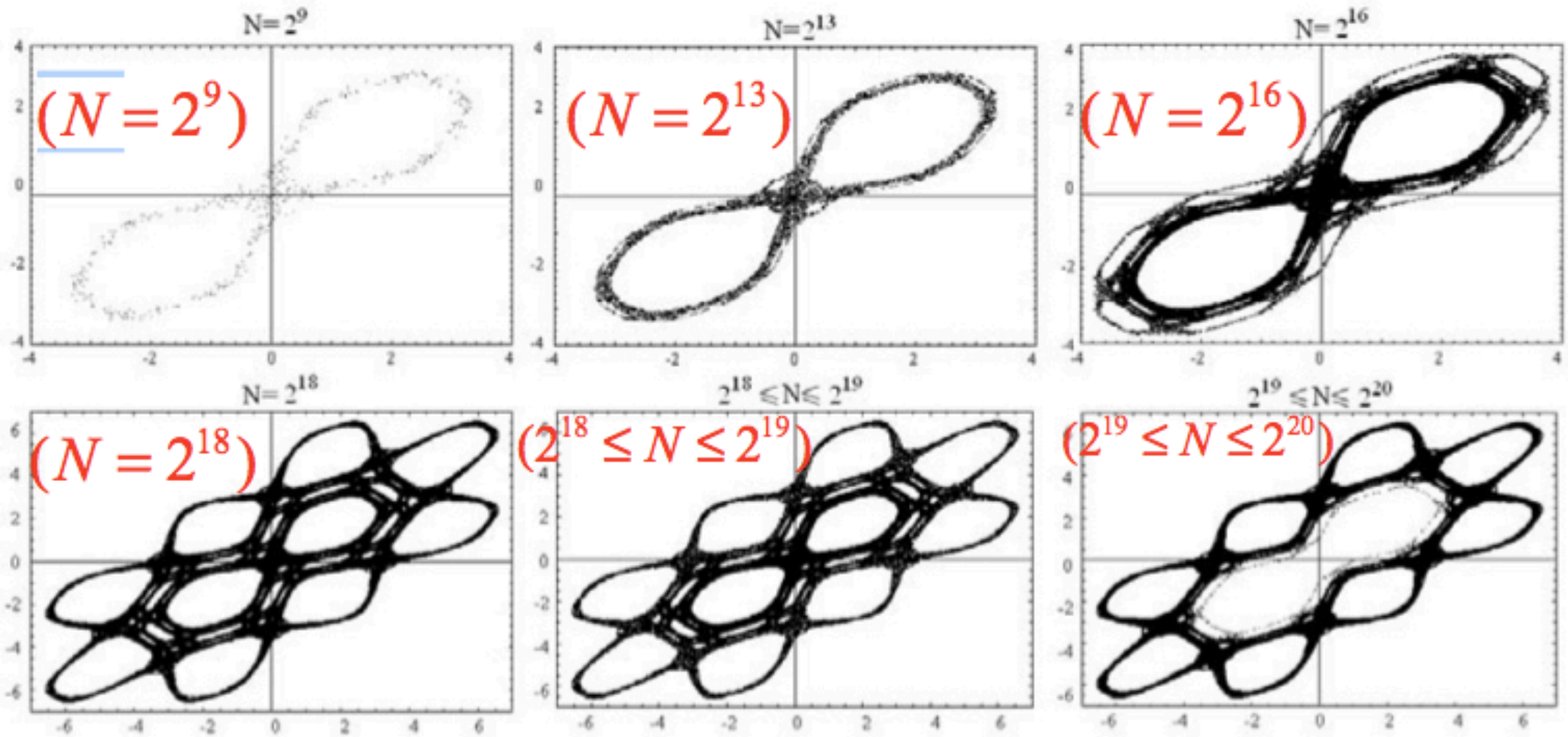
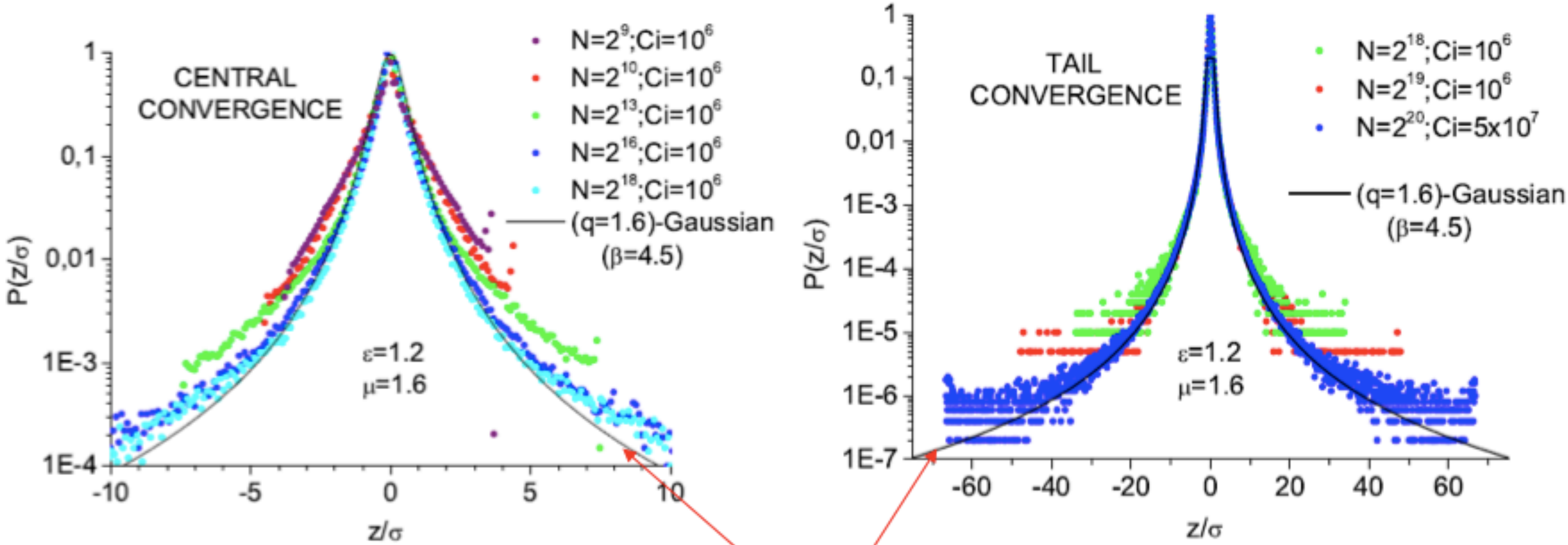


FIG. 10. Structure of phase space plot of Mc. Millan perturbed map for parameter values $\mu = 1.6$ and $\epsilon = 1.2$, starting form a randomly chosen initial condition in a square $(0, 10^{-6}) \times (0, 10^{-6})$, and for $i = 1 \ldots N$ $(N = 2^{10}, 2^{13}, N^{16}, N^{18})$ iterates.

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2012), in press



$$p \propto e_q^{-\beta(z/\sigma)^2}$$

with $(q,\beta) = (1.6, 4.5)$

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2012), in press

Non-Maxwellian behavior and quasistationary regimes near the modal solutions of the Fermi-Pasta-Ulam β system

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(Received 9 September 2011; published 30 March 2012)

In a recent paper [M. Leo, R. A. Leo, and P. Tempesta, J. Stat. Mech. (2011) P03003], it has been shown that the $\pi/2$-mode exact nonlinear solution of the Fermi-Pasta-Ulam β system, with periodic boundary conditions, admits two energy density thresholds. For values of the energy density ϵ below or above these thresholds, the solution is stable. Between them, the behavior of the solution is unstable, first recurrent and then chaotic. In this paper, we study the chaotic behavior between the two thresholds from a statistical point of view, by analyzing the distribution function of a dynamical variable that is zero when the solution is stable and fluctuates around zero when it is unstable. For mesoscopic systems clear numerical evidence emerges that near the second threshold, in a large range of the energy density, the numerical distribution is fitted accurately with a q-Gaussian distribution for very large integration times, suggesting the existence of a quasistationary state possessing a weakly chaotic behavior. A normal distribution is recovered in the thermodynamic limit.

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i{}^2 + \frac{1}{2}\sum_{i=1}^{N}(x_{i+1}-x_i)^2 + \frac{\beta}{4}\sum_{i=1}^{N}(x_{i+1}-x_i)^4$$

with $x_{N+1} = x_1$ and $\beta \geqslant 0$. All quantities are dimensionless.

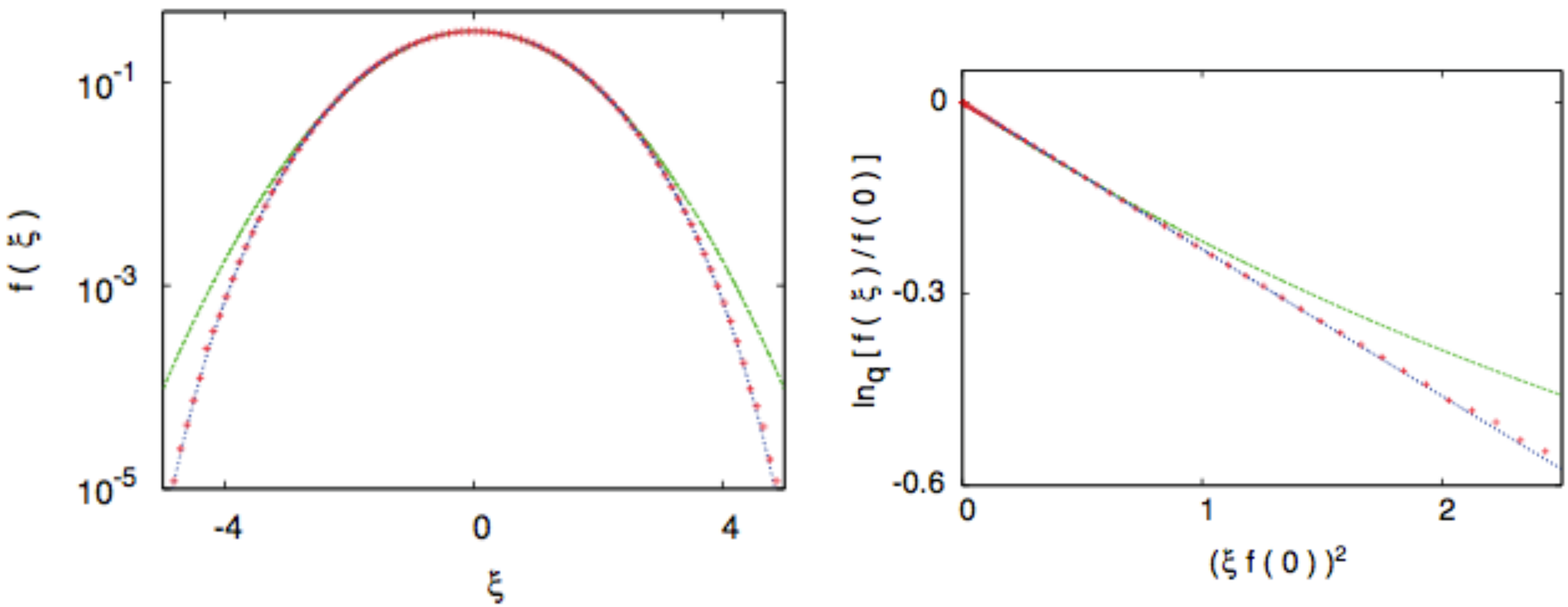
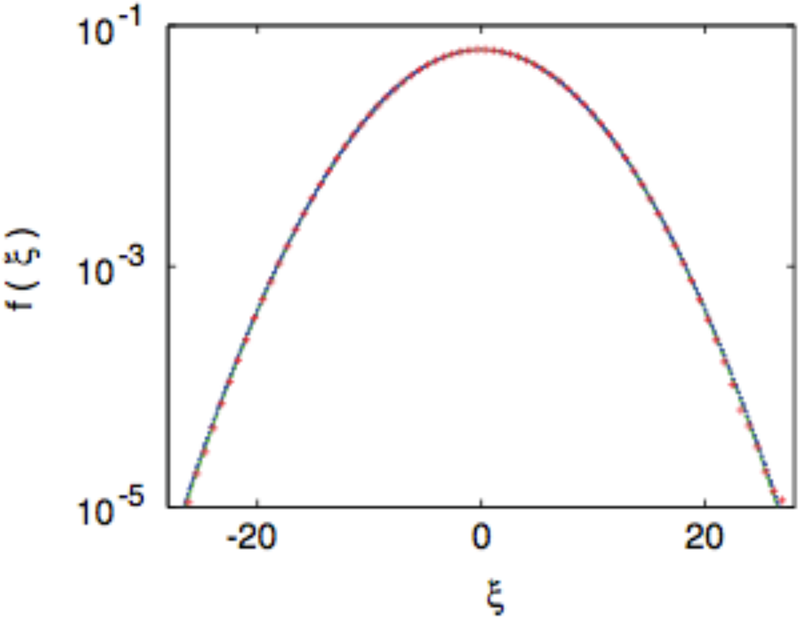


FIG. 2. (Color online) Log-linear representation of the numerical distribution $f(\xi)$ (red pluses), Gaussian distribution (green dashed upper curve), and q-Gaussian distribution (blue dotted curve) with $q = 0.9225 \pm 0.0007$, for $N = 16$ and $\epsilon = \epsilon_2 = 0.135\,266\,73$. We get $\chi_g^2 = 2.78 \times 10^{-4}$ and $\chi_{qg}^2 = 5.72 \times 10^{-7}$.

M. Leo, R.A. Leo, P. Tempesta and C. T.
Phys Rev E **85**, 031149 (2012)



M. Leo, R.A. Leo, P. Tempesta and C. T.
Phys Rev E **85**, 031149 (2012)

FIG. 7. (Color online) log-linear representation of the three distributions for $N = 256$ and $\epsilon = 0.1477$: Gaussian and q-Gaussian distributions overlap with the numerical one (red crosses). We get $\chi_g^2 = 2.42 \times 10^{-7}$ and $\chi_{qg}^2 = 2.21 \times 10^{-7}$.

