

Nonadditive entropy and nonextensive statistical mechanics - Some central concepts and recent applications

Constantino Tsallis^{1,2} and Ugur Tirnakli^{3,4}

¹ Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil

² Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA

³ Department of Physics, Faculty of Science, Ege University, 35100 Izmir, Turkey

⁴ Division of Statistical Mechanics and Complexity, Institute of Theoretical and Applied Physics (ITAP) Kaygiseki Mevkii, 48740 Turunc, Mugla, Turkey

E-mail: tsallis@cbpf.br, ugur.tirnakli@ege.edu.tr

Abstract. We briefly review central concepts concerning nonextensive statistical mechanics, based on the nonadditive entropy $S_q = k \frac{1 - \sum_i p_i^q}{q-1}$ ($q \in \mathcal{R}$; $S_1 = -k \sum_i p_i \ln p_i$). Among others, we focus on possible realizations of the q -generalized Central Limit Theorem, including at the edge of chaos of the logistic map, and for quasi-stationary states of many-body long-range-interacting Hamiltonian systems.

1. Introduction

1.1. Entropy

The ubiquitous concept of *energy* is associated with the *possibilities* for the configurations of a mechanical system (e.g., the eigenvalues of the Hamiltonian of a quantum system defined in a specific Hilbert space). The concept of *entropy* emerges in an even larger domain, since it can be defined for any system, mechanical or not, which admits a set of *probabilities* for its possible configurations. For instance, if we are dealing with a quantum mechanical system, the set of probabilities typically is that corresponding to the eigenvectors of the Hilbert space. Epistemologically speaking, entropy is one of the most subtle concepts in physics. Entropy and energy together constitute the basis on which statistical mechanics — one of the pillars of contemporary physics — is constructed.

The entropy, initially defined by Clausius for thermodynamics, connects the macroscopic and microscopic worlds. Its most elementary form is the logarithmic one, first introduced by Boltzmann and refined by Gibbs, von Neumann, Shannon, Jaynes and others. For a finite discrete set of probabilities $\{p_i\}$ is given by

$$S_{BG} = -k \sum_{i=1}^W p_i \ln p_i \quad \left(\sum_{i=1}^W p_i = 1 \right), \quad (1)$$

where *BG* stands for *Boltzmann-Gibbs*. The conventional constant k is typically taken to be the Boltzmann universal constant for thermostistical systems, or taken to be unity in information

theory. For the particular case of equal probabilities, i.e., $p_i = 1/W, \forall i$, we have

$$S_{BG} = k \ln W, \quad (2)$$

carved on stone in Boltzmann grave in Vienna. Expression (1) enables the construction of a remarkably useful physical theory, referred to as *Boltzmann-Gibbs statistical mechanics*.

Many entropic forms have been introduced, and reintroduced, since Boltzmann (see [1, 2] and references therein for details). In 1988 [3], a more general form, namely

$$S_q = k \frac{1 - \sum_{i=1}^W p_i^q}{q - 1} \quad \left(\sum_{i=1}^W p_i = 1; q \in \mathcal{R}; S_1 = S_{BG} \right), \quad (3)$$

was proposed as the basis for generalizing *BG* statistical mechanics (into a theory now known as *nonextensive statistical mechanics*¹). For the particular case of equal probabilities, expression (3) becomes

$$S_q = k \frac{W^{1-q} - 1}{1 - q}. \quad (4)$$

By introducing the q – *logarithmic* function

$$\ln_q x \equiv \frac{x^{1-q} - 1}{1 - q} \quad (x > 0; q \in \mathcal{R}; \ln_1 x = \ln x), \quad (5)$$

expressions (3) and (4) can be respectively rewritten as follows:

$$S_q = k \sum_{i=1}^W p_i \ln_q \frac{1}{p_i} = -k \sum_{i=1}^W p_i^q \ln_q p_i, \quad (6)$$

and

$$S_q = k \ln_q W. \quad (7)$$

The axiomatics associated with this entropy have been quite explored and interesting characterizations have emerged. For details we may address the reader to [4, 5, 6, 7, 8], among others.

1.2. Additivity versus extensivity

We adopt for *entropy additivity* the definition given in Penrose’s classical book [9], namely that an entropy S is said *additive* if, for any two *probabilistically independent* systems A and B , i.e., for $p_{i,j}^{A+B} = p_i^A p_j^B, \forall (i, j)$, we have that

$$S(A + B) = S(A) + S(B), \quad (8)$$

where $S(A + B) \equiv S(\{p_{i,j}^{A+B}\})$, $S(A) \equiv S(\{p_i^A\})$, and $S(B) \equiv S(\{p_j^B\})$.

From definition (3) it is straightforward to prove that, for any two probabilistically independent systems A and B ,

$$\frac{S_q(A + B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1 - q) \frac{S_q(A)}{k} \frac{S_q(B)}{k}. \quad (9)$$

¹ The word *nonextensive* is to be associated with the fact that the total energy of long-range-interacting mechanical systems is nonextensive, in contrast with the case of short-range-interacting systems, whose total energy is *extensive* in the thermodynamical sense.

Therefore, S_{BG} is additive, whereas S_q ($q \neq 1$) is nonadditive ².

Entropic extensivity is a concept in some sense more subtle than additivity. An entropy S of a given system constituted by N elements is said to be *extensive* if

$$0 < \lim_{N \rightarrow \infty} \frac{S(N)}{N} < \infty, \quad (10)$$

i.e., if $S(N) \propto N$ for $N \gg 1$. We see therefore that additivity only depends on the specific mathematical connection between the macroscopic entropy functional and the probabilities of the configurations of the system. Extensivity depends on this but *also* on the specific system, more precisely on the nature of the correlations of its elements, and therefore of its collective configurations.

The distinction between additivity and extensivity has already been illustrated in simple probabilistic systems [10]. It has also been shown for the so-called *block entropy* (entropy of a subsystem of the entire system) of strongly quantum entangled fermionic and bosonic systems [11].

For the probabilistic system it has been shown [10] that

$$S_{q_{ent}}(N) \propto N \quad (N \gg 1), \quad (11)$$

where *ent* stands for *entropy*, N is the number of (strongly correlated) binary random variables, and

$$q_{ent} = 1 - \frac{1}{d}, \quad (12)$$

$d = 1, 2, 3, \dots$ characterizing the width of an infinitely long strip of nonvanishing probabilities of a probability triangle asymptotically satisfying the Leibnitz rule (i.e., asymptotically scale-invariant).

For the (one-dimensional) fermionic system it has been shown that, at criticality at vanishing temperature, we have [11]

$$\lim_{N \rightarrow \infty} S_{q_{ent}}(N, L) \propto L \quad (L \gg 1), \quad (13)$$

where L is the number of first-neighboring spins (or analogous elements) within an infinitely long ($N \rightarrow \infty$) chain, and

$$q_{ent} = \frac{\sqrt{9 + c^2} - 3}{c}, \quad (14)$$

where c is the *central charge* ($c = 1/2$ for the Ising ferromagnet, and $c = 1$ for the isotropic XY ferromagnet, in the presence of a critical transverse magnetic field in both cases). We verify that q_{ent} monotonically increases from zero to one when c increases from zero to infinity.

