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Self-organized percolation in multi-layered structures

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Abstract. We present a self-organized model for the growth of two- and three-dimensional percolation clusters in multi-layered structures. Anisotropy in the medium is modeled by randomly allocating layers of different physical properties. A controlling mechanism for the growing aggregate perimeter is introduced in such a manner that the system self-tunes to a stationary regime that corresponds to the percolation threshold. The critical probability for infinite growth is studied as a function of the anisotropy of the medium.

Keywords: percolation problems (theory), self-organized criticality (theory), fractal growth (theory), disordered systems (theory)

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Contents

1. Introduction

Percolation models provide a useful tool for the quantitative description of several phenomena in natural systems [1, 2]. Some relevant applications may be encountered in soil physics [3], oil recovery [4], fluid transport in porous media [5]–[8] and growth of branched structures [9]–[13], among many other areas [14]–[22].

Indeed, whereas most studies have been dedicated to isotropic percolation, disordered systems such as layered fractured rocks are in general anisotropic, i.e. physical properties such as density or porosity may vary over the different layers [23]–[26]. It is well known that geometrical and transport properties in diverse kinds of rocks and geological fields must be effectively characterized in terms of their degree of anisotropy [7]. Moreover, many systems in nature are known to be anisotropic in geometrical, mechanical, electrical or optical properties. Anisotropy has been incorporated into a two-dimensional percolation model by attributing different concentrations p to the different layers, whereby the critical percolation threshold p_c has been found to be a function of the amount of anisotropy in the system [25].

Taking inspiration from the self-organized branching process proposed by Zapperi *et al* [27], a percolation model has been proposed for the mechanism of growth of percolation clusters in which self-organization leads the system spontaneously to criticality [28, 29]. In the self-organized percolation (SOP) model the number of sites or bonds in the growth front of the aggregate is maintained close to a threshold value by imposing a controlling mechanism on the probability of occupation p of the lattice sites. In contrast to previous models, in which p is fixed all over the simulation, the SOP model allows for variations in p which are controlled by the mass of the perimeter of the growing cluster. In this manner, the system is spontaneously driven to a stationary state that corresponds to the percolation threshold of the lattice topology [28]. In fact, in the SOP process, criticality and self-organization [27, 30] are ingredients of the same dynamics: the system self-tunes around a critical state without the need of adjusting any model parameter. An advantageous method for calculating p_c is, thus, provided, since the SOP model does not require the trial and error scheme intrinsic to traditional numerical models for percolation [31]. In this way, the SOP model has been successfully applied for the prediction of the percolation threshold in a variety of physical systems, ranging from the

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growth of branching polymers to ferromagnetic Ising systems and the spreading of fire [28], [32]–[38].

In the present work, the self-organized percolation model is adapted in order to account for anisotropy. As in multi-layered percolation [25], sites which belong to different layers of the lattice representing the medium are allocated with different probabilities. However, these probabilities are allowed to vary in time according to the controlling mechanism of the SOP model. The percolation threshold is then studied as a function of the amount of anisotropy for two- and three-dimensional systems.

The paper is organized as follows. In section 2, we present a description of the anisotropic SOP model. In section 3, we present and discuss our results. A list of conclusions is given in section 4.

2. The model

We first consider a square lattice of dimensions $L \times L$, the axes of which are labeled x and y. Stratification is introduced by attributing to the concentration p_y of each layer in the y coordinate one of the values p_1 or p_2, which are chosen with equal probability 1/2. The layer is of type 1 (2) if its probability is p_1 (p_2), whereas we consider $p_2 \leq p_1$. We assume that p_y are independent random variables. All sites (x, y) along the layer y have the same probability, $p_y = p_1$ or p_2, whereas the case $p_1 = p_2$ recovers the set of regular isotropic percolations [25]. For convenience, we define the transformed variables

$$\bar{p} = (p_1 + p_2)/2, \tag{1}$$

$$\Delta = |p_1 - p_2|/2, \tag{2}$$

where the parameter Δ gives a measure of the amount of anisotropy in the system [25].