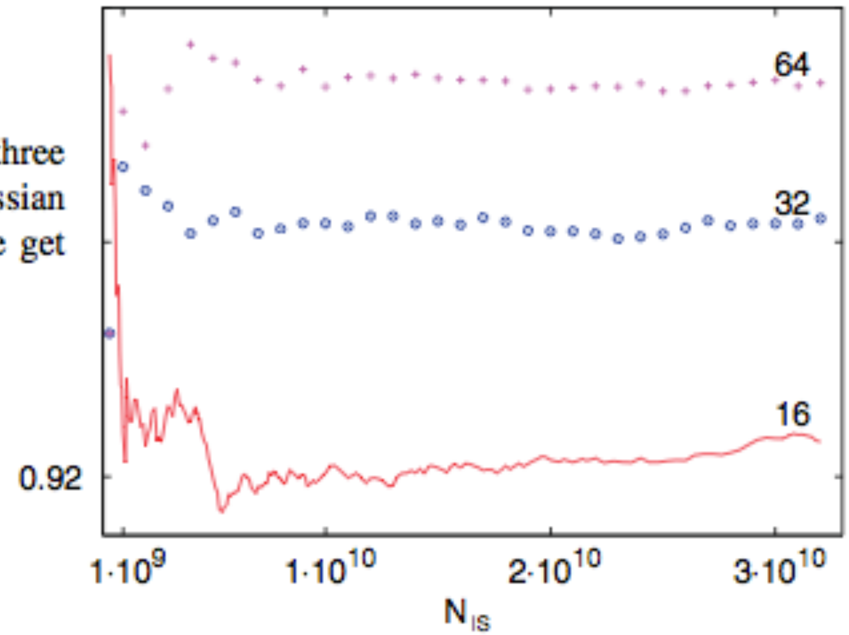


FIG. 9. (Color online) q vs the number of integration steps $N_{\rm IS}$ for $\epsilon = 0.095$, $N = 16$ (red solid curve), $N = 32$ (blue ○), and $N = 64$ (purple pluses). The integration time is $t = 0.02N_{\rm IS}$. The asymptotic values of q are respectively $q_{16} = 0.9255 \pm 0.0003$, $q_{32} = 0.9640 \pm 0.0004$, $q_{64} = 0.9871 \pm 0.0004$.

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Quantum potentials with q-Gaussian ground states

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ABSTRACT

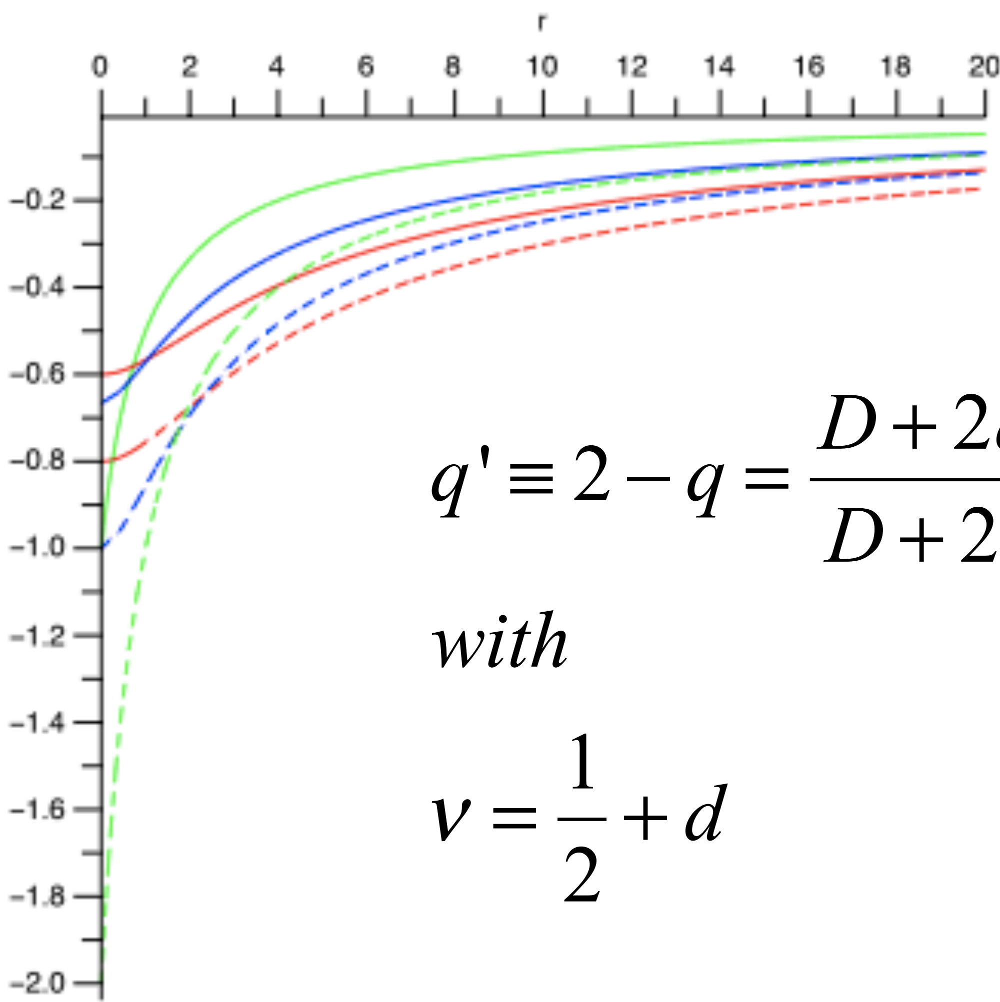
We determine families of spherically symmetrical D-dimensional quantum potential functions $V(r)$ having ground-state wavefunctions that exhibit, either in configuration space or in momentum space, the form of an isotropic q-Gaussian. These wavefunctions admit a maximum-entropy description in terms of S_q power-law entropies. We show that the potentials with a ground state of the q-Gaussian form in momentum space admit the Coulomb potential $-1/r$ as a particular instance. Furthermore, all these potentials behave asymptotically as the Coulomb potential for large r for all values of the parameter q such that $0 < q < 1$.

Table 1
Forms of the potential function $V_\nu(r)$ and corresponding values of the parameter q, as a function of the space dimension D, for different half-integer values of the parameter ν.

ν	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
q	$\frac{D}{D+1}$	$\frac{D+2}{D+3}$	$\frac{D+4}{D+5}$	$\frac{D+6}{D+7}$
$V_\nu(r)$	$-\frac{1}{r}$	$-\frac{1}{2}\left[\frac{1+D}{1+r}\right]$	$-\frac{1}{2}\left[\frac{(3+D)(r+1)}{r^2+3r+3}\right]$	$-\frac{1}{2}\left[\frac{(5+D)(r^2+3r+3)}{r^3+6r^2+15r+15}\right]$

Table 1
Forms of the potential function $V_\nu(r)$ and corresponding values of the parameter q, as a function of the space dimension D, for different half-integer values of the parameter ν.

ν	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
q	$\frac{D}{D+1}$	$\frac{D+2}{D+3}$	$\frac{D+4}{D+5}$	$\frac{D+6}{D+7}$
$V_\nu(r)$	$-\frac{1}{r}$	$-\frac{1}{2}\left[\frac{1+D}{1+r}\right]$	$-\frac{1}{2}\left[\frac{(3+D)(r+1)}{r^2+3r+3}\right]$	$-\frac{1}{2}\left[\frac{(5+D)(r^2+3r+3)}{r^3+6r^2+15r+15}\right]$



$$q' \equiv 2 - q = \frac{D+2d+2}{D+2d+1}$$

with

$$\nu = \frac{1}{2} + d$$

EXTREME EVENTS IN FINANCIAL RECORDS:

epl A Letters Journal Exploring the Frontiers of Physics

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Universal behaviour of interoccurrence times between losses in financial markets: An analytical description

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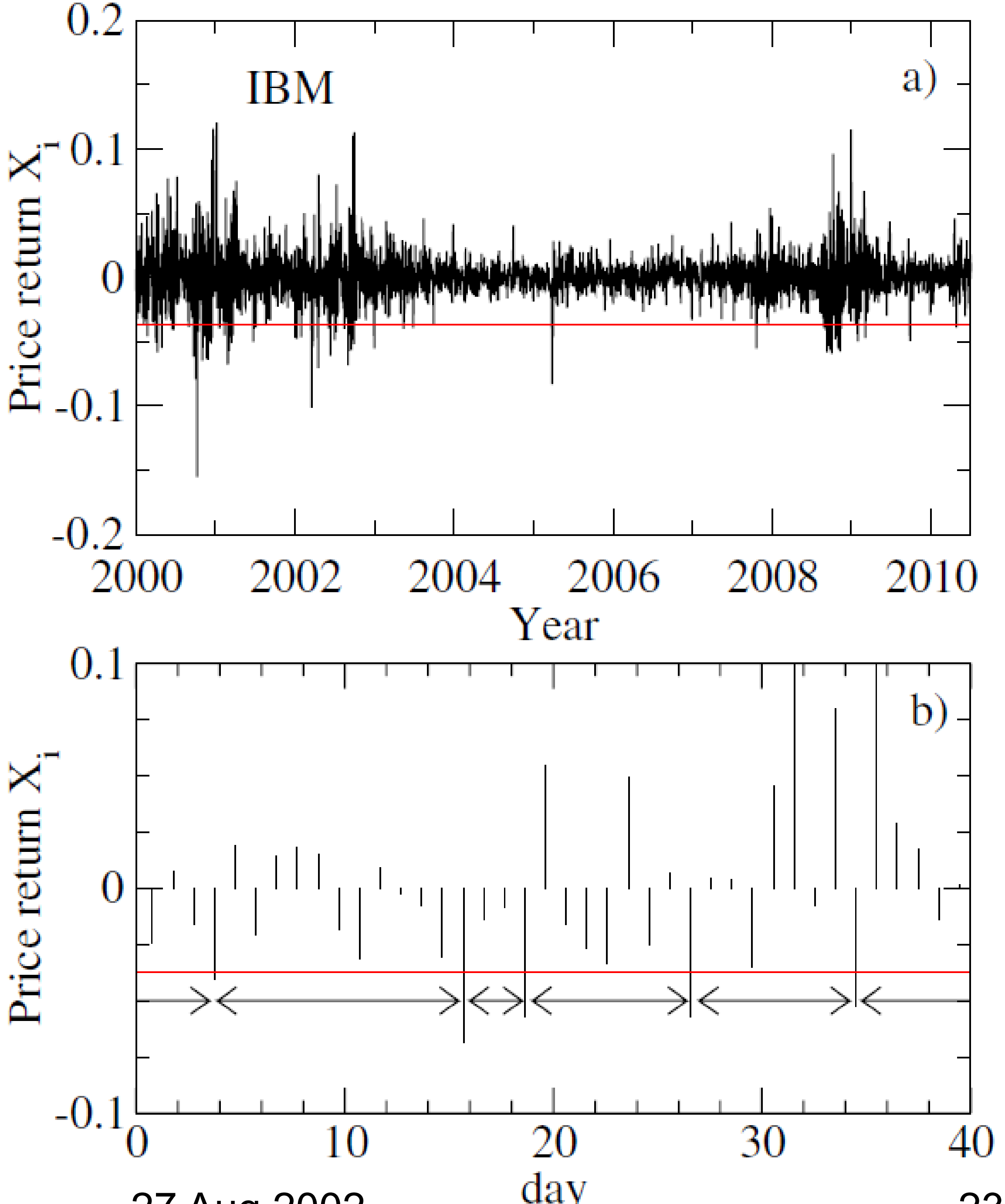
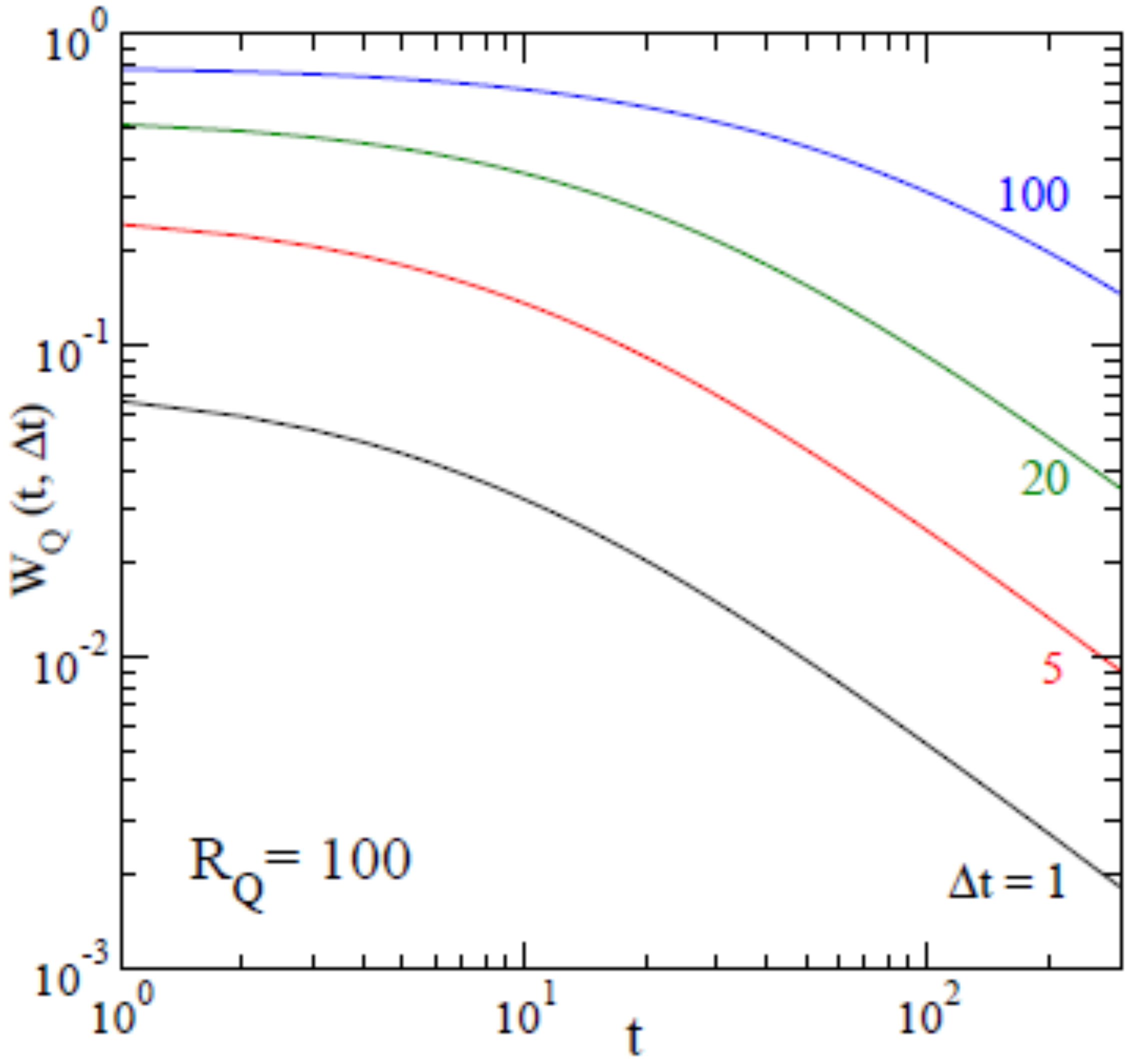