For the (two-dimensional) bosonic system the results are qualitatively the same. However they have been established only numerically, not analytically.

The generic scenario which emerges is that, for a vast class of systems (but certainly not all), a value q_{ent} exists such that $S_{q_{ent}}(N) \propto N$ ($N \rightarrow \infty$), where N is the number of elements of the system under consideration (which might be the entire system, or only a large part of it). For standard systems, we have that $q_{ent} = 1$; for various classes of anomalous systems, we have $q_{ent} \neq 1$. The situation is depicted in Fig. 1.

² We remark that additivity is obtained whenever $(1 - q)/k \rightarrow 0$. We see that this can occur in two different manners: $q \rightarrow 1$ for fixed k , and $k \rightarrow \infty$ for fixed q . The latter corresponds to the infinite temperature T limit, since in all thermostatical systems T always appears in the form kT , thus having the dimension of an energy.

SYSTEMS	ENTROPY S_{BG} (additive)	ENTROPY S_q ($q < 1$) (nonadditive)
Short-range interactions, weakly entangled blocks, etc	EXTENSIVE	NONEXTENSIVE
Long-range interactions (QSS), strongly entangled blocks, etc	NONEXTENSIVE	EXTENSIVE

Figure 1. Entropic additivity and entropic extensivity are different concepts. The BG entropy is additive, whereas S_q (for $q \neq 1$) is nonadditive. Extensivity depends on the system: for standard systems, S_{BG} is extensive, whereas it can be nonextensive for anomalous systems; S_q is the other way around.

1.3. Central limit theorems

The *Central Limit Theorem* within theory of probabilities basically states that the *sum* of a large number N of *independent* (or quasi-independent in some specific sense) random variables whose variance is *finite* converge, after appropriate centering and rescaling, to a Gaussian (i.e., $p(x) \propto e^{-\beta x^2}$). This distribution constitutes an attractor in the space of distributions, and is therefore thought to be the reason for the ubiquity of Gaussian distributions in nature. If the single distribution has a *divergent* variance instead (and also satisfies some supplementary mathematical conditions), the attractors are the celebrated Lévy distributions. The situation changes drastically if strong correlations exist among the N random variables. Depending on the nature of the correlations very many types of attractors might emerge. There is however a special class of strong correlations, referred to as *q-independence* [12], for which the attractors are *q-Gaussians* (i.e., $p(x) \propto e_q^{-\beta x^2}$, where the *q-exponential* function is defined as the inverse of the *q-logarithmic* one defined in Eq. (5)), if a specific generalized variance is *finite*. If this variance *diverges* instead, the attractors are the so-called (q, α) -stable distributions: see [13] for full details. The schematic description of these four theorems is presented in Fig. 2.

The physical-mathematical interpretation of the class of strong correlations named as *q-independence* is not yet fully elucidated. However, it might well be that *q-independence* between N random variables implies (strict or asymptotic) probabilistic scale-invariance in the sense that

$$\int dx_N h_N(x_1, x_2, \dots, x_N) \sim h_{N-1}(x_1, x_2, \dots, x_{N-1}) \quad (N \rightarrow \infty). \quad (15)$$

Although probably necessary, this property is surely not sufficient. Indeed, (strictly or asymptotically) scale-invariant probabilistic models (with finite values for the appropriately

generalized variance) have been analytically solved, some of them yielding, in the $N \rightarrow \infty$ limit, q -Gaussians, whereas other models yield distributions which numerically are amazingly close to q -Gaussians, but which definitively are *not* exactly q -Gaussians. Models that yield q -Gaussians are available in [15, 16]; models that have been proved [17] to be not exactly q -Gaussians are presented in [18, 19].

Like Gaussians, q -Gaussians also are ubiquitous³ in natural, artificial and social systems. What could be the cause of such fact? It could very well be precisely the theorem appearing in Fig. 2 which corresponds to q -Gaussian attractors. In what follows we shall exhibit various nearly q -Gaussian distributions: in Section 2 for dissipative one-dimensional dissipative maps, in Section 3 for long-range-interacting many-body classical Hamiltonians. These two systems share a crucial property, namely that they have a maximal Lyapunov exponent which approaches zero, thus excluding strong chaos. Finite-size or finite-precision effects are present in them: we mimic this property in Section 4 with a simple mathematical model. Finally we conclude in Section 5 by mentioning various systems presented in the literature which also appear to exhibit q -Gaussians.

2. Unimodal one-dimensional dissipative maps

Let us here concentrate on a paradigmatic one-dimensional dissipative dynamical system, namely the logistic map, defined as $x_{t+1} = 1 - ax_t^2$, where a is the map parameter ($0 \leq a \leq 2$), $-1 \leq x_t \leq 1$, and $t = 0, 1, 2, \dots$. Our object of interest is the sum

$$y = \sum_{i=N_0+1}^{N_0+N} (x_i - \langle x \rangle) \quad (16)$$

in the vicinity of chaos threshold $a_c = 1.4011\dots$, where N_0 is the number of transient steps (typically $N_0 \gg 1$) and

$$\langle x \rangle = \frac{1}{n_{ini}} \frac{1}{N} \sum_{j=1}^{n_{ini}} \sum_{i=1}^N x_i^{(j)} \quad (17)$$

is the average over a large number of N iterates as well as a large number n_{ini} of randomly chosen initial values $x_1^{(j)}$ of iterates of the map. This problem has been addressed firstly in [20] and a closer look has been given very recently in [21]. Here, we try to further clarify the study and analyse it in a more compact manner.

Generically, the problem at hand is the form of the probability distribution of the random variable given in Eq. (16). Indeed, the ordinary Central Limit Theorem (CLT) (yielding the Gaussian form) is applicable when the Lyapunov exponent is positive, but, when approaching the edge of chaos from above (i.e., for $a > a_c$), an infinite number of values of a accumulate which violate this condition. The situation becomes then quite subtle, as we shall review here. Essentially, strong correlations between the iterates of the map emerge. In [21] it is shown that the problem is much more complex than the ordinary CLT, which is the case when the map is at the chaotic regime (e.g., for $a = 2$). One needs now to be careful on how we approach the chaos threshold point (a_c) and how the number of iterations (N) to be used increases to infinity. In mathematical language, this means that two limits are to be performed simultaneously, namely, $(a - a_c) \rightarrow 0$ and $1/N \rightarrow 0$. It is argued in [21] that the limit distributions appear to be of q -Gaussian type if these two limits are performed simultaneously in the following special way (which satisfies an appropriate scaling relation). We first choose a value of a above and close to

³ The word *ubiquitous* is here used not in the strict sense of being everywhere, but only in the loose sense of being found very frequently.