At time $t = 0$, only the central site of the lattice is occupied. Each one of the available nearest neighbors of the seed is assigned a uniform random number $0 < r < 1$, and the neighbor is subsequently occupied if its probability p_y is larger than r. If, however, an available nearest neighbor is not occupied, then it remains non-occupied until the end of the calculations. The algorithm is applied, at each time step t, to the $N(t)$ sites that were occupied at iteration $t - 1$. The $N(t)$ perimeter sites are also called the *active* sites of the growing process [28, 29, 31]. However, in order to ensure that $N(t)$ will not increase exponentially in time and that, at the same time, the cluster will never stop growing, we introduce a controlling mechanism by allowing the probability $\bar{p}$ to vary in time according to the following equation:

$$\bar{p}(t + 1) = \bar{p}(t) + k[N_L - N(t)], \tag{3}$$

where k is a kinetic coefficient and N_L is a threshold parameter [28, 29]. At each time step t, equation (3) is applied to obtain $\bar{p}$ from $N(t)$, and the probabilities p_1 and p_2 are, thereafter, updated by using equations (1) and (2). In this manner, in contrast to previous models [25, 31], where the site probabilities are kept constant over time, our model allows for time variations in $\bar{p}$ in order to compensate for the increase or decrease in the number $N(t)$ of active sites in the growth front. As a consequence of equation (3), the cluster growth is kept at a minimum rate [28, 29]. Although p_1 and p_2 are allowed to vary, the respective labels (1 or 2) of the lattice layers do not change in time. The values of the

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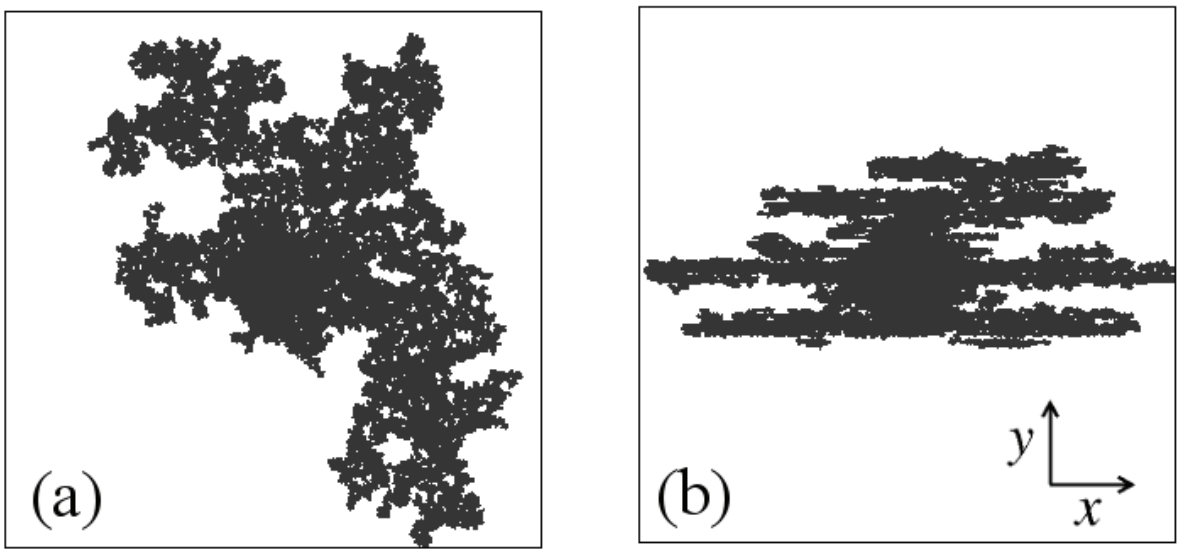

Figure 1. Typical realizations in a square lattice of dimensions 2000×2000 with (a) $\Delta = 0.0$ (isotropic) and (b) $\Delta = 0.2$. In case (b), the lattice is stratified, i.e. each value of the coordinate y (vertical axis) is randomly labeled 1 or 2. At $t = 0$, $p_1 = p_2 = 1.0$ for both layer types, whereas only the central site of the lattice is occupied. The growth algorithm described in section 2 is, then, applied, whereby p_1 and p_2 are recursively calculated from $\bar{p}$ and Δ, using equations (1)–(3). In the simulations, $k = 10^{-5}$ and $N_L = 200$.

probabilities p_1 and p_2 are obviously limited to the range $[0, 1]$, so whenever calculations yield a value smaller than 0 or larger than 1 for one of the probabilities p_1 or p_2, the values 0 and 1, respectively, are imposed as the solution for that probability. The simulation is performed until the growing cluster reaches one of the lattice borders.