27 Aug 2002 23 Oct 2002

Risk function



$$W(t;\Delta t) \equiv \frac{\int_t^{t+\Delta t} P_Q(r)dr}{\int_t^{\infty} P_Q(r)dr} = 1 - \left[1 + \frac{\beta(q-1)\Delta t}{1+\beta(q-1)t}\right]^{\frac{q-2}{q-1}} = 1 - \frac{e_{1/(2-q)}^{-[\beta(2-q)](t+\Delta t)}}{e_{1/(2-q)}^{-[\beta(2-q)]\,t)}}$$

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Research on the Key Technology of Image Guided Surgery

***Abstract**. It research on the key technology on IGS (image-guided surgery). It proposes medical image segmentation based on PCNN and the virtual endoscopic scenes real-time rendering method based on GPU parallel computing technology, which improves the display quality of IGS's virtual scene and real-time rendering speed. These methods are very important for IGS's applications.*

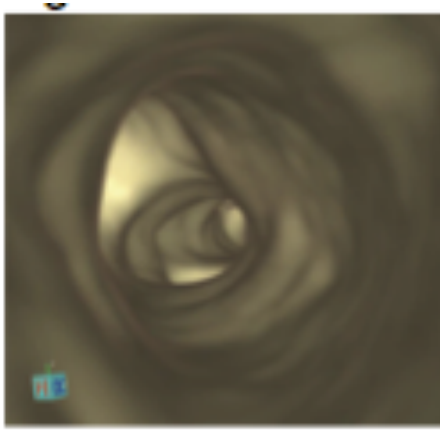

Fig.12. Bronchus

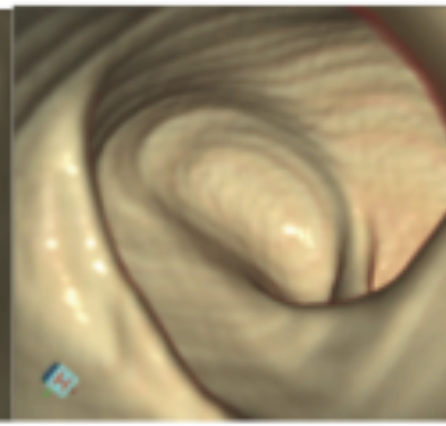

Fig.13.Colon

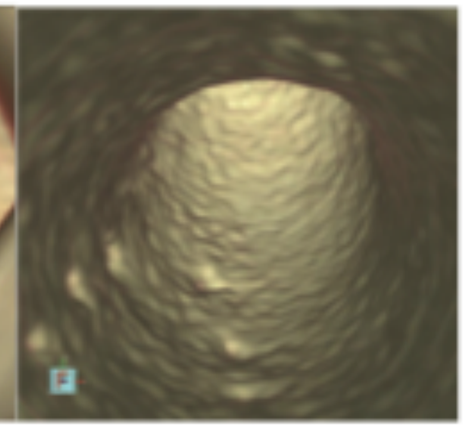

Fig.14.blood vessel

Table 1. Speed Comparison between Traditional Algorithms and Present Algorithm(uint: fps)

CT Image	Bronchus	Colon	blood vessel
Image Extent	512*51*217	512*512*252	512*512*355
Ray casting	9.8	6.4	3.5
Our algorithm	36.2	35.7	32.1

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Restricted random walk model as a new testing ground for the applicability of q-statistics

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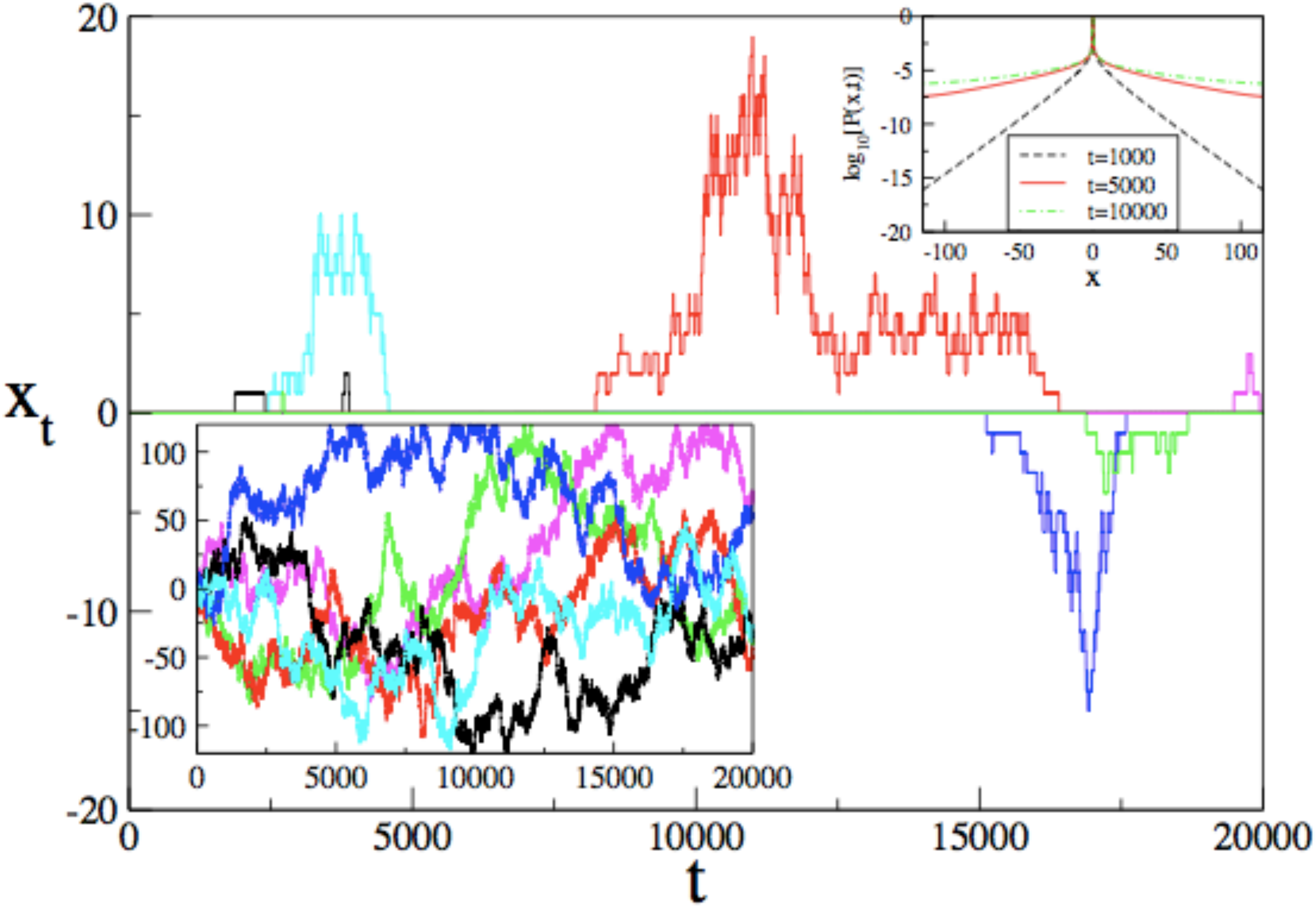


Fig. 1: Six representative trajectories for $L = 120$. Main panel: The restricted RW model ($p = 5 \cdot 10^{-6}$). Lower Inset: The standard RW model ($p = 1$). Non-ergodic behavior of the restricted RW model can easily be seen. Upper Inset: The time evolution of $P(x,t)$ at three different t values ($t = 1000$, 5000 and 10000 from bottom to top).

U. Tirnakli, H.J. Jensen and C. T., EPL **96**, 40008 (2012)

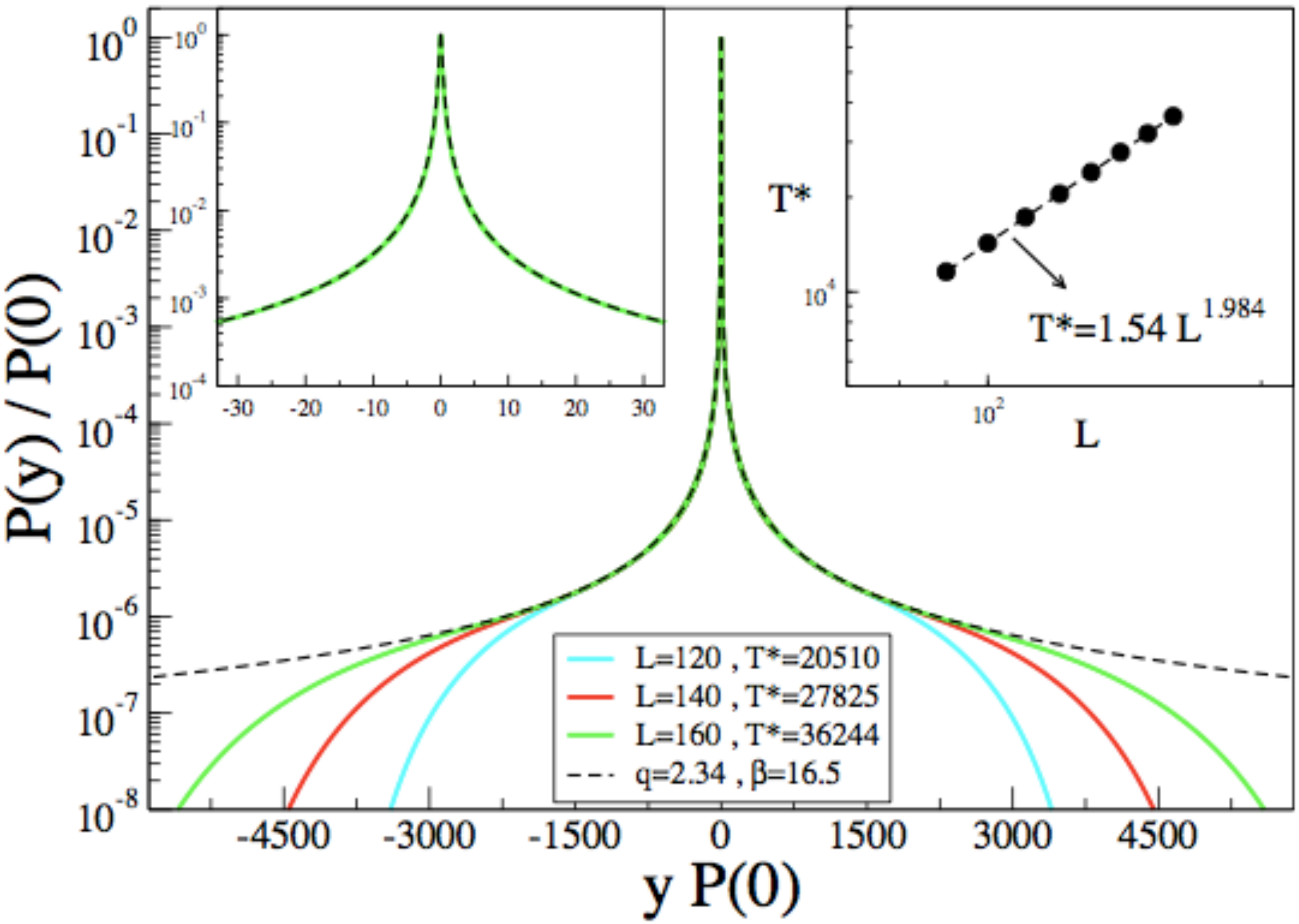


Fig. 3: The case $a = 1$ for $L = 120$, 140 and 160 with $p = 5 \cdot 10^{-6}$. The main panel shows the probability function $P(y, T^*)$. The center of the function is shown in detail in the left inset. The time T^* is chosen to optimize the fit to the q-Gaussian. The scaling of T^* is given in the right inset.