	$q=1$ [<i>independent</i>]	$q \neq 1$ (i.e., $Q \equiv 2q-1 \neq 1$) [<i>globally correlated</i>]
$\sigma_Q < \infty$ ($\alpha = 2$)	$\mathbb{F}(x) = \text{Gaussian } G(x)$, with same σ_1 of $f(x)$	$\mathbb{F}(x) = G_q(x) \equiv G_{(3q-1)/(1+q_1)}(x)$, with same σ_Q of $f(x)$ $G_q(x) \sim \begin{cases} G(x) & \text{if } x \ll x_c(q, 2) \\ f(x) \sim C_q / x ^{2/(q-1)} & \text{if } x \gg x_c(q, 2) \end{cases}$, with $\lim_{q \rightarrow 1} x_c(q, 2) = \infty$
$\sigma_Q \rightarrow \infty$ ($0 < \alpha < 2$)	$\mathbb{F}(x) = \text{Levy distribution } L_\alpha(x)$, with same $ x \rightarrow \infty$ behavior $L_\alpha(x) \sim \begin{cases} G(x) & \text{if } x \ll x_c(1, \alpha) \\ f(x) \sim C_\alpha / x ^{1+\alpha} & \text{if } x \gg x_c(1, \alpha) \end{cases}$ with $\lim_{\alpha \rightarrow 2} x_c(1, \alpha) = \infty$	$\mathbb{F}(x) = L_{q,\alpha}$, with same $ x \rightarrow \infty$ asymptotic behavior $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)}, \alpha}(x) \sim C_{q,\alpha}^* / x ^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} & \text{(intermediate regime)} \\ G_{\frac{2\alpha q - \alpha + 3}{\alpha + 1}, 2}(x) \sim C_{q,\alpha}^L / x ^{(1+\alpha)/(1+\alpha q - \alpha)} & \text{(distant regime)} \end{cases}$

Figure 2. $N^{1/[\alpha(2-q)]}$ -scaled attractors $\mathcal{F}(x)$ when summing $N \rightarrow \infty$ q -independent identical random variables with symmetric distribution $f(x)$ with Q -variance $\sigma_Q \equiv \int_{-\infty}^{\infty} dx x^2 [f(x)]^Q / \int_{-\infty}^{\infty} dx [f(x)]^Q$ ($Q \equiv 2q - 1$; $q_1 = (1 + q)/(3 - q)$; $q \geq 1$). *Top left:* The attractor is the Gaussian sharing with $f(x)$ the same variance σ_1 (standard CLT). *Bottom left:* The attractor is the α -stable Lévy distribution which shares with $f(x)$ the same asymptotic behavior, i.e., the coefficient C_α (Lévy-Gnedenko CLT, or α -generalization of the standard CLT). *Top right:* The attractor is the q -Gaussian which shares with $f(x)$ the same $(2q - 1)$ -variance, i.e., the coefficient C_q (q -generalization of the standard CLT, or q -CLT). *Bottom right:* The attractor is the (q, α) -stable distribution which shares with $f(x)$ the same asymptotic behavior, i.e., the coefficient $C_{q,\alpha}^L$ (q -generalization of the Lévy-Gnedenko CLT and α -generalization of the q -CLT). The case $\alpha < 2$, for both $q = 1$ and $q \neq 1$ (more precisely $q > 1$), further demands specific asymptotics for the attractors to be those indicated; essentially the divergent q -variance must be due to fat tails of the power-law class, excepting for possible logarithmic corrections (for the $q = 1$ case see, for instance, [14] and references therein).

a_c . We then calculate the quantity n defined as follows:

$$n = -\frac{\log |a - a_c|}{\log \delta}, \tag{18}$$

where $\delta = 4.6692011\dots$ is the Feigenbaum constant. We then denote by k the nearest integer value of $2n$, and define N^* through

$$N^* = 2^k. \tag{19}$$

The q -Gaussians numerically appear to gradually emerge when we choose $N = N^*$ and keep making $(a - a_c) \rightarrow 0$ (hence $N = N^* \rightarrow \infty$). Let us refer to this region as the q -Gaussian

probability distribution functions (PDFs) one and analyse its borders. To do this, typical values for a have been used: they are given in Table 1, as well as their related parameters; a values can easily be taken so that $2n$ values would be obtained with the same precision. Each group with the same precision enables the construction of a linear curve in the space $1/N$ vs $(a - a_c)^s$, where $s \equiv \ln 4 / \ln \delta \simeq 0.9$: see Fig. 3. Numerical inspection has shown that no other q -Gaussian linear curves occur at the left of the largest slope and at the right of the smallest slope in Fig. 3. All q -Gaussian lines appear to exist only between these two extremes; see examples in Fig 4 as well as Fig. 5 where (q, β) pairs of all studied cases are plotted.

Table 1. The values of map parameter a and N^* used in this work. The values of $2n$ (obtained from the scaling relation), and the values of q and β (estimated from simulations) are also listed. In this work, the critical value a_c is approximated as $a_c = 1.4011551890920505$.

a	$a - a_c$	$2n$	$N^* = 2^k$	q	β
1.40159888	0.00044369	10.02	2^{10}	1.70	6.6
1.40125021721	0.00009503	12.02	2^{12}		
1.40117554121	0.00002035	14.02	2^{14}		
1.40115954790	0.00000436	16.02	2^{16}		
1.40152683	0.00037164	10.25	2^{10}	1.70	6.5
1.40123478	0.00007959	12.25	2^{12}		
1.401172235	0.00001705	14.25	2^{14}		
1.4011588398	0.00000365	16.25	2^{16}		
1.4014862	0.00033101	10.40	2^{10}	1.70	6.8
1.401226075	0.00007088	12.40	2^{12}		
1.401170372	0.00001518	14.40	2^{14}		
1.401158441	0.00000325	16.40	2^{16}		
1.401464065	0.00030888	10.49	2^{10}	1.68	6.7
1.401221341	0.00006615	12.49	2^{12}		
1.4011693567	0.00001417	14.49	2^{14}		
1.4011582234	0.00003034	16.49	2^{16}		
1.40145934	0.00030415	10.51	2^{11}	1.61	6.8
1.40122033	0.00006514	12.51	2^{13}		
1.40116914	0.00001395	14.51	2^{15}		
1.401158177	0.00000299	16.51	2^{17}		
1.40138924	0.00023405	10.85	2^{11}	1.65	6.5
1.401205317	0.00005013	12.85	2^{13}		
1.401165925	0.00001074	14.85	2^{15}		
1.4011574883	0.00000230	16.85	2^{17}		
1.40136531	0.00021012	10.99	2^{11}	1.63	6.5
1.40120019	0.00004500	12.99	2^{13}		
1.401164827	0.00000964	14.99	2^{15}		
1.4011572532	0.00000206	16.99	2^{17}		

The q -Gaussian-like PDFs are lost as soon as the scaling relation (Eq. (18)) is ignored, i.e., if values for N larger or smaller than N^* are used. Two new regions emerge. If we use values for N that are sensibly smaller than N^* , or, in other words, the value of a that is being used is too close to the critical point, peaked PDF's are observed (see also [22]). On the other extreme, if we use values for N that are sensibly larger than N^* , we observe in most of the cases Gaussian