A systematic study of the SOP model for different k and N_L parameters was presented in [29, 35] for the case of isotropic percolation, i.e. when $p_1 = p_2$. In this case, the value of $\bar{p}$ is found to evolve in time into a stationary value that fluctuates around the critical percolation threshold for the square lattice, $p_c \approx 0.59$, independently of the values of the model parameters. In the stationary state, $\bar{p}$ undergoes fluctuations around p_c, the frequency and magnitude of which depend on k and N_L. As explained in [29], large values of k lead to high frequency oscillations in $N(t)$. On the other hand, the threshold number N_L must be large enough to prevent the remaining active sites in the growth front from being killed by fluctuations, and small enough to avoid too long transient times. As in previous applications [28, 29, 32, 33, 36, 37], the value of k taken in the calculations of the present work is 10^{-5}, whereas N_L is of the order of 10^2.

3. Results and discussion

Figures 1(a) and (b) show typical simulation outcomes in a square lattice with 2000×2000 sites, using $\Delta = 0$ and $\Delta = 0.2$, respectively, and starting, in both cases, with $\bar{p} = 1.0$ at $t = 0$. As expected, the first simulation steps yield a compact cluster for both values of Δ. However, once the growing number $N(t)$ of active sites at the cluster perimeter exceeds

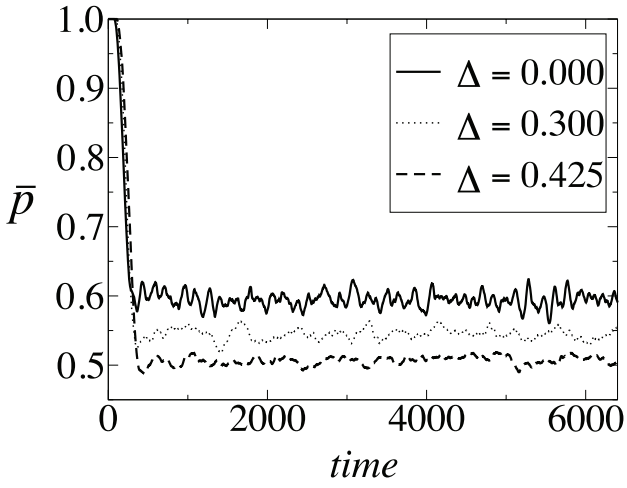

Figure 2. Different realizations on the square lattice with stratification. The figure shows the time evolution of $\bar{p}$ for different values of Δ, obtained using $k = 10^{-5}$ and $N_L = 300$. In each case, $\bar{p}$ is found to converge into a stationary value $p_c(\Delta)$, which corresponds to the critical probability for percolation in the stratified lattice.

the threshold number N_L, $\bar{p}$ starts decreasing due to equation (3). After a given number of time steps, which depends both on k and N_L [29, 35], a critical probability p_c is obtained around which $\bar{p}$ fluctuates (figure 2) in such a way as to maintain $N(t)$ around N_L. In the case of no stratification ($\Delta = 0$, figure 1(a)), the stationary state is $\bar{p} = p_1 = p_2 \simeq 0.59$, i.e. the threshold p_c for isotropic percolation in the square lattice [28, 29]. However, as $\Delta \to 0.5$, the stationary values of p_2 and p_1 approach 0 and 1, respectively, whereas $p_c \to 0.5$. Consequently, as the anisotropy in the system increases, the growing clusters become increasingly narrowed in the y direction, as illustrated in figure 1(b).

We perform simulations for different values of Δ on a lattice with 10^8 sites, by varying Δ from 0 to 0.5 in steps of 0.025. Figure 3 shows the phase diagram $p_c(\Delta)$ obtained from our model simulations (circles). Also shown are the results obtained by Dayan *et al* [25] (stars) using the 'cluster perimeter method' [9, 39, 40]. As seen in figure 3, the agreement between this method and our model of automatic search for the critical point $\bar{p} \to p_c$ is excellent. For each value of Δ, the curve $p_c(\Delta)$ separates two different regimes, one containing systems of finite clusters ($\bar{p} < p_c$) and one with systems of an infinite cluster ($\bar{p} > p_c$). Indeed, in both models, $p_c(\Delta)$ is not linear, in contrast to the prediction of [24].

For completeness, we also investigated the 'alternating' percolation model in which the stratification is such that a layer y is labeled 1 (2) if it is even (odd) [25]. In this case, smaller values of $p_c(\Delta)$ than the ones for the random model are found in the interval $0 < \Delta < 0.5$, as also found in [25].