U. Tirnakli, H.J. Jensen and C. T., EPL **96**, 40008 (2012)

CORRELATIONS IN COUPLED LOGISTIC MAPS AT THE EDGE OF CHAOS IN THE PRESENCE OF GLOBAL NOISE

We consider a linear chain of N coupled maps with periodic boundary conditions in a noisy environment:

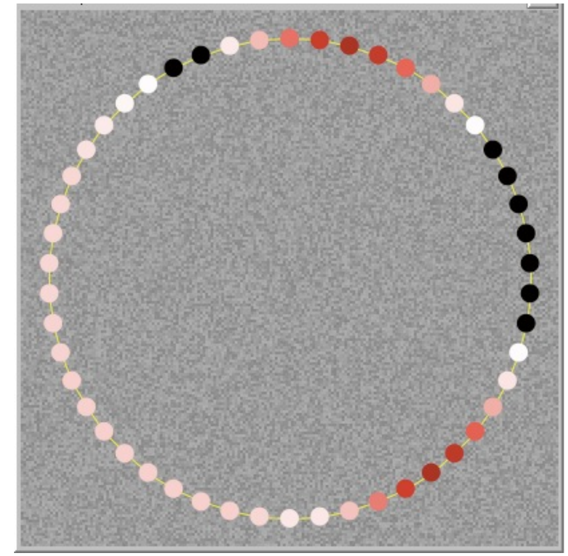
$$x_{t+1}^{i} = (1-\varepsilon) f(x_t^i) + \frac{\varepsilon}{2}[f(x_t^{i-1}) + f(x_t^{i+1})] + \sigma_t$$

$\boxed{\sigma_t \in [0, \sigma_{\max}]}$ additive noise

with $\varepsilon \in [0,1]$ coupling strength

and $f(x_t^i) = 1 - \mu (x_t^i)^2$ $\quad \mu \in [0,2]$ $\quad$ i-th logistic map ($i = 1...N$)

[zero noise: N.B. Ouchi and K. Kaneko, Chaos **10**, 359 (2000)]



$\boxed{\text{edge of chaos: } \mu = 1.4011551}$

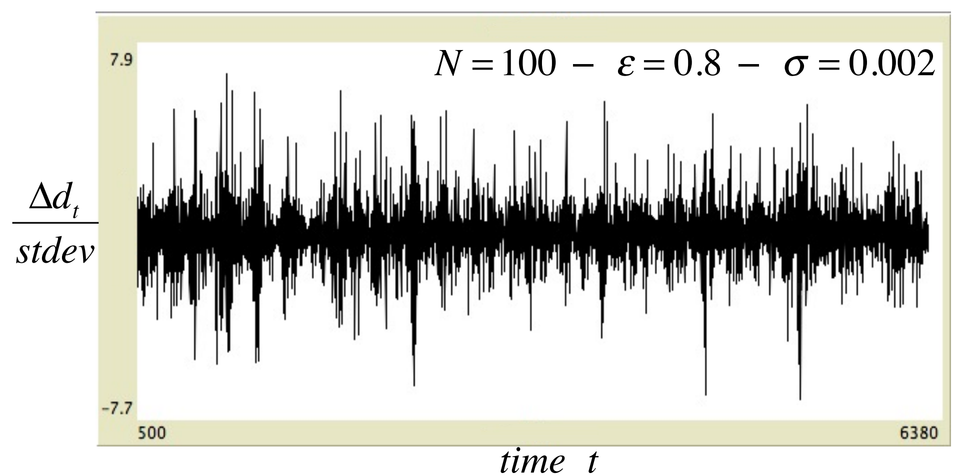
global parameter

$$d_t = \Sigma_{i=1}^{N} |x_t^i - < x_t^i > |$$

time returns

$$\Delta d_t = d_{t+\tau} - d_t$$

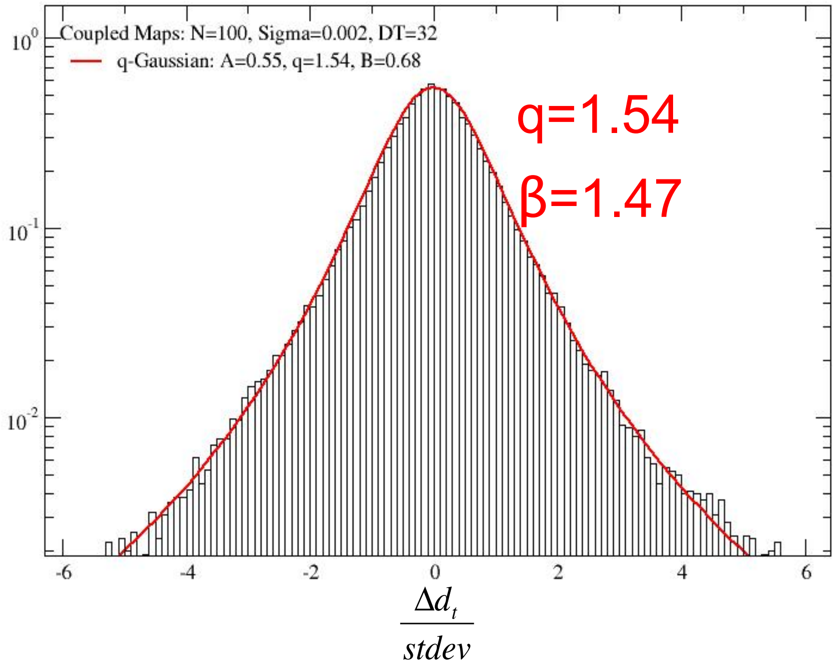
Intermittency in the normalized returns time-series



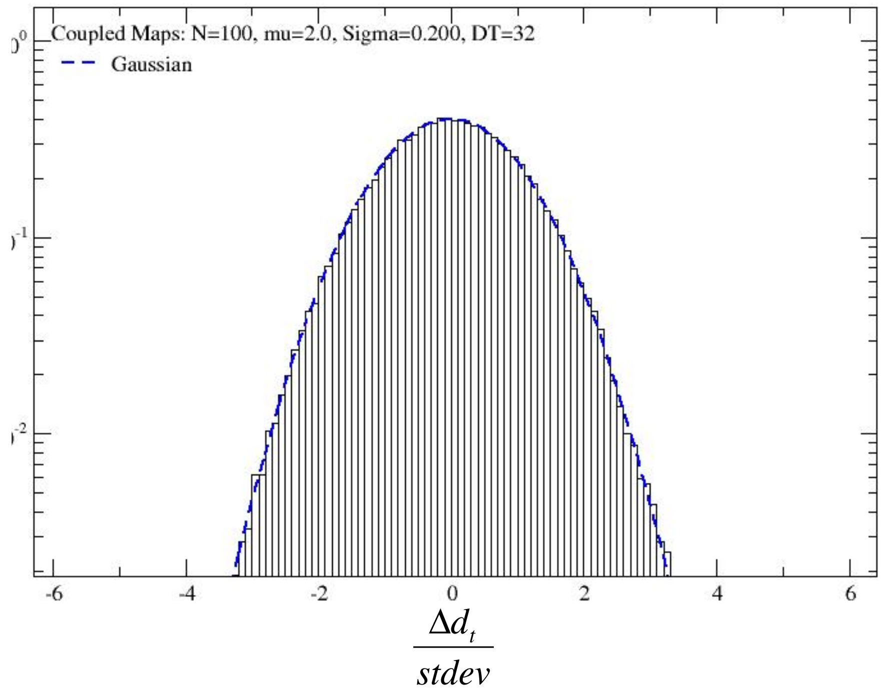
A. Pluchino, A. Rapisarda and C. T. (2012)

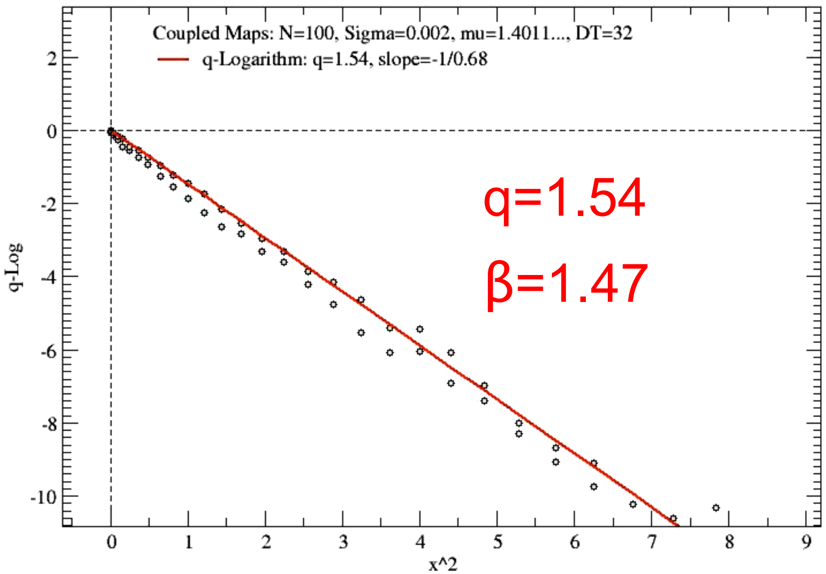
$$N = 100;\ \varepsilon = 0.8;\ \sigma_{\max} = 0.002;\ \tau = 32$$

Edge of chaos: $\mu = 1.4011551$

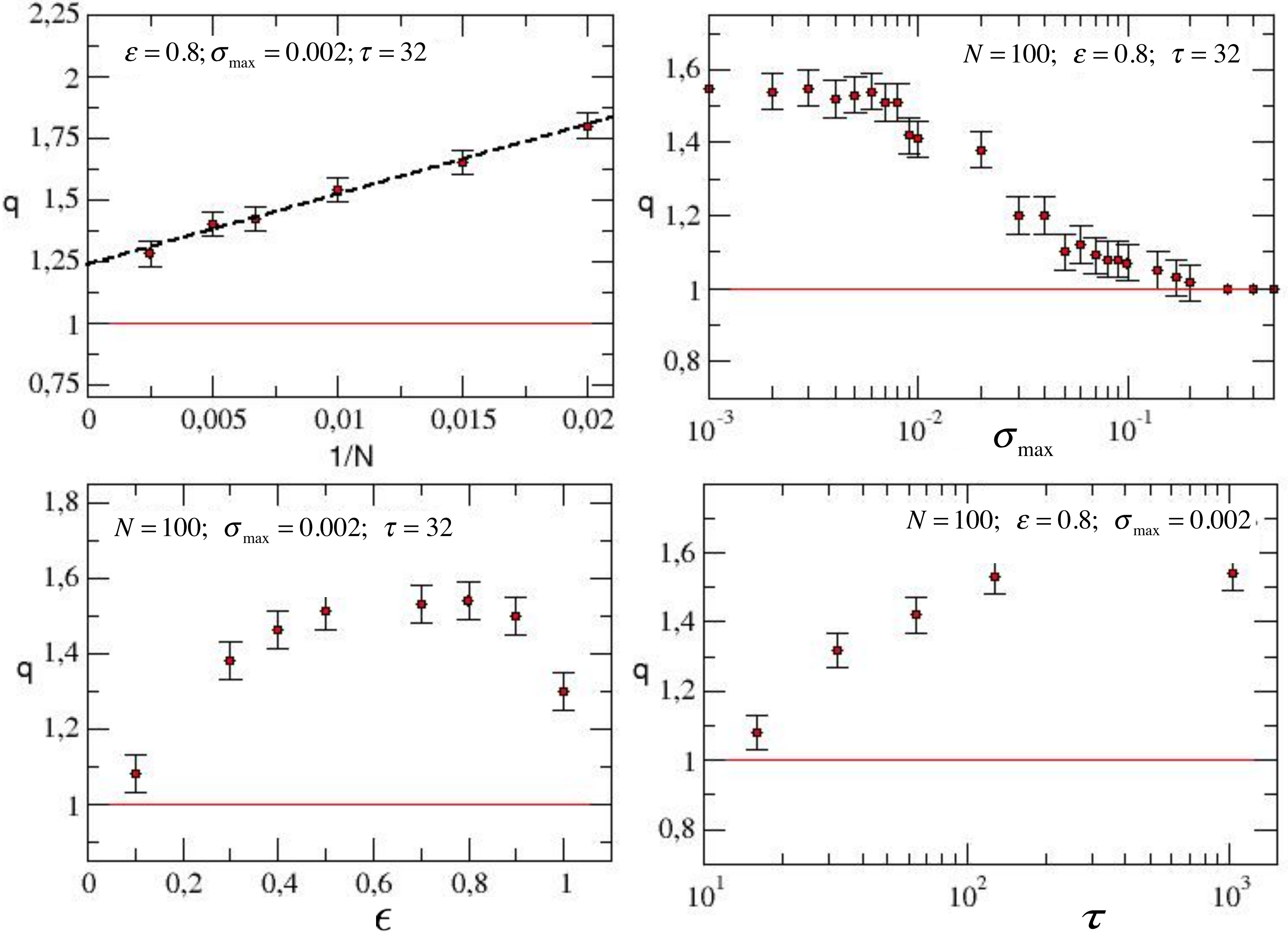


Chaotic Regime: $\mu = 2.0$





A. Pluchino, A. Rapisarda and C. T. (2012)



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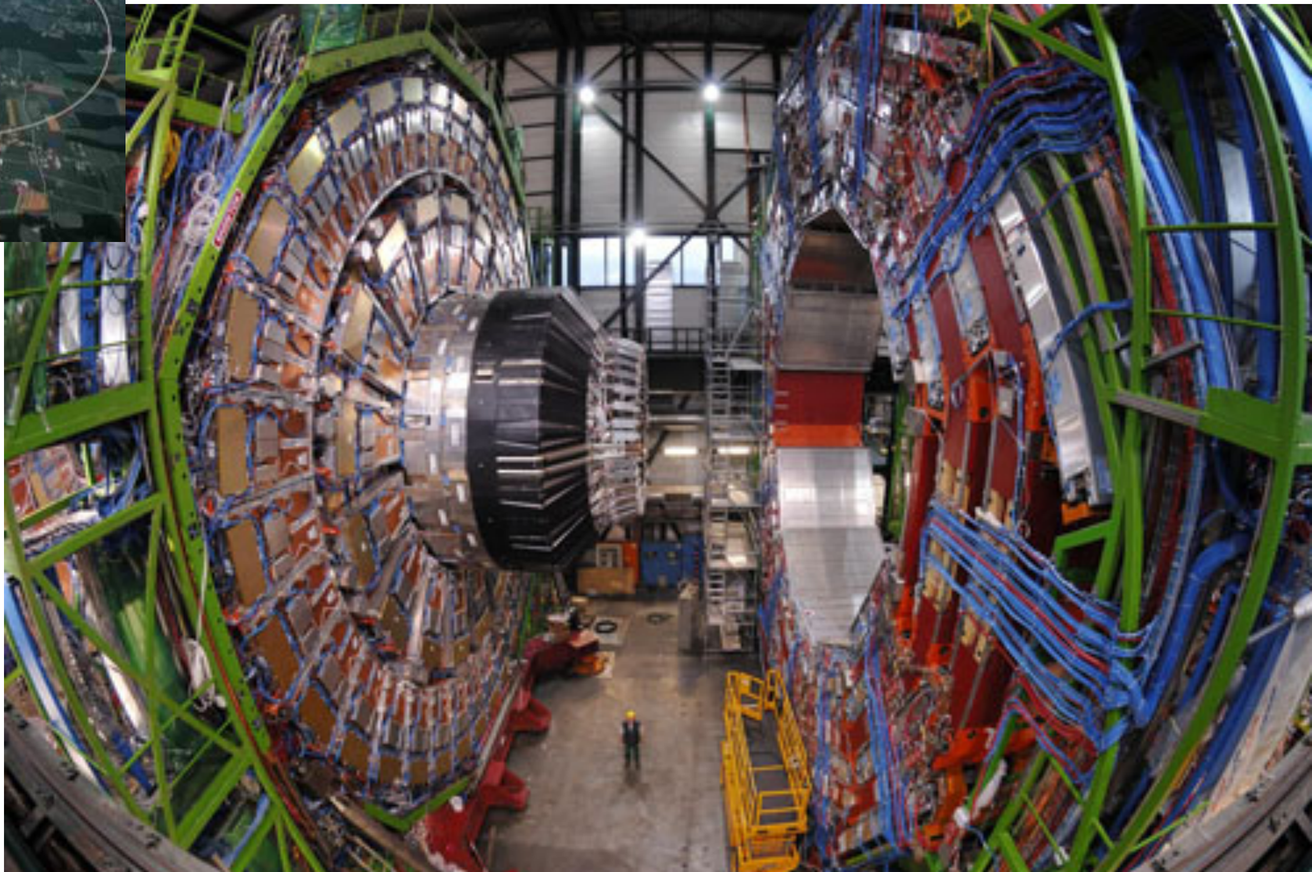
LHC (Large Hadron Collider)