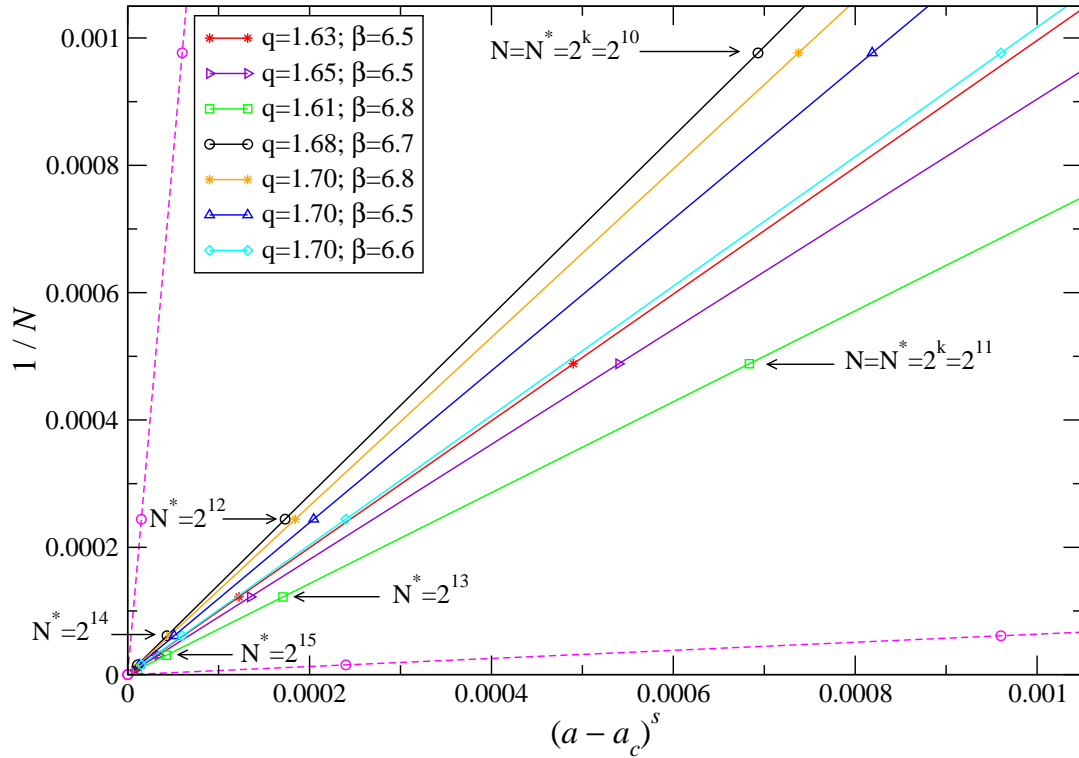


Figure 3. View of the parameter region $1/N$ vs $(a - a_c)^s$ with $s = \ln 4 / \ln \delta \simeq 0.9$ coming directly from the scaling relation (18). Typical values of a are shown, and their corresponding values of N^* are determined from the scaling relation (18). Any other possible choices for N^* yield lines that remain between the lines with the largest and the smallest slopes shown in the figure. If one approaches the critical point (origin) along any of these lines in this region, the probability distribution function (PDF) appears to gradually approach, excepting for a small oscillating contribution, a q -Gaussian. The regions at the left of the largest slope and at the right of the smallest slope are not accessible as far as N^* values are concerned. Four of the seven q -Gaussian examples are presented in Fig. 4; the peaked (almost vertical line) and the Gaussian (almost horizontal line) examples are presented in Fig. 6.

PDF's, i.e., $P(y) = e^{-y^2/(2\sigma^2)}/\sqrt{2\pi\sigma^2}$. Two representative examples for the two regions outside the q -Gaussian one are indicated with dashed lines (magenta) in Fig. 3 and illustrated in Fig. 6.

3. Long-range-interacting many-body classical Hamiltonian systems

Let us consider the following classical Hamiltonian of N interacting planar classical rotators on a d -dimensional (simple hypercubic) lattice [23]:

$$\mathcal{H} = \sum_{i=1}^N \frac{p_i^2}{2} + \frac{1}{2} \sum_{i,j} \frac{1 - \cos(\theta_i - \theta_j)}{r_{ij}^\alpha} \quad (\alpha \geq 0), \quad (20)$$

where r_{ij} runs over all possible distances within the d -dimensional lattice. The particular case $\alpha = 0$ is referred to in the literature as the *HMF* model [24], and has been intensively studied in the last decade (in its standard representation the coupling constant is divided by N , which artificially makes the total energy extensive in the thermodynamical sense). Its dynamical molecular approach has exhibited a variety of interesting phenomena: see, for instance, [1] and