We next extend the model in order to study self-organized percolation in three dimensions. Stratification of the cubic lattice is modeled in two different ways. The first model, called stratification in *sheets*, consists of randomly labeling different coordinates y as 1 or 2, in such a manner that all sites (x, y_n, z) on the sheet $y = y_n$ have the same probability p_1 or p_2, respectively. Sedimentary rocks constitute one of the many examples in nature where such stratification occurs [2]. The results obtained for this model are

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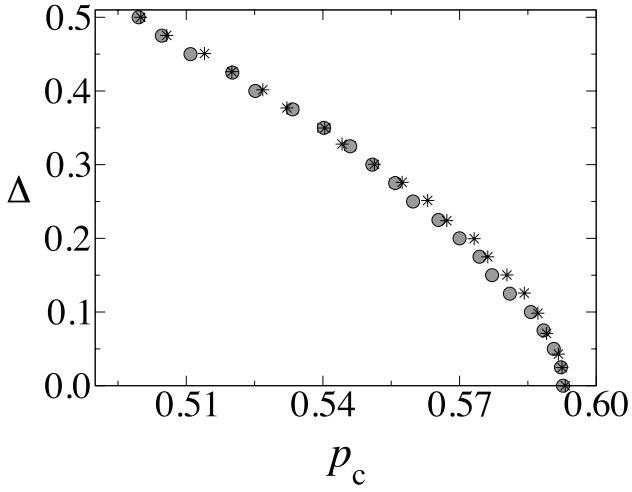

Figure 3. Circles represent the critical probability p_c as a function of Δ for the square lattice calculated with the method described in section 2. Results have been averaged over 4000 time steps in a total of 100 realizations. Good agreement is found between our results and the ones obtained by Dayan *et al* [25] (stars).

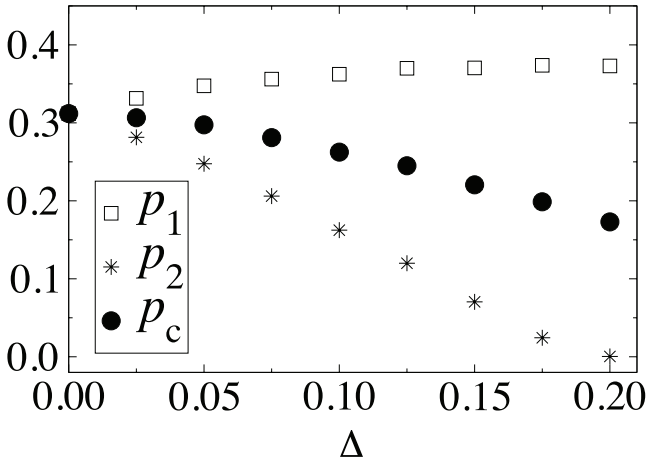

Figure 4. Self-organized percolation in three-dimensional multi-layered systems. The results are for the cubic lattice with stratification in sheets, i.e. each layer y is randomly labeled type 1 or 2. The main plot shows the stationary state values of the probabilities p_1 (empty squares), p_2 (stars) and $\bar{p}$ ($\equiv p_c$) (filled circles) for different values of Δ. The calculations have been performed with $k = 10^{-5}$ and $N_L = 300$.

shown in figure 4. The main plot of this figure shows the average stationary values of p_1 (empty squares), p_2 (stars) and p_c (filled circles) obtained from simulations in a lattice with 1.25×10^8 sites. As can be seen in this figure, for $\Delta = 0$ the critical probability $p_c \approx 0.31$ for the isotropic percolation in the cubic lattice is recovered [1]. As Δ increases from zero, the cluster grows with increasing preference in the sheets of higher probability (p_1), and becomes, thus, increasingly narrowed in the direction of stratification. We find that the lower probability (p_2) vanishes when $\Delta \approx 0.19$.

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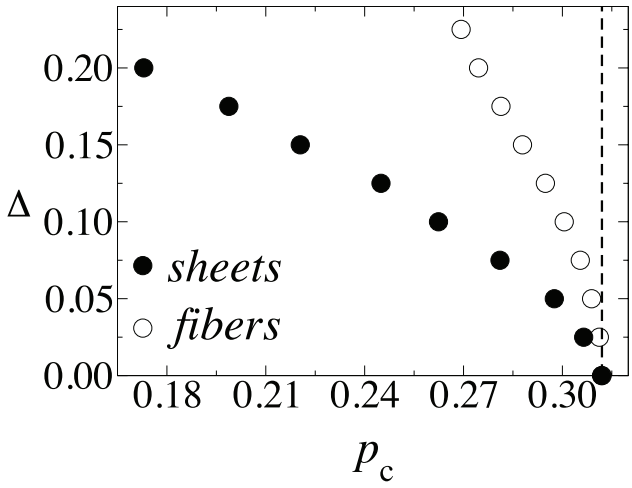

Figure 5. Critical probability p_c as a function of Δ for the cubic lattice calculated with the SOP model [28, 29] using two types of stratification: (i) 'sheets' (filled circles) and (ii) 'fibers' (empty circles), i.e. with one of the coordinates (x, y, z) randomly labeled 1 or 2. For $\Delta = 0$, the value $p_c \approx 0.31$ (dashed line) for the homogeneous cubic lattice is recovered in both cases, as expected. The calculations were performed with $k = 10^{-5}$ and $N_L = 300$.