CMS (Compact Muon Solenoid) detector

~ 2500 scientists/engineers from 183 institutions of 38 countries



Eur. Phys. J. C (2011) 71:1655
DOI 10.1140/epjc/s10052-011-1655-9

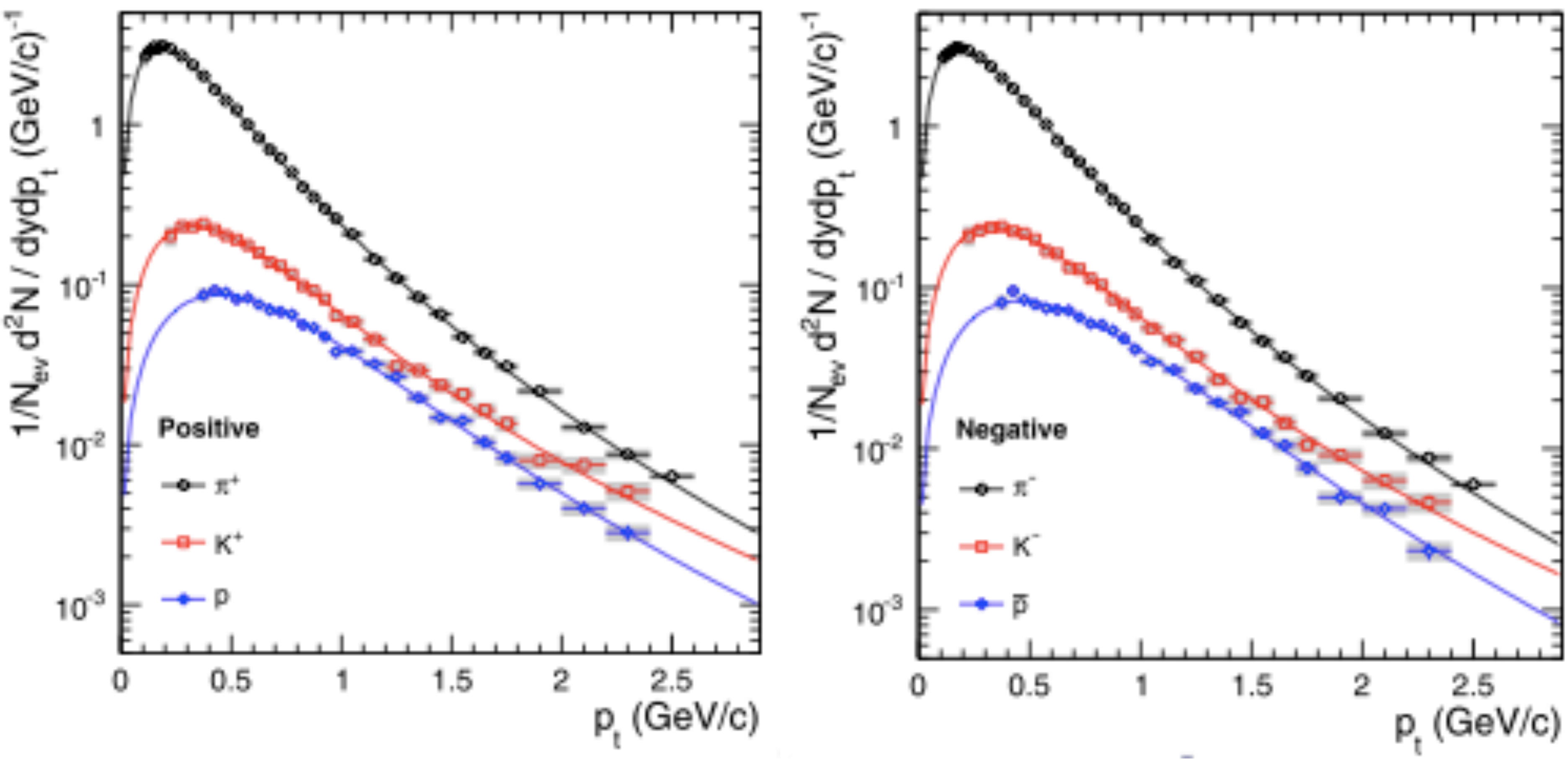
THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Experimental Physics

Production of pions, kaons and protons in pp collisions at $\sqrt{s} = 900$ GeV with ALICE at the LHC

The ALICE Collaboration

K. Aamodt[77], N. Abel[43], U. Abeysekara[75], A. Abrahantes Quintana[42], A. Abramyan[112], D. Adamová[85], M.M. Aggarwal[25], G. Aglieri Rinella[40], A.G. Agocs[18], S. Aguilar Salazar[63], Z. Ahammed[53], A. Ahmad[2], N. Ahmad[2], S.U. Ahn[38,b], R. Akimoto[99], A. Akindinov[66], D. Aleksandrov[68], B. Alessandro[104], R. Alfaro Molina[63], A. Alici[13], E. Almaráz Aviña[63], J. Alme[8], T. Alt[43,c], V. Altini[5], S. Altinpinar[31], C. Andrei[17], A. Andronic[31], G. Anelli[40], V. Angelov[43,c], C. Anson[27], T. Antičić[113], F. Antinori[40,d], S. Antinori[13], K. Antipin[36], D. Antończyk[36], P. Antonioli[14], A. Anzo[63], L. Aphecetche[71], H. Appelshäuser[36], S. Arcelli[13], R. Arceo[63], A. Arend[36], N. Armesto[91], R. Arnaldi[104], T. Aronsson[72], I.C. Arsene[77,e], A. Asryan[97], A. Augustinus[40], R. Averbeck[31], T.C. Awes[74], J. Äystö[49], M.D. Azmi[2], S. Bablok[8], M. Bach[35], A. Badalà[24], Y.W. Baek[38,b], S. Bagnasco[104], R. Bailhache[31,f], R. Bala[105], A. Baldisseri[88], A. Baldit[26], J. Bán[56], R. Barbera[23], G.G. Barnaföldi[18], L.S. Barnby[12], V. Barret[26], J. Bartke[29], F. Barile[5], M. Basile[13], V. Basmanov[93], N. Bastid[26], B. Bathen[70], G. Batigne[71], B. Batyunya[34], C. Baumann[70,f], I.G. Bearden[28], B. Becker[20,g], I. Belikov[98], R. Bellwied[33], E. Belmont-Moreno[63], A. Belogianni[4], L. Benhabib[71], S. Beole[103], I. Berceanu[17], A. Bercuci[31,h], E. Berdermann[31], Y. Berdnikov[39], L. Betev[40], A. Bhasin[48], A.K. Bhati[25], L. Bianchi[103], N. Bianchi[37], C. Bianchin[78], J. Bielčík[80], J. Bielčíková[85], A. Bilandzic[3], L. Bimbot[76], E. Biolcati[103], A. Blanc[26], F. Blanco[23,i], F. Blanco[61], D. Blau[68], C. Blume[36], M. Boccioli[40], N. Bock[27], A. Bogdanov[67], H. Bøggild[28], M. Bogolyubsky[82], J. Bohm[95], L. Boldizsár[18], M. Bombara[55], C. Bombonati[78,k], M. Bondila[49], H. Borel[88], A. Borisov[50], C. Bortolin[78,ao], S. Bose[52], L. Bosisio[100], F. Bossú[103], M. Botje[3], S. Böttger[43], G. Bourdaud[71], B. Boyer[76], M. Braun[97], P. Braun-Munzinger[31,32,c], L. Bravina[77], M. Bregant[100,l], T. Breitner[43], G. Bruckner[40], R. Brun[40], E. Bruna[72], G.E. Bruno[5], D. Budnikov[93], H. Buesching[36], P. Buncic[40], O. Busch[44], Z. Buthelezi[22], D. Caffarri[78], X. Cai[111], H. Caines[72], E. Calvo[38], E. Camacho[64], P. Camerini[100], M. Campbell[40], V. Canoa Roman[40]



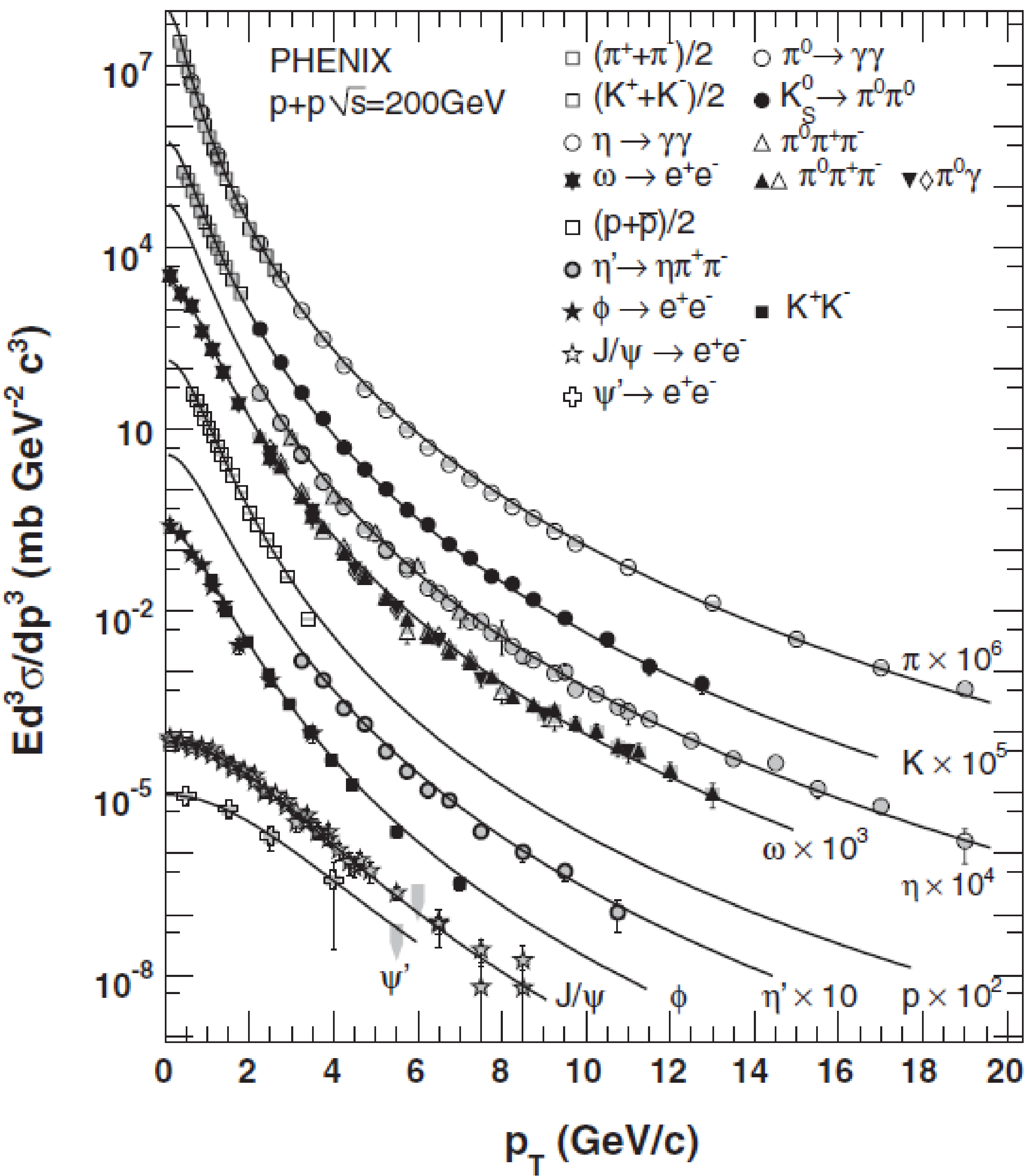
$1/N_{ev}\, d^2N / dydp_t\, (GeV/c)^{-1}$
1
10^{-1}
10^{-2}
10^{-3}
0
0.5
1
1.5
2
2.5
p_t (GeV/c)
Positive
π^+
K^+
p
Negative
π^-
K^-
$\bar{p}$

PHYSICAL REVIEW D **83,** 052004 (2011)

Measurement of neutral mesons in $p + p$ collisions at $\sqrt{s} = 200$ GeV and scaling properties of hadron production