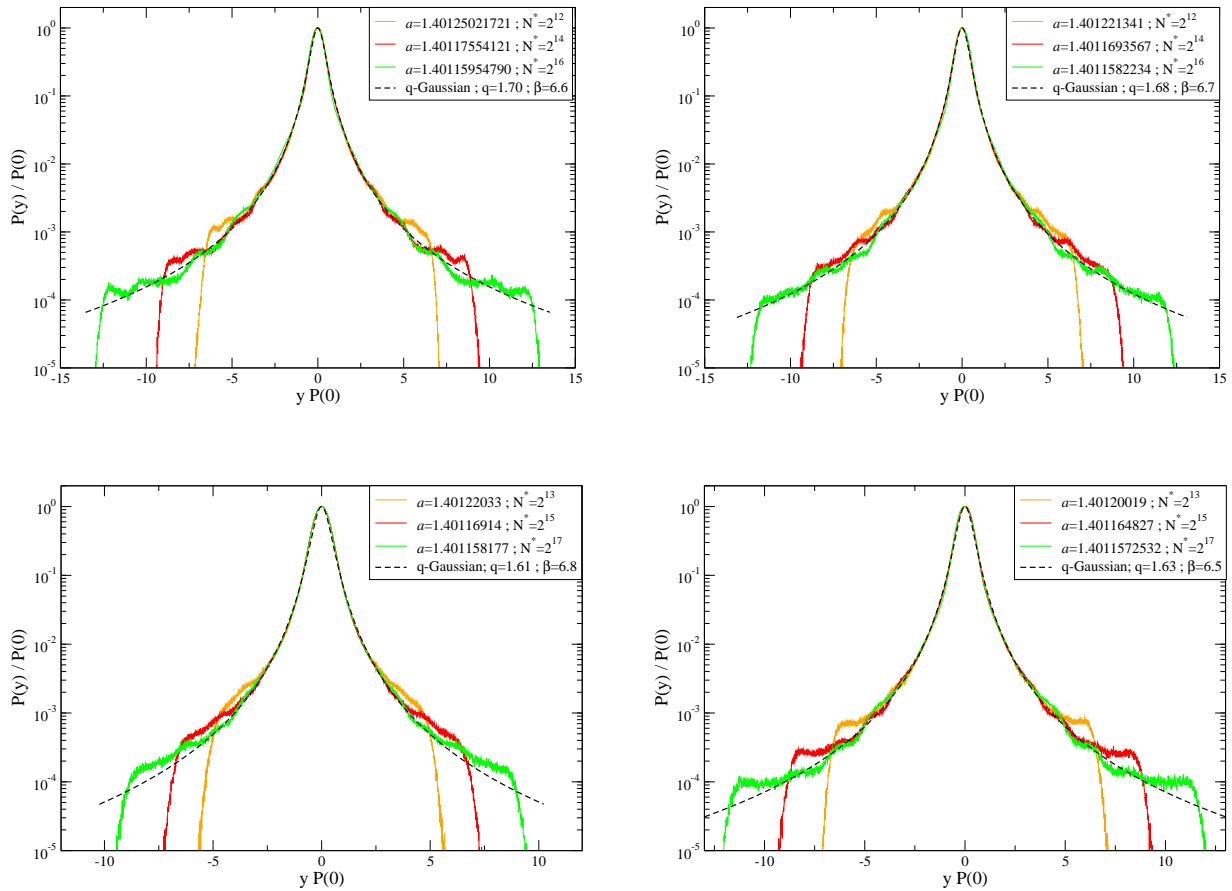


Figure 4. Data collapse of PDF's for the cases $N = N^* = 2^k$ for four of the seven representative examples given in Table 1. The (possible) q -Gaussians are approached through finite- N effects analogous to those exhibited in Fig. 9.

references therein. In particular, for the isolated system (microcanonical ensemble) at energy per particle equal to 0.69, long-standing quasi-stationary states (QSS) emerge when certain classes of initial conditions (usually called *water-bag* initial conditions) are used. Within the water-bag conditions, one may consider initial magnetization equal to zero (usually referred to as $M = 0$), or equal to its maximal value (usually referred to as $M = 1$), or values in between. Such choices influence the specific trajectory of the full system within its $2N$ -dimensional phase space (Gibbs Γ space). It has been shown that, for $M = 1$ initial conditions (possibly for virtually all values of initial M), *ergodicity is broken*. Indeed the *summed* (over n equidistant instants) one-velocity marginal PDF differs when we take ensemble-average or time-average: see [25] and references therein. Many of the time-averaged PDF's numerically approach a q -Gaussian (see Fig. 7 for one such example). Although no analytical proof is available at the present time, this might be a consequence of the q -Central Limit Theorem [12], within which q -Gaussians are the attractors in the space of PDF's. The finite-size effects are illustrated in Fig. 7.

4. Simple mathematical model for crossover between q -Gaussians

As a mathematical simple model for finite-size or finite-precision effects exhibiting the gradual approach to q -Gaussians, let us consider the following differential equation [1, 26]:

$$\frac{dy}{dx} = -ay^r - by^q \quad (b \geq a \geq 0; q > r; y(0) = 1). \quad (21)$$

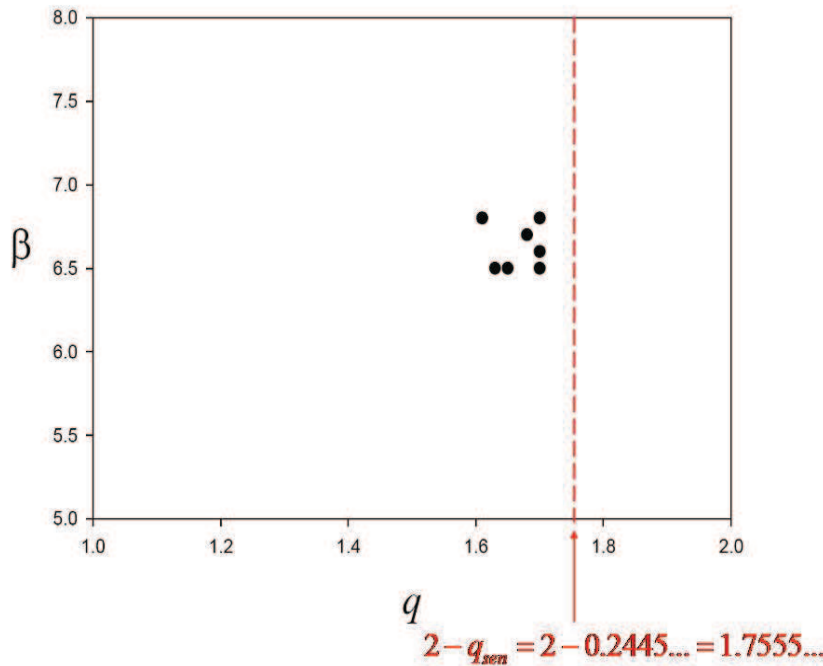


Figure 5. The parameters (q, β) corresponding to the seven q -Gaussians indicated in Fig. 3. These specific seven examples appear to exclude the value $2 - q_{sen} = 1.7555\dots$, which could have been a plausible result. At the present numerical precision, even if quite high, it is not possible to infer whether the analytical result corresponding to the present observations would be only one or a set of q -Gaussians, assuming that exact q -Gaussians are involved, on top of which a small oscillating component possibly exists.

If $a = 0$ the solution is

$$y = e_q^{-bx}. \quad (22)$$

If $b = 0$ the solution is

$$y = e_r^{-ax}. \quad (23)$$

If $b > a > 0$ a crossover occurs from the q -exponential solution for x not too large to the r -exponential solution for x large enough; *increasing size or increasing precision* for specific models (such as the logistic map at its edge of chaos, or the *HMF* Hamiltonian model at its *QSS* state) acts analogously to *decreasing values of a* towards the limit $a = 0$, for fixed b . Let us address this interesting case. From (21) we obtain

$$x = - \int_1^y \frac{du}{au^r + bu^q} = \frac{r}{b(-1+q)(q-r)} - \frac{\left(r + (q-r) {}_2F_1 \left[\frac{-1+q}{q-r}, 1, 1 + \frac{-1+q}{q-r}, -\frac{a}{b} \right] \right)}{b(-1+q)(q-r)} - \frac{ry^{1-q}}{b(-1+q)(q-r)} + \frac{y^{1-q} \left(r + (q-r) {}_2F_1 \left[\frac{-1+q}{q-r}, 1, 1 + \frac{-1+q}{q-r}, -\frac{ay^{-q+r}}{b} \right] \right)}{b(-1+q)(q-r)} \quad (24)$$

where ${}_2F_1$ is the hypergeometric function. In general, this function does not admit an *explicit* expression in the form $y(x)$. An exception is the $r = 1$ case, which yields

$$y = \frac{1}{\left[\left(\frac{b}{a} + 1 \right) e^{(q-1)ax} - \frac{b}{a} \right]^{\frac{1}{q-1}}}. \quad (25)$$