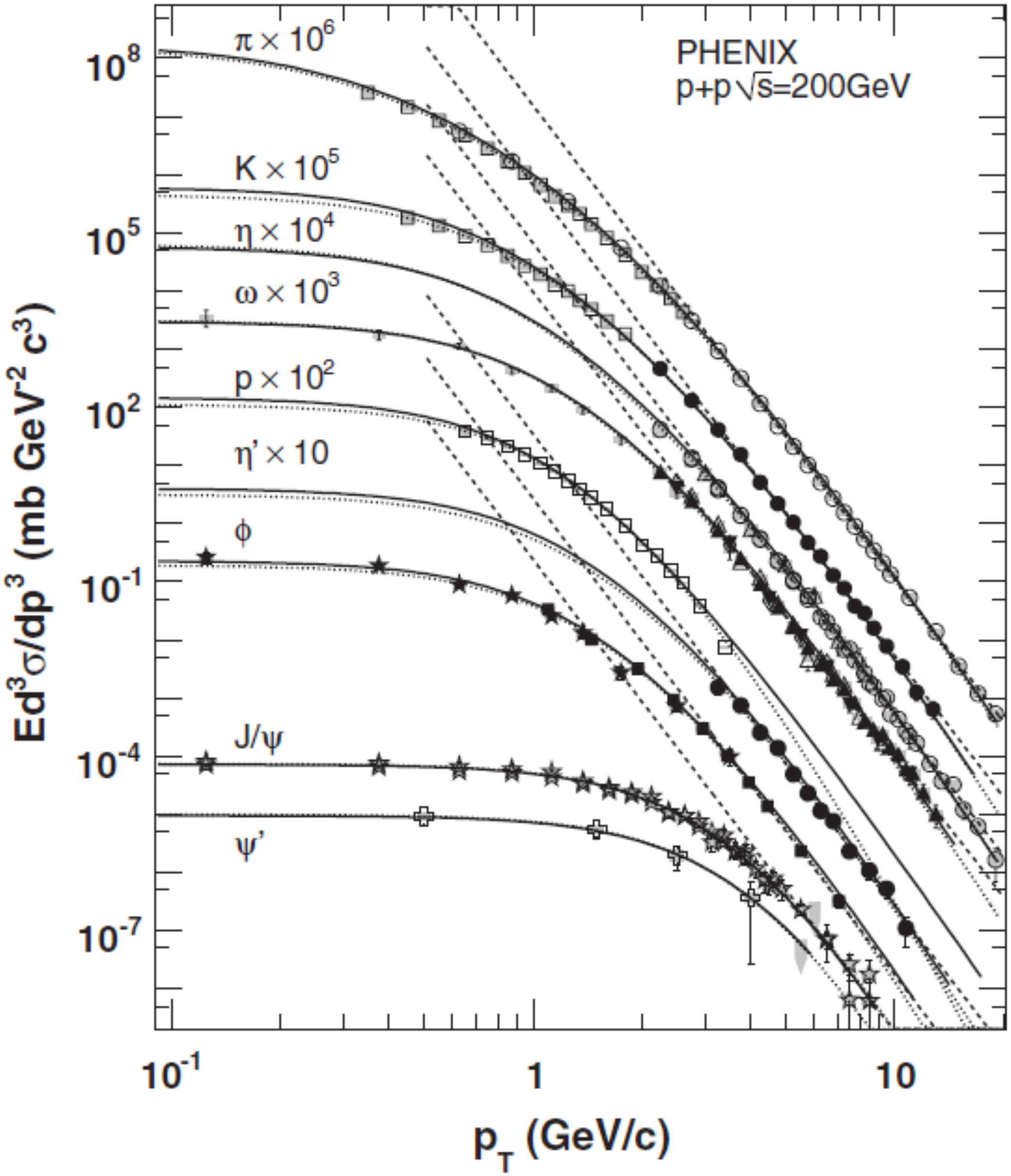
A. Adare,[11] S. Afanasiev,[25] C. Aidala,[12,36] N. N. Ajitanand,[53] Y. Akiba,[47,48] H. Al-Bataineh,[42] J. Alexander,[53] K. Aoki,[30,47] L. Aphecetche,[55] R. Armendariz,[42] S. H. Aronson,[6] J. Asai,[47,48] E. T. Atomssa,[31] R. Averbeck,[54] T. C. Awes,[43] B. Azmoun,[6] V. Babintsev,[21] M. Bai,[5] G. Baksay,[17] L. Baksay,[17] A. Baldisseri,[14] K. N. Barish,[7] P. D. Barnes,[33] B. Bassalleck,[41] A. T. Basye,[1] S. Bathe,[7] S. Batsouli,[43] V. Baublis,[46] C. Baumann,[37] A. Bazilevsky,[6] S. Belikov,[6,*] R. Bennett,[54] A. Berdnikov,[50] Y. Berdnikov,[50] A. A. Bickley,[11] J. G. Boissevain,[33] H. Borel,[14] K. Boyle,[54] M. L. Brooks,[33] H. Buesching,[6] V. Bumazhnov,[21] G. Bunce,[6,48] S. Butsyk,[33,54] C. M. Camacho,[33] S. Campbell,[54] B. S. Chang,[62] W. C. Chang,[2] J.-L. Charvet,[14] S. Chernichenko,[21] J. Chiba,[26] C. Y. Chi,[12] M. Chiu,[22] I. J. Choi,[62] R. K. Choudhury,[4] T. Chujo,[58,59] P. Chung,[53] A. Churyn,[21] V. Cianciolo,[43] Z. Citron,[54] C. R. Cleven,[19] B. A. Cole,[12] M. P. Comets,[44] P. Constantin,[33] M. Csanád,[16] T. Csörgő,[27] T. Dahms,[54] S. Dairaku,[30,47] K. Das,[18] G. David,[6] M. B. Deaton,[1] K. Dehmelt,[17] H. Delagrange,[55] A. Denisov,[21] D. d'Enterria,[12,31] A. Deshpande,[48,54] E. J. Desmond,[6] O. Dietzsch,[51] A. Dion,[54] M. Donadelli,[51] O. Drapier,[31] A. Drees,[54] K. A. Drees,[5] A. K. Dubey,[61] A. Durum,[21] D. Dutta,[4] V. Dzhordzhadze,[7] Y. V. Efremenko,[43] J. Egdemir,[54] F. Ellinghaus,[11] W. S. Emam,[7] T. Engelmore,[12] A. Enokizono,[32] H. En'yo,[47,48] S. Esumi,[58] K. O. Eyser,[7] B. Fadem,[38] D. E. Fields,[41,48] M. Finger, Jr.,[8,25] M. Finger,[8,25] F. Fleuret,[31] S. L. Fokin,[29] Z. Fraenkel,[61,*] J. E. Frantz,[54] A. Franz,[6] A. D. Frawley,[18] K. Fujiwara,[47] Y. Fukao,[30,47] T. Fusayasu,[40] S. Gadrat,[34] I. Garishvili,[56] A. Glenn,[11] H. Gong,[54] M. Gonin,[31] J. Gosset,[14] Y. Goto,[47,48] R. Granier de Cassagnac,[31] N. Grau,[12,24] S. V. Greene,[59] M. Grosse Perdekamp,[22,48] T. Gunji,[10] H. -Å. Gustafsson,[35,*] T. Hachiya,[20] A. Hadj Henni,[55] C. Haegemann,[41] J. S. Haggerty,[6] H. Hamagaki,[10] R. Han,[45] H. Harada,[20] E. P. Hartouni,[32] K. Haruna,[20] E. Haslum,[35] R. Hayano,[10] M. Heffner,[32] T. K. Hemmick,[54] T. Hester,[7] X. He,[19] H. Hiejima,[22] J. C. Hill,[24] R. Hobbs,[41] M. Hohlmann,[17] W. Holzmann,[53] K. Homma,[20] B. Hong,[28] T. Horaguchi,[10,47,57] D. Hornback,[56] S. Huang,[59] T. Ichihara,[47,48] R. Ichimiya,[47] H. Iinuma,[30,47] Y. Ikeda,[58] K. Imai,[30,47] J. Imrek,[15] M. Inaba,[58] Y. Inoue,[49,47] D. Isenhower,[1] L. Isenhower,[1] M. Ishihara,[47] T. Isobe,[10] M. Issah,[53] A. Isupov,[25] D. Ivanischev,[46] B. V. Jacak,[54,†] J. Jia,[12] J. Jin,[12] O. Jinnouchi,[48] B. M. Johnson,[6] K. S. Joo,[39] D. Jouan,[44] F. Kajihara,[10] S. Kametani,[10,47,60] N. Kamihara,[47,48] J. Kamin,[54] M. Kaneta,[48] J. H. Kang,[62] H. Kanou,[47,57] J. Kapustinsky,[33] D. Kawall,[36,48] A. V. Kazantsev,[29] T. Kempel,[24] A. Khanzadeev,[46] K. M. Kijima,[20] J. Kikuchi,[60] B. I. Kim,[28] D. H. Kim,[39] D. J. Kim,[62] E. Kim,[52] S. H. Kim,[62] E. Kinney,[11] K. Kiriluk,[11] Á. Kiss,[16] E. Kistenev,[6] A. Kiyomichi,[47] J. Klay,[32] C. Klein-Boesing,[37] L. Kochenda,[46] V. Kochetkov,[21] B. Komkov,[46] M. Konno,[58] J. Koster,[22] D. Kotchetkov,[7] A. Kozlov,[61] A. Král,[13] A. Kravitz,[12] J. Kubart,[8,23] G. J. Kunde,[33] N. Kurihara,[10]

PHENIX @ RHIC



PHENIX
p+p √s=200GeV
(π⁺+π⁻)/2
π⁰→ γγ
(K⁺+K⁻)/2
K⁰S → π⁰π⁰
η → γγ
π⁰π⁺π⁻
ω → e⁺e⁻
π⁰π⁺π⁻
π⁰γ
(p+p̄)/2
η'→ ηπ⁺π⁻
φ → e⁺e⁻
K⁺K⁻
J/ψ → e⁺e⁻
ψ'→ e⁺e⁻
π × 10⁶
K × 10⁵
ω × 10³
η × 10⁴
ψ'
J/ψ
φ
η' × 10
p × 10²
Ed³σ/dp³ (mb GeV⁻² c³)
p_T (GeV/c)

$q\ ;\ 1.10$



q ; 1.10

FIG. 13. The p_T spectra of various hadrons measured by PHENIX fitted to the power law fit (dashed lines) and Tsallis fit (solid lines). See text for more details.

Nonlinear Relativistic and Quantum Equations with a Common Type of Solution

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Generalizations of the three main equations of quantum physics, namely, the Schrödinger, Klein-Gordon, and Dirac equations, are proposed. Nonlinear terms, characterized by exponents depending on an index q, are considered in such a way that the standard linear equations are recovered in the limit $q \to 1$. Interestingly, these equations present a common, solitonlike, traveling solution, which is written in terms of the q-exponential function that naturally emerges within nonextensive statistical mechanics. In all cases, the well-known Einstein energy-momentum relation is preserved for arbitrary values of q.

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

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A generalized nonlinear Schrödinger equation: Classical field-theoretic approach

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PACS 11.10.Ef – Lagrangian and Hamiltonian approach
PACS 11.10.Lm – Nonlinear or nonlocal theories and models
PACS 02.30.Jr – Partial differential equations

Abstract – An exact classical field theory, for a recently proposed nonlinear generalization of the Schrödinger equation, is presented. In this generalization, a nonlinearity depending on an index q appears in the kinetic term, such that the free-particle linear Schrödinger equation is recovered in the limit $q \to 1$. It is shown that besides the usual $\Psi(\vec{x},t)$, a new field $\Phi(\vec{x},t)$ must be introduced, which becomes $\Psi^*(\vec{x},t)$ only when $q \to 1$. In analogy to the linear case, $\rho(\vec{x},t) = \frac{1}{2V}[\Psi(\vec{x},t)\Phi(\vec{x},t) + \Psi^*(\vec{x},t)\Phi^*(\vec{x},t)]$ is interpreted as the probability density for finding the particle at time t, in a given position $\vec{x}$ inside an arbitrary finite volume V, for any q. The possible physical consequences are discussed, and, in particular, the important property that the fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ do not interact with light.

ALGUNS RESULTADOS RECENTES

TEORIA DOS GRANDES DESVIOS

A ENTROPIA DO BURACO NEGRO

Physics Reports 478 (2009) 1–69

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The large deviation approach to statistical mechanics

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ABSTRACT

The theory of large deviations is concerned with the exponential decay of probabilities of large fluctuations in random systems. These probabilities are important in many fields of study, including statistics, finance, and engineering, as they often yield valuable information about the large fluctuations of a random system around its most probable state or trajectory. In the context of equilibrium statistical mechanics, the theory of large deviations provides exponential-order estimates of probabilities that refine and generalize Einstein's theory of fluctuations. This review explores this and other connections between large deviation theory and statistical mechanics, in an effort to show that the mathematical language of statistical mechanics is the language of large deviation theory. The first part of the review presents the basics of large deviation theory, and works out many of its classical applications related to sums of random variables and Markov processes. The second part goes through many problems and results of statistical mechanics, and shows how these can be formulated and derived within the context of large deviation theory. The problems and results treated cover a wide range of physical systems, including equilibrium many-particle systems, noise-perturbed dynamics, nonequilibrium systems, as well as multifractals, disordered systems, and chaotic systems. This review also covers

	PHYSICS (Statistical mechanics)	MATHEMATICS (Theory of large deviations)
$q = 1$ (quasi-independent)	$p_N \propto e^{-\beta H_N}$	$P_N(x) : e^{-N\, r(x)}$ $(N \to \infty)$
$q \neq 1$ (strongly correlated)	$p_N \propto e_q^{-\beta_q H_N}$	$P_N(x) : e_q^{-N\, r_q(x)}$ $(N \to \infty)$???

STANDARD THEORY - AN EXAMPLE

N independent coins $\rightarrow n$ heads $(n = 0,1,2,\ldots,N)$

$$\left[\textit{hence}\;\; N \rightarrow \infty \;\;\Rightarrow\;\; \textit{Gaussian}\right]$$

$P(N = 10; n < 2) >> P(N = 100; n < 20) >> P(N = 1000; n < 200)$

more precisely, $\lim_{N\rightarrow\infty} P(N;\; n/N < x) = 0 \quad (0 \leq x < 1/2)$

$$P(N;\; n/N < x) = \sum_{n \,|\, n/N < x} \frac{1}{2^N} \frac{N!}{n!(N-n)!} \approx e^{-N\, r_1(x)} \;\; (N \rightarrow \infty)$$