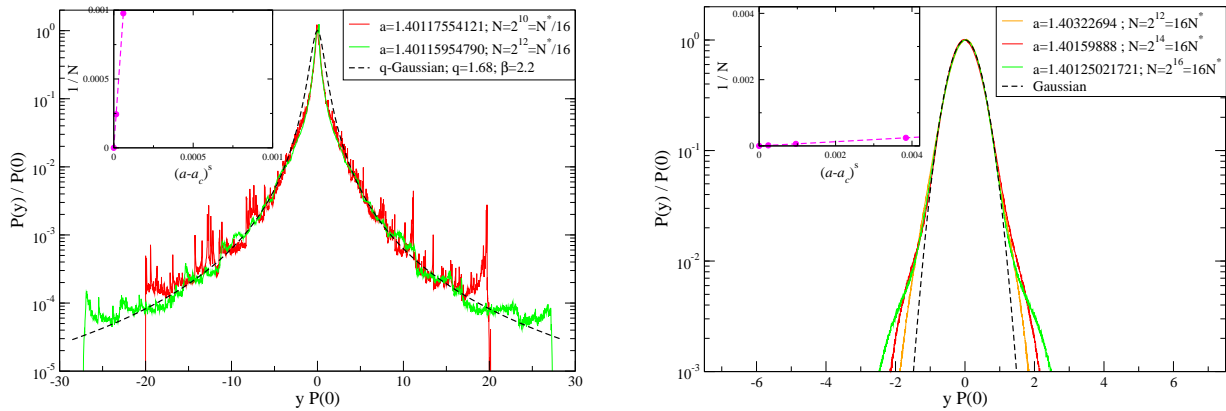


Figure 6. Representative examples from the peaked region (left) and the Gaussian region (right).

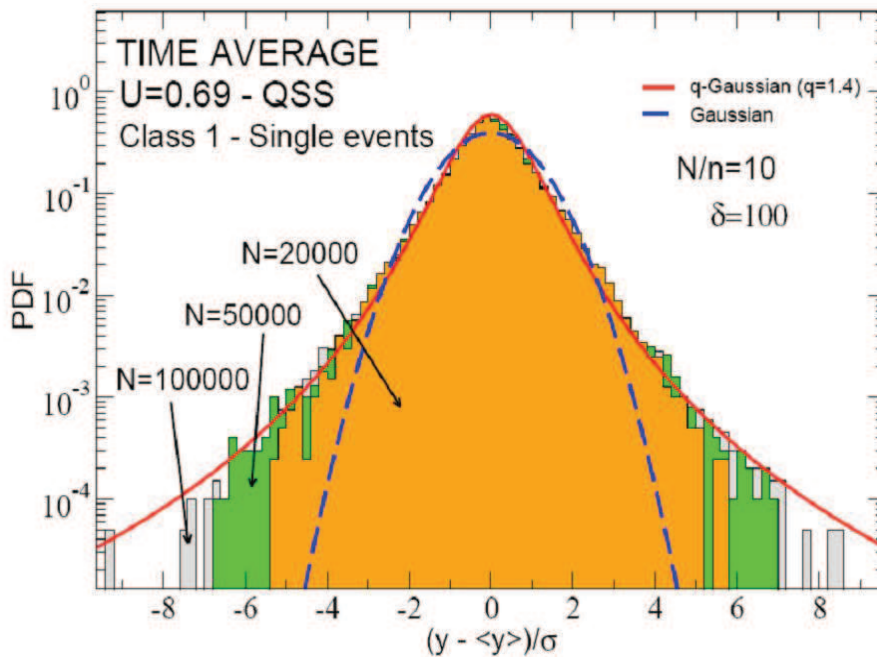


Figure 7. Velocity distribution for the HMF model at its quasi-stationary state (QSS) [Data from Fig. 2 of [25]]. The finite- N effects are visible, gradually approaching a (possible) q -Gaussian when both N and n are diverging. An analogous behavior is shown in Fig. 9. For comparison, a Gaussian is shown as well.

The $r = 0$ case must be handled through the explicit $x(y)$ form, namely

$$x = \frac{1}{a} \left\{ {}_2F_1 \left[\frac{1}{q}, 1, 1 + \frac{1}{q}, -\frac{b}{a} \right] - y {}_2F_1 \left[\frac{1}{q}, 1, 1 + \frac{1}{q}, -\frac{b}{a} y^q \right] \right\}. \quad (26)$$

Let us address now the case of the q -Gaussians. Following the form of Eq. (21), we consider

$$\frac{dy}{d(x^2)} = -a_r y^r - (a_q - a_r) y^q \quad (a_q \geq a_r \geq 0; q > r; y(0) = 1). \quad (27)$$

If $a_r = 0$, or equivalently if $r = q$, the solution is given by the q -Gaussian $y = e_q^{-a_q x^2}$. If $a_r = a_q$, the solution is given by the r -Gaussian $y = e_r^{-a_r x^2}$. For the case $a_q > a_r > 0$ and $q > r$, we obtain a crossover between these two solutions, the $|x| \rightarrow \infty$ asymptotic one being the r -Gaussian behavior.

For $r = 1$ and $q > 1$, the solution is given by

$$y = \frac{1}{\left[1 - \frac{a_q}{a_1} + \frac{a_q}{a_1} e^{(q-1)a_1 x^2}\right]^{\frac{1}{q-1}}}. \quad (28)$$

The general behavior of these solutions is given in Fig. 8. It is evident from this figure that this solution is not the most appropriate one for the behavior observed in the neighborhood of the logistic map edge of chaos. We notice concomitantly that the appropriate solution for the logistic map seems to be very close to the one with $r = 0$ and $q > 1$, whose solution is given by

$$x^2 = \frac{1}{a} \left\{ {}_2F_1\left[\frac{1}{q}, 1, 1 + \frac{1}{q}, -\frac{b}{a}\right] - y {}_2F_1\left[\frac{1}{q}, 1, 1 + \frac{1}{q}, -\frac{b}{a} y^q\right] \right\}, \quad (29)$$

with $a \equiv a_0$ and $b \equiv a_q - a_0$. Indeed, this solution seems to be a very good approximation for the behavior of PDFs obtained numerically for the logistic map. This can be seen immediately whenever the representative example of the solution given in Fig. 9 is compared to the case given in Fig. 4a.

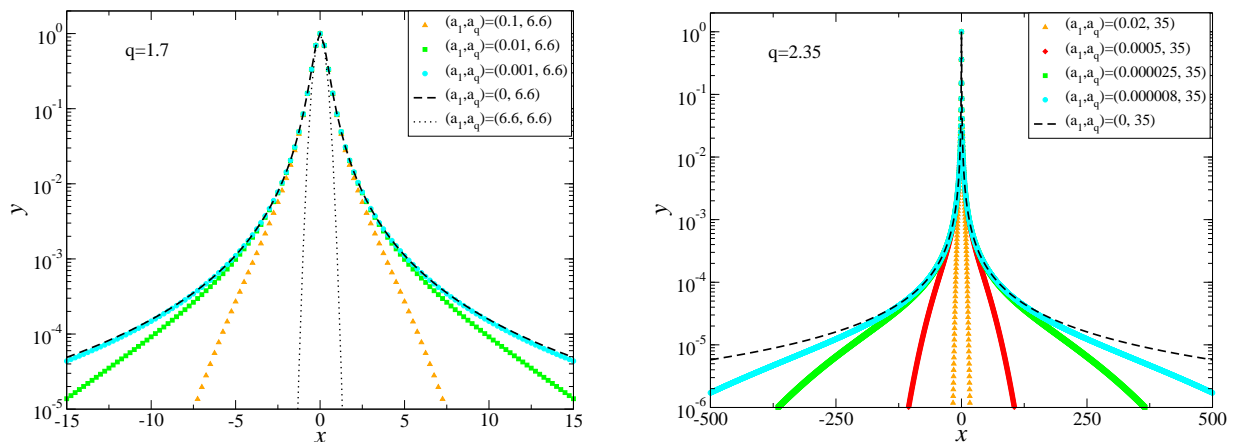


Figure 8. Crossover from q -Gaussian to successively distant Gaussians, i.e., illustrations of the case $r = 1$ and $q > 1$. *Left:* $q = 1.7$; for comparison, a Gaussian is shown as well. *Right:* $q = 2.35$.