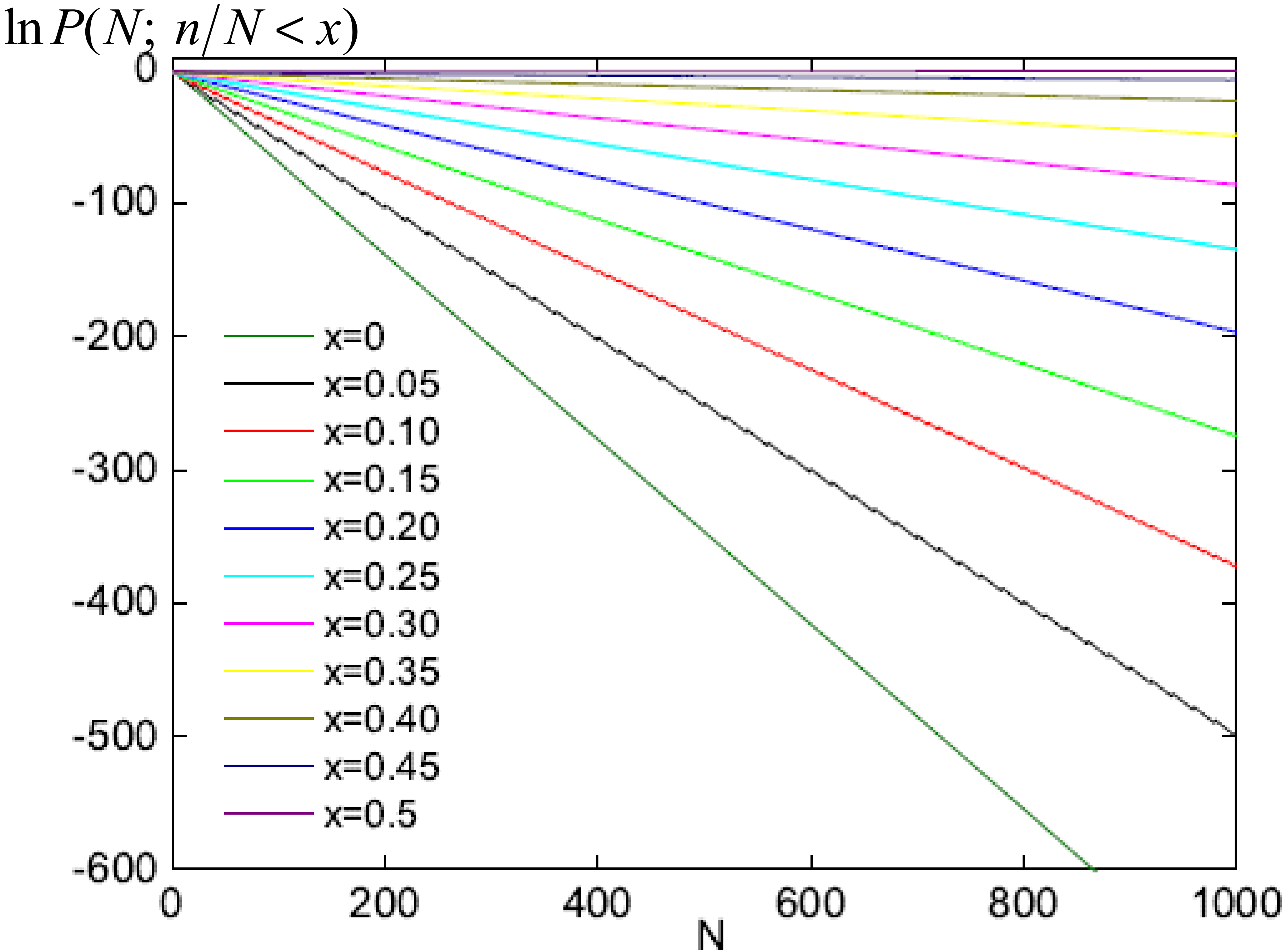
where the *rate function* is given by

$$r_1(x) = \ln 2 + x \ln x + (1-x)\ln(1-x)$$

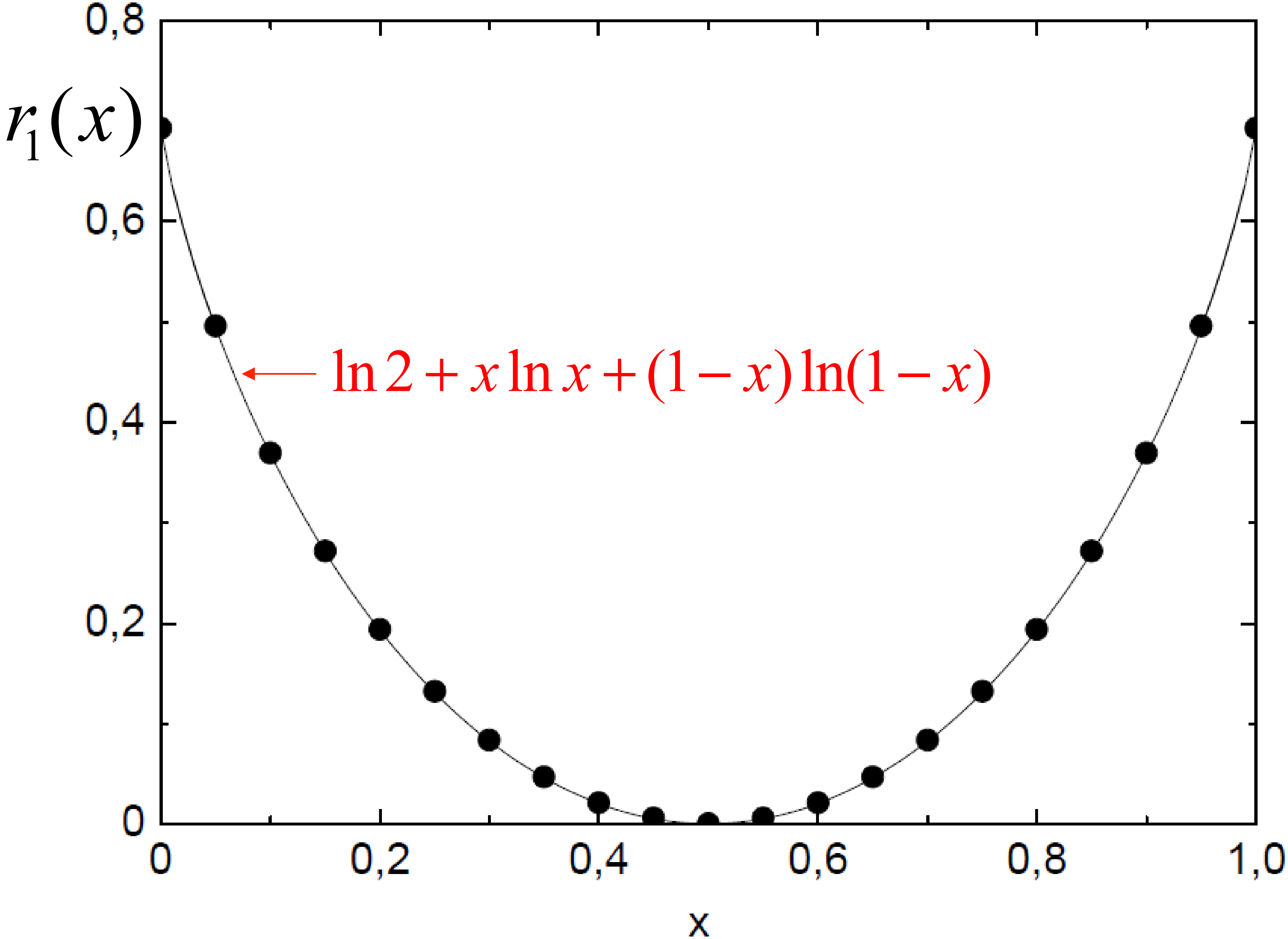
REMARK: Relative entropy $I_1 = -\sum_{i=1}^{W} p_i \ln \frac{p_i^{(0)}}{p_i}$

$$p_i^{(0)} = \frac{1}{W} \;\; \text{yields} \;\; I_1 = \ln W + \sum_{i=1}^{W} p_i \ln p_i$$

$$r_1(x) = I_1[W = 2;\; p_1 \rightarrow x;\; p_2 \rightarrow (1-x)]$$



G. Ruiz and C. T., 1110.6303 [cond-mat.stat-mech]



G. Ruiz and C. T., 1110.6303 [cond-mat.stat-mech]

GENERALIZING THE STANDARD THEORY- AN EXAMPLE

N strongly correlated coins $\rightarrow n$ heads $(n = 0, 1, 2, \ldots, N)$

$$\left[\textit{such that } N \rightarrow \infty \;\Rightarrow\; Q\text{-}\textit{Gaussian}\right]$$

$$P(N;\; n/N < x) = \sum_{n \mid n/N < x} p_{N,n}$$

$$p_{N,n} \equiv \frac{p_Q(y_{N,n})}{\sum_{n=0}^{N} p_Q(y_{N,n})} \qquad \text{with} \quad p_Q(z) \propto \left[1 + (Q-1)z^2\right]^{-\frac{1}{Q-1}}$$

and

$$y_{N,n} \equiv \Delta_N \left(\frac{n}{N} - \frac{1}{2}\right) \quad \left[\Delta_N \equiv \delta(N+1)^{\gamma};\; \delta > 0;\; 0 < \gamma < 1\right]$$

Probabilistic model $\{p_{N,n}\}$ is determined by $(1 \le Q < 3,\; 0 < \gamma \le 1,\; \delta > 0)$:

$$\lim_{N \to \infty} \{p_{N,n}\} = Q\text{-}\textit{Gaussian} \quad (\forall \gamma, \forall \delta)$$

We numerically verify that, for $N >> 1$,

$$\mathrm{P}\left(\mathrm{N} \mid n/N < x\right) \approx e_q^{-N\, r_q(x;\, Q,\, \gamma,\, \delta)} \quad (\text{for } Q \geq 1) \quad \left[\propto 1/N^{1/(q-1)} \text{ if } Q > 1\right]$$

with $q = \dfrac{Q-1}{\gamma(3-Q)} + 1 \geq 1$ hence $\dfrac{1}{\gamma(q-1)} = \dfrac{2}{Q-1} - 1 \quad (0 < x < 1/2)$

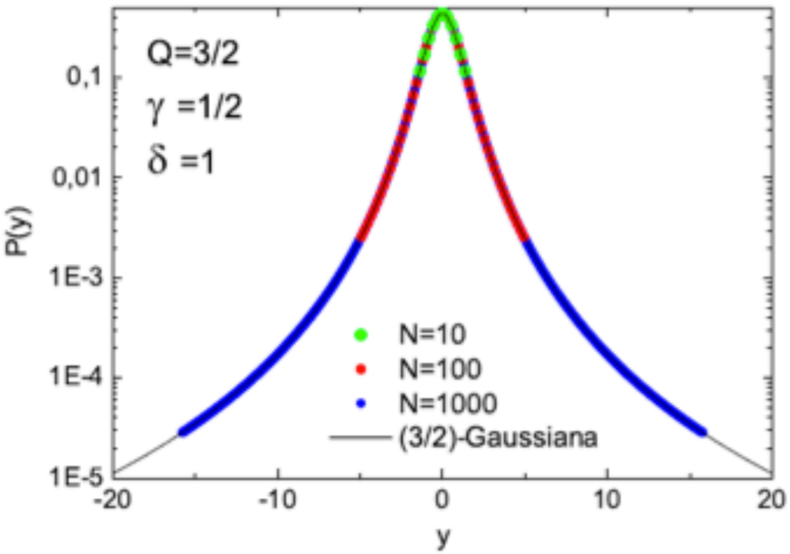
and $q(x=0) = 2 - \dfrac{1}{q(0 < x < 1/2)} \quad (\forall \gamma;\ \forall \delta)$

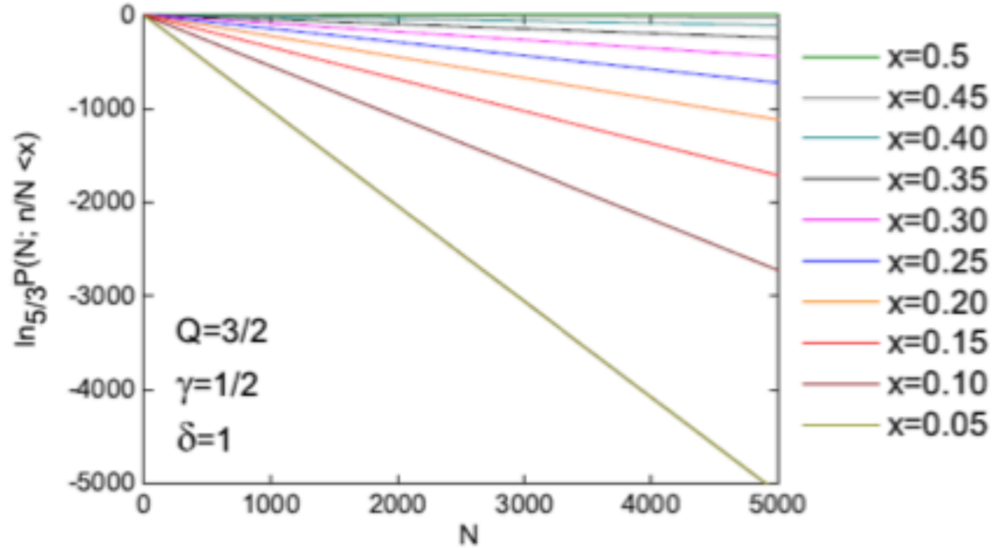
REMARK: Relative entropy $I_q = -\sum_{i=1}^{W} p_i \ln_q \dfrac{p_i^{(0)}}{p_i}$

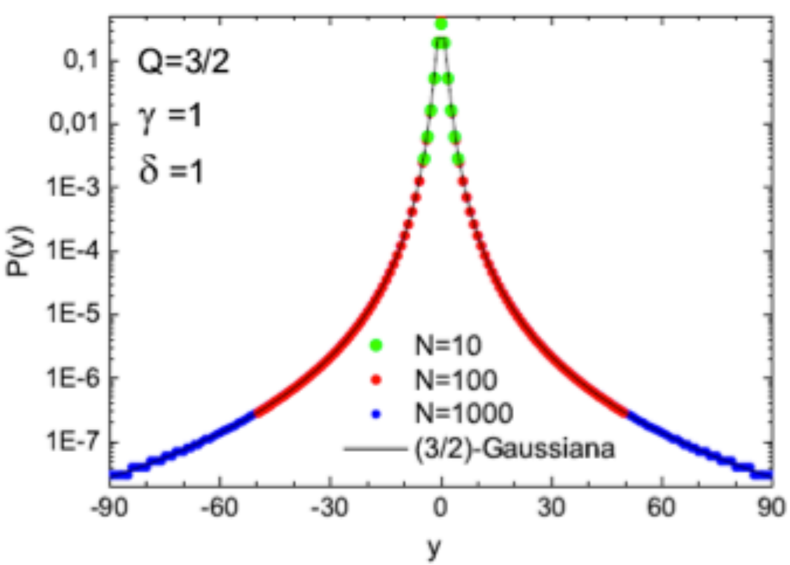
$$p_i^{(0)} = \frac{1}{W} \text{ yields } I_q = W^{q-1}\left[\ln_q W - S_q\right]$$

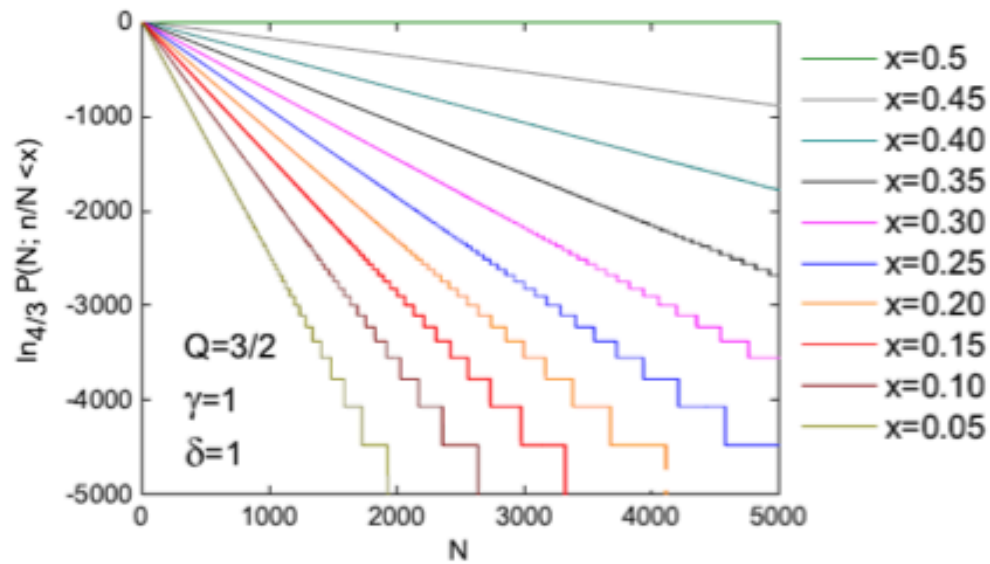
$$I_q[W=2;\ p_1 \to x;\ p_2 \to (1-x)] = 2^{q-1}\left[\ln_q 2 - \frac{x^q + (1-x)^q - 1}{q-1}\right]$$

G. Ruiz and C. T., 1110.6303 [cond-mat.stat-mech]

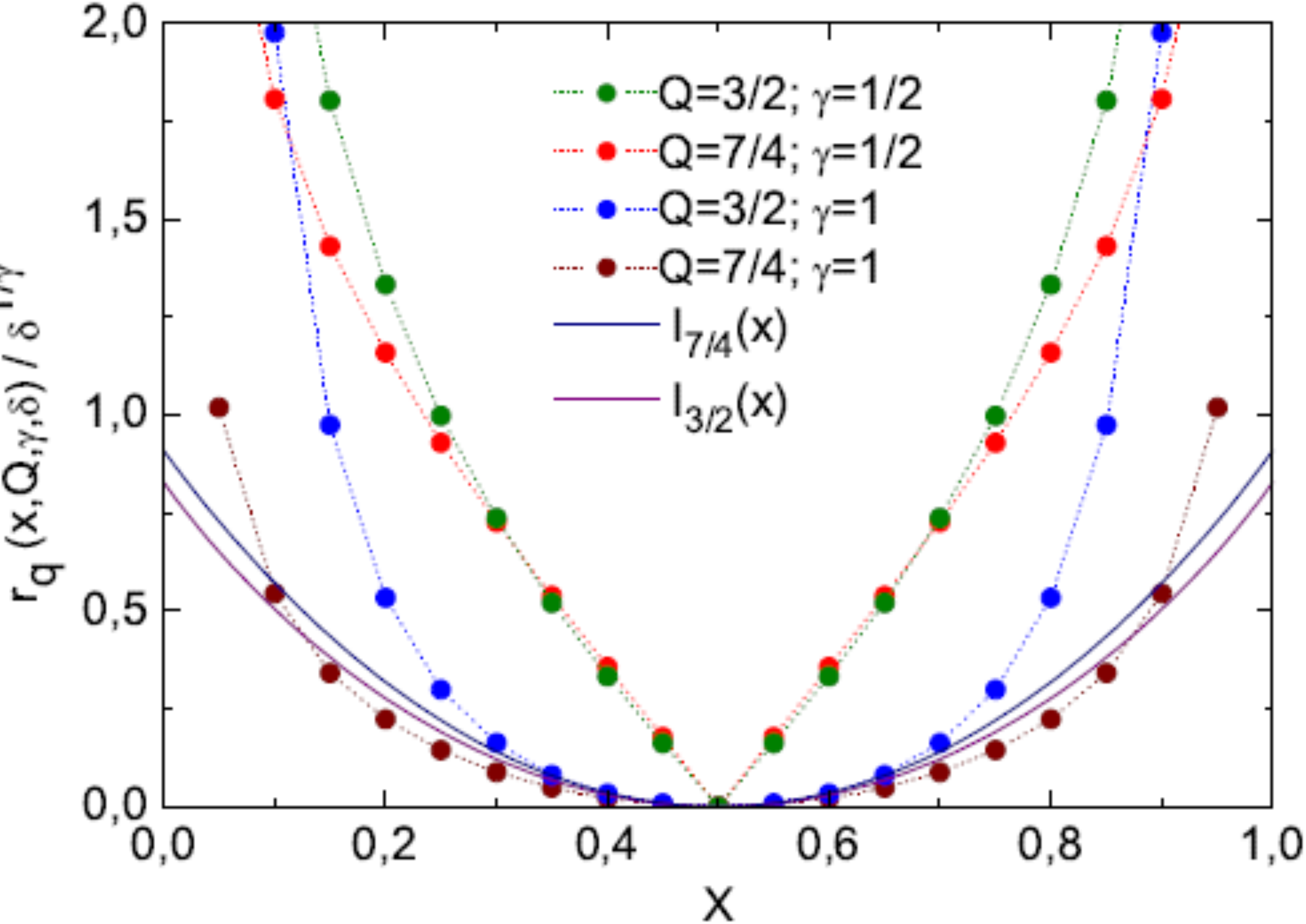








G. Ruiz and C. T., 1110.6303 [cond-mat.stat-mech]



G. Ruiz and C. T., 1110.6303 [cond-mat.stat-mech]

ALGUNS RESULTADOS RECENTES

TEORIA DOS GRANDES DESVIOS

A ENTROPIA DO BURACO NEGRO

When entropy does not seem extensive

Earlier speculations about the entropy of black holes has prompted an ingenious calculation suggesting that entropy may (in special circumstances) be the same inside and outside an arbitrary boundary.

EVERYBODY who knows about entropy knows that it is an extensive property, like mass or enthalpy. That, of course, is why the entropy of some substance will be quoted as so much per gram, or mole. If you then take two grams, or two moles, of the same material under the same conditions, the entropy will be twice as much. And there should be no confusion about the units; the simple Carnot definition of a change of entropy in a reversible process is the heat transfer divided by the absolute temperature, so that the units of entropy are simply those of energy divided by temperature, joules per degree (kelvin) in the SI system. The definitions of the Gibbs and Helmholtz free energies would be dimensionally discordant for that reason were it not that entropy (S) always turns up multiplied by temperature T. So much will readily be agreed.

Of course, there is more than that to entropy, which is also a measure of disorder. Everybody also agrees on that. But how is disorder measured? By the number of ways in which the constituents of some material (the atoms and molecules) can be rearranged without changing its properties and without energetic consequences. But now there comes a snag.

Like any extensive property, the combined entropy of two separate chunks of material should be the *sum* of the two entropies, but the number of rearrangements of the combined system must be the *product* of the numbers of ways in which the two parts separately can be rearranged. How to reconcile that with extensivity? By supposing entropy is proportional not to the number of rearrangements (technically called 'complexions'), but with the logarithm thereof. And because entropy decreases as disorder increases, the constant of proportionality must be a negative (real) number.

From that it follows that $S = S_0 - K \log N$, where K is a positive constant with the dimensions of entropy, N is a number (without dimensions) measuring disorder and S_0 is an arbitrary constant entropy. All that is simply a precis of the standard introductory chapter in statistical mechanics textbooks, most of which go on to show how to calculate the properties of assemblages of, say, diatomic molecules from a knowledge of their individual behaviour. Because the number of complexions of a particular state of an assemblage is invariably a function of the number (n) of molecules it contains, usually in the form of $n!$, because n is usually large and because $\log(n!)$ can then be approximated by $n \log n$, the extensive property of entropy then follows simply from the appearance of the leading factor n: entropy is proportional to the number of molecules.

That is what the textbooks say. It also makes sense of what is known of the thermodynamics of the real world. In a sample of a diatomic gas, for example, there are vibrations (one) and rotations (two) as well as three rectilinear degrees of freedom. But the problem is to tell how the energy available is distributed among the different degrees of freedom. The arithmetic simplifies marvellously because (in this case) each molecule and each of its degrees of freedom is independent. The best measure of disorder works out at $N = Z^n$, where n is the number of molecules, and where Z, which must be a

[...]

well suited to the discussion of systems in which one part (say the black hole) is singled out for attention while the remainder (the Universe outside it) is dealt with in less detail, perhaps because some averaging process is appropriate, or because the whole problem may not be calculable at all. (In Dirac's notation, the density matrix corresponding to some state of the whole Universe would be represented as $|I\rangle\langle I|$, where "I" is simply the name for a particular state of the Universe.) What matters, where entropy is concerned, is that the density matrix, like all matrices, has eigenvalues from which the entropy can be calculated.

So imagine that the Universe is partitioned into two parts by means of a closed boundary of some kind and filled with a

[...]

A bit of quantum mechanics goes into the argument as well, notably the notion of the density matrix — an artificially constructed operator (on quantum states) that is

[...]

dealt with explicitly, as other entropy calculations are made. And that could be exceedingly important.

John Maddox

Jacob D. Bekenstein
Stephen W. Hawking
Gerard 't Hooft
Leonard Susskind
Stephen Lloyd
Juan M. Maldacena
…

When entropy does not seem extensive
John Maddox, Nature **365**, 103 (1993)

Everybody who knows about entropy knows that it is an extensive property, like mass or enthalpy. [...] Of course, there is more than that to entropy, which is also a measure of disorder. Everybody also agrees on that. But how is disorder measured? [...] So why is the entropy of a black hole proportional to the square of its radius, and not to the cube of it? To its surface area rather than to its volume?

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Black holes and thermodynamics*

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(Received 30 June 1975)

A black hole of given mass, angular momentum, and charge can have a large number of different unobservable internal configurations which reflect the possible different initial configurations of the matter which collapsed to produce the hole. The logarithm of this number can be regarded as the entropy of the black hole and is a measure of the amount of information about the initial state which was lost in the formation of the black hole. If one makes the hypothesis that the entropy is finite, one can deduce that the black holes must emit thermal radiation at some nonzero temperature. Conversely, the recently derived quantum-mechanical result that black holes do emit thermal radiation at temperature $\kappa\hbar/2\pi kc$, where κ is the surface gravity, enables one to prove that the entropy is finite and is equal to $c^3A/4G\hbar$, where A is the surface area of the event horizon or boundary of the black hole. Because black holes have negative specific heat, they cannot be in stable thermal equilibrium except when the additional energy available is less than 1/4 the mass of the black hole. This means that the standard statistical-mechanical canonical ensemble cannot be applied when gravitational interactions are important. Black holes behave in a completely random and time-symmetric way and are indistinguishable, for an external observer, from white holes. The irreversibility that appears in the classical limit is merely a statistical effect.

ENTROPIES

$$S_{BG} = k_B \sum_{i=1}^{W} p_i \ln \frac{1}{p_i} \qquad \rightarrow \textit{additive}$$

$$S_q = k_B \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i} \qquad (S_1 = S_{BG}) \quad \rightarrow \textit{nonadditive} \text{ if } q \neq 1$$

C. T. (1988)

$$S_\delta = k_B \sum_{i=1}^{W} p_i \left(\ln \frac{1}{p_i} \right)^\delta \qquad (S_1 = S_{BG}) \quad \rightarrow \textit{nonadditive} \text{ if } \delta \neq 1$$

C. T. (2009)

$$S_{q,\delta} = k_B \sum_{i=1}^{W} p_i \left(\ln_q \frac{1}{p_i} \right)^\delta \qquad (S_{q,1} = S_q;\ S_{1,\delta} = S_\delta;\ S_{1,1} = S_{BG})$$

$$\rightarrow \textit{nonadditive} \text{ if } (q,\delta) \neq (1,1)$$

C. T. and L.J.L. Cirto (2011)

1202.2154 [cond-mat.stat-mech]

EXTENSIVITY OF THE ENTROPY $(N \to \infty)$

If $W(N) \sim \mu^N \quad (\mu > 1)$

$$\Rightarrow \; S_{BG}(N) = k_B \ln W(N) \; \propto N$$

If $W(N) \sim N^\rho \quad (\rho > 0)$

$$\Rightarrow \; S_q(N) \; = k_B \ln_q W(N) \propto [W(N)]^{1-q} \propto N^{\rho(1-q)}$$

$$\Rightarrow \; S_{q=1-1/\rho}(N) \; \propto N$$

If $W(N) \sim \nu^{N^\gamma} \quad (\nu > 1;\; 0 < \gamma < 1)$

$$\Rightarrow \; S_\delta(N) = k_B \left[\ln W(N) \right]^\delta \propto N^{\gamma\,\delta}$$

$$\Rightarrow \; S_{\delta=1/\gamma}(N) \; \propto N$$

Hawking, string theory, etc, yield

$$S_{BG}^{black\ hole}(N) \equiv k_B \ln W(N) \propto L^2 \propto N^{2/3} \quad (N \propto L^3)$$

More generally, we have

$$S_{BG}(N) = k_B \ln W(N) \propto L^{d-1} \propto N^{\frac{d-1}{d}} \quad (d > 1)$$

hence

$$W(N) \propto \Phi(N)\ \nu^{N^{\frac{d-1}{d}}} \quad \left(\text{with } \lim_{N\to\infty} \frac{\ln \Phi(\mathrm{N})}{N^{\frac{d-1}{d}}} = 0 \right)$$

hence the entropy which is extensive is S_δ with $\delta = \dfrac{d}{d-1}$

i.e., $$S_\delta(N) = k_B \sum_{i=1}^{W(N)} p_i \left(\ln \frac{1}{p_i} \right)^{\frac{d}{d-1}} \quad (d > 1)$$

Consequently $$S_{\delta=3/2}^{black\ hole}(N) = k_B \sum_{i=1}^{W(N)} p_i \left(\ln \frac{1}{p_i} \right)^{\frac{3}{2}} \propto N \propto L^3 \quad !!!$$

SYSTEMS $W(N)$	ENTROPY S_{BG} (ADDITIVE)	ENTROPY S_q $(q \neq 1)$ (NONADDITIVE)	ENTROPY S_δ $(\delta \neq 1)$ (NONADDITIVE)
$\sim \mu^N$ $(\mu > 1)$	EXTENSIVE	NONEXTENSIVE	NONEXTENSIVE
$\sim N^\rho$ $(\rho > 0)$	NONEXTENSIVE	EXTENSIVE	NONEXTENSIVE
$\sim \nu^{N^\gamma}$ $(\nu > 1;$ $0 < \gamma < 1)$	NONEXTENSIVE	NONEXTENSIVE	EXTENSIVE