5. Final remarks

Many q -Gaussian-like distributions have been observed in recent years in natural, artificial and social systems. Obviously, only for a mathematically formulated model, the hope exists to possibly prove analytically that the relevant distribution precisely, and not only approximatively,

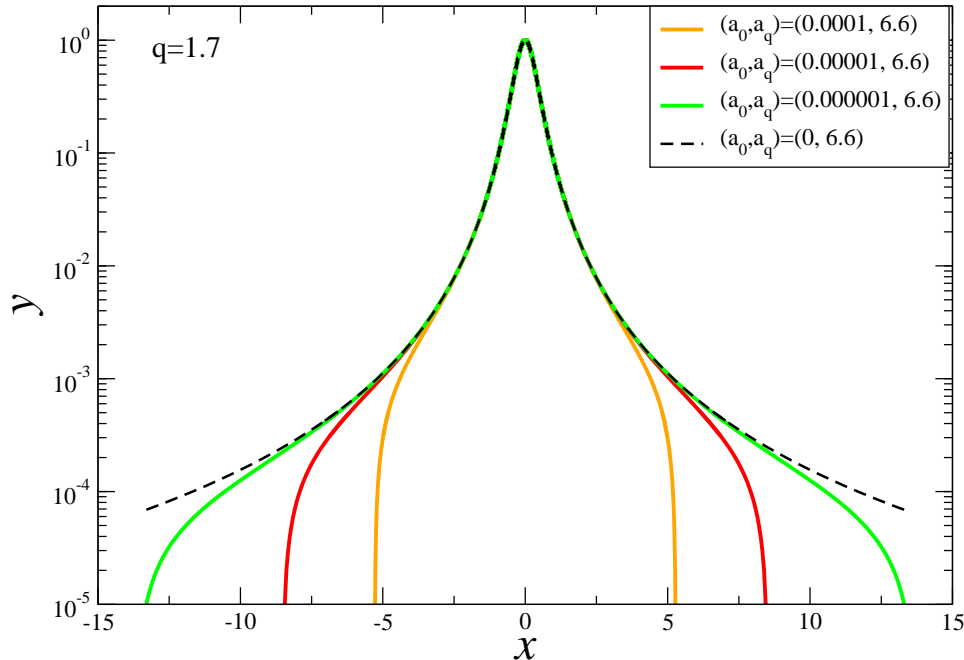


Figure 9. Crossover from 1.7-Gaussian with $\beta = 6.6$ to successively distant r -Gaussians with $r = 0$.

is a q -Gaussian. In all other cases, we can only expect for increasingly high-precision indications from real experiments or observations. Computational evidence can and does provide important hints, however never a proof.

This said, let mention in what follows some of the many other systems where q -Gaussians have been used to approach the observed PDF's: (i) The velocity distribution of (cells of) *Hydra viridissima* follows a $q = 3/2$ PDF [27]; (ii) The velocity distribution of (cells of) *Dictyostelium discoideum* follows a $q = 5/3$ PDF in the vegetative state and a $q = 2$ PDF in the starved state [28]; (iii) The velocity distribution in defect turbulence [29]; (iv) The velocity distribution of cold atoms in a dissipative optical lattice [32]; (v) Velocity distribution during silo drainage [30, 31]; (vi) The velocity distribution in a driven-dissipative 2D dusty plasma, with $q = 1.08 \pm 0.01$ and $q = 1.05 \pm 0.01$ at temperatures of 30000 K and 61000 K respectively [33]; (vii) The spatial (Monte Carlo) distributions of a trapped $^{136}\text{Ba}^+$ ion cooled by various classical buffer gases at 300 K [34]; (viii) The distributions of price returns at the stock exchange [35, 36, 37]; (ix) The distributions of returns of magnetic field fluctuations in the solar wind plasma as observed in data from Voyager 1 [38] and from Voyager 2 [39]; (x) The distributions of returns of the avalanche sizes in the Ehrenfest's dog-flea model [40]; (xi) The distributions of returns of the avalanche sizes in the self-organized critical Olami-Feder-Christensen model, as well as in real earthquakes [41]; (xii) The distributions of angles in the *HMF* model [42]; (xiii) The distribution of stellar rotational velocities in the Pleiades [43]. Some indirect evidence is available as well: although no q -Gaussian distribution has been directly observed in some relevant physical quantity, a q -exponential relaxation has been seen in various paradigmatic spin-glass substances through neutron spin echo experiments [44].

Clearly, the simplest hypothesis which would explain the ubiquity of q -Gaussians is the

validity of the q -Central Limit theorem. The involved random variables would, in such case, be expected to be q -independent [12]. The present belief is that (probabilistic) scale-invariance is necessary but not sufficient for q -independence. Further studies are needed to clarify the applicability of such ideas to the systems mentioned above, as well as possibly others.

Acknowledgments

We thank insightful remarks by C. Beck, E.M.F. Curado, A. Pluchino, A. Rapisarda, A. Robledo and B.C.C. dos Santos. This work has been partially supported by CNPq and Faperj (Brazilian agencies), and by TUBITAK (Turkish agency) under the Research Project number 104T148.

References

- [1] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics - Approaching a Complex World* (Springer, New York, 2009).
- [2] C. Tsallis, *Entropy*, in *Encyclopedia of Complexity and Systems Science*, ed. R.A. Meyers (Springer, Berlin, 2009), 11 volumes [ISBN: 978-0-387-75888-6].
- [3] C. Tsallis, *Possible generalization of Boltzmann-Gibbs statistics*, J. Stat. Phys. **52**, 479-487 (1988).
- [4] R.J.V. dos Santos, *Generalization of Shannon's theorem for Tsallis entropy*, J. Math. Phys. **38**, 4104 (1997).
- [5] S. Abe, *Axioms and uniqueness theorem for Tsallis entropy*, Phys. Lett. A **271**, 74 (2000).
- [6] F. Topsoe, *Factorization and escorting in the game-theoretical approach to non-extensive entropy measures*, Physica A **365**, 91-95 (2006).
- [7] F. Topsoe, *Towards operational interpretations of generalized entropies*, in the present volume.
- [8] A. Ohara, H. Matsuzoe and S. Amari, *A dually flat structure on the space of escort distributions*, in the present volume.
- [9] O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), page 167.
- [10] C. Tsallis, M. Gell-Mann and Y. Sato, *Asymptotically scale-invariant occupancy of phase space makes the entropy S_q extensive*, Proc. Natl. Acad. Sc. USA **102**, 15377-15382 (2005).
- [11] F. Caruso and C. Tsallis, *Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics*, Phys. Rev. E **78**, 021101 (2008).
- [12] S. Umarov, C. Tsallis and S. Steinberg, *On a q -central limit theorem consistent with nonextensive statistical mechanics*, Milan J. Math. **76**, 307-328 (2008).
- [13] S. Umarov, C. Tsallis, M. Gell-Mann and S. Steinberg, *Generalization of symmetric α -stable Lévy distributions for $q \neq 1$* , unpublished (2009).
- [14] J.P. Bouchaud and A. Georges, *Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications*, Phys. Rep. **195**, 127 (1990).
- [15] A. Rodriguez, V. Schwammle and C. Tsallis, *Strictly and asymptotically scale-invariant probabilistic models of N correlated binary random variables having q -Gaussians as $N \rightarrow \infty$ limiting distributions*, JSTAT P09006 (2008).
- [16] R. Hanel, S. Thurner and C. Tsallis, *Limit distributions of scale-invariant probabilistic models of correlated random variables with the q -Gaussian as an explicit example*, Eur. Phys. J. B (2009), DOI: 10.1140/epjb/e2009-00330.
- [17] H.J. Hilhorst and G. Schehr, *A note on q -Gaussians and non-Gaussians in statistical mechanics*, J. Stat. Mech. (2007) P06003.
- [18] L.G. Moyano, C. Tsallis and M. Gell-Mann, *Numerical indications of a q -generalised central limit theorem*, Europhys. Lett. **73**, 813-819 (2006).
- [19] W.J. Thistleton, J.A. Marsh, K.P. Nelson and C. Tsallis, *q -Gaussian approximants mimic non-extensive statistical-mechanical expectation for many-body probabilistic model with long-range correlations*, Cent. Eur. J. Phys. **7**, 387-394 (2009).
- [20] U. Tirnakli, C. Beck and C. Tsallis, *Central limit behavior of deterministic dynamical systems*, Phys. Rev. E **75**, 040106 (2007).
- [21] U. Tirnakli, C. Tsallis and C. Beck, *A closer look at time averages of the logistic map at the edge of chaos*, Phys. Rev. E **79**, 056209 (2009).
- [22] A. Robledo and M.A. Fuentes, unpublished (2009).
- [23] C. Anteneodo and C. Tsallis, *Breakdown of the exponential sensitivity to the initial conditions: Role of the range of the interaction*, Phys. Rev. Lett. **80**, 5313 (1998).
- [24] M. Antoni and S. Ruffo, Phys. Rev. E **52**, 2361 (1995).

- [25] A. Pluchino, A. Rapisarda and C. Tsallis, *Comment on “Ergodicity and central limit theorem in systems with long-range interactions” by Figueiredo A. et al*, Europhys. Lett. **85**, 60006 (2009).
- [26] C. Tsallis, G. Bemsiki and R.S. Mendes, *Is re-association in folded proteins a case of nonextensivity?*, Phys. Lett. A **257**, 93 (1999).
- [27] A. Upadhyaya, J.-P. Rieu, J.A. Glazier and Y. Sawada, *Anomalous diffusion and non-Gaussian velocity distribution of Hydra cells in cellular aggregates*, Physica A **293**, 549 (2001).
- [28] A.M. Reynolds, *Can spontaneous cell movements be modelled as Lévy walks?*, Physica A **389**, 273-277 (2010).
- [29] K.E. Daniels, C. Beck and E. Bodenschatz, *Defect turbulence and generalized statistical mechanics*, Physica D **193**, 208 (2004).
- [30] R. Arevalo, A. Garcimartin and D. Maza, *Anomalous diffusion in silo drainage*, Eur. Phys. J. E **23**, 191-198 (2007)[DOI:10.1140/epje/i2006-10174-1].
- [31] R. Arevalo, A. Garcimartin and D. Maza, *A non-standard statistical approach to the silo discharge*, in *Complex Systems - New Trends and Expectations*, eds. H.S. Wio, M.A. Rodriguez and L. Pesquera, Eur. Phys. J.-Special Topics **143** (2007) [DOI: 10.1140/epjst/e2007-00087-9].
- [32] P. Douglas, S. Bergamini and F. Renzoni, *Tunable Tsallis distributions in dissipative optical lattices*, Phys. Rev. Lett. **96**, 110601 (2006); G.B. Bagci and U. Tirnakli, *Self-organization in dissipative optical lattices*, Chaos **19**, 033113 (2009).
- [33] B. Liu and J. Goree, *Superdiffusion and non-Gaussian statistics in a driven-dissipative 2D dusty plasma*, Phys. Rev. Lett. **100**, 055003 (2008).
- [34] R.G. DeVoe, *Power-law distributions for a trapped ion interacting with a classical buffer gas*, Phys. Rev. Lett. **102**, 063001 (2009).
- [35] L. Borland, *Closed form option pricing formulas based on a non-Gaussian stock price model with statistical feedback*, Phys. Rev. Lett. **89**, 098701 (2002).
- [36] L. Borland, *A theory of non-gaussian option pricing*, Quantitative Finance **2**, 415 (2002).
- [37] S.M.D. Queiros, *On non-Gaussianity and dependence in financial in time series: A nonextensive approach*, Quant. Finance **5**, 475-487 (2005).
- [38] L.F. Burlaga and A.F.-Vinas, *Triangle for the entropic index q of non-extensive statistical mechanics observed by Voyager 1 in the distant heliosphere*, Physica A **356**, 375 (2005).
- [39] L.F. Burlaga and N.F. Ness, *Compressible “turbulence” observed in the heliosheath by Voyager 2*, Astrophys. J. **703**, 311-324 (2009).
- [40] B. Bakar and U. Tirnakli, *Analysis of self-organized criticality in Ehrenfest’s dog-flea model*, Phys. Rev. E **79**, 040103(R) (2009).
- [41] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra and A. Rapisarda, *Analysis of self-organized criticality in the Olami-Feder-Christensen model and in real earthquakes*, Phys. Rev. E **75**, 055101(R)(2007).
- [42] L.G. Moyano and C. Anteneodo, *Diffusive anomalies in a long-range Hamiltonian system*, Phys. Rev. E **74**, 021118 (2006).
- [43] J.C. Carvalho, R. Silva, J.D. do Nascimento and J.R. de Medeiros, *Power law statistics and stellar rotational velocities in the Pleiades*, Europhys. Lett. **84**, 59001 (2008).
- [44] R.M. Pickup, R. Cywinski, C. Pappas, B. Farago and P. Fouquet, *Generalized spin glass relaxation*, Phys. Rev. Lett. **102**, 097202 (2009).