The second three-dimensional model describes the percolation process with *fiber* stratification. In this case, each pair (x, y) of the cubic lattice is attributed one of the labels 1 or 2, whereas all sites (x_n, y_n, z) along the line $(x = x_n, y = y_n)$ have equal probability p_1 or p_2, respectively. One possible application of this model is the study of percolation in composites filled with fibers [2, 14]. In this model, as Δ increases the cluster growth takes place mainly along preferred lines (or fibers) of higher probability p_1. This type of stratification is evidently a case intermediate between the isotropic case and the stratification in sheets. As shown in figure 5, we find that $p_c^s(\Delta) < p_c^f(\Delta) < 0.31$ for $\Delta > 0$, where p_c^s and p_c^f refer to the critical points for stratification in sheets and fibers, respectively.

It should be emphasized that, although the stationary state obtained in the SOP model in fact corresponds to the critical point for percolation, the structures obtained in the simulations do not display the same statistical properties of the fractal percolating clusters of corresponding dimensionality due to the fluctuations around the critical point that are intrinsic to the SOP dynamics [28]. In fact, since the SOP model introduces a controlling mechanism to the original dynamical algorithm of growing percolation clusters [31], it can be adapted to dimensions $d > 2$ more easily than, for example, the self-avoiding walk method for obtaining the percolation threshold [25]. Further, the SOP model can be extended to the study of d-dimensional anisotropic percolation, with $d > 3$.

The quantities N_L and k controlling the growth dynamics dictated by the feedback mechanism, equation (3), of the SOP model are both observable and controllable. Thus, practical implementation of the SOP process in order to reach a self-organized critical state where the percolation probability is around the threshold p_c does indeed pose an experimentally feasible task. One example of possible application is in polymerization

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experiments, where the growth rate of the total polymer mass can be controlled by means of a device which monitors the concentrations of the different species of monomers at the inlet, as well as the total concentration of monomers at the outlet solution from the polymerization reactor [28]. As shown recently, the SOP model also suggests a conceivable experimental scheme for the generation of percolation hydrocarbons [20]. The process of vascularization (angiogenesis) of certain kinds of tumors, which display percolation-like scaling [41], can represent a possible biological application of the SOP model [33]. Further practical applications may involve the growth of percolating clusters in disordered or stratified lattices.

4. Conclusions

In conclusion, a model for self-organized percolation (SOP) of growing clusters in anisotropic random systems was presented. The model couples the dynamical model for the growth of anisotropic percolation clusters [31] with global controlling rules that regulate the growth rate of a percolation aggregate, leading the system to a self-organized critical state that corresponds to the percolation threshold [28, 29]. The SOP process was studied in the stratified square lattice with randomly allocated layers of two different types (concentrations p_1 or p_2) [25]. The dependence of the percolation threshold p_c on the amount of anisotropy, namely $\Delta = (p_2 - p_1)/2$, was studied and good agreement with results from traditional percolation methods [25] was found. Further, first results for the cubic lattice were presented using two types of stratification. By using the SOP model, it was possible to predict the threshold anisotropy amount in the cubic lattice, above which one of the lattice layers becomes statistically inactive (i.e. $p_2 = 0$). This threshold value is $\Delta \approx 0.19$. Finally, we showed that the critical probability $p_c(\Delta)$ in three-dimensional lattices with stratification in 'fibers' is larger than in the case of stratification in 'sheets'.

Thus, the SOP model has proven to be an efficient method for obtaining the critical point in anisotropic random media. As a result of the dynamic rules of the SOP model, the system is driven spontaneously to the percolation threshold with no need of tuning model parameters or repeating simulations with different values of p [3, 31]. In fact, previously studied percolation models, such as those in [27, 42, 43], also display this property. Further, the SOP model can be easily adapted to different algorithms of percolation [28], lattice types [34] and also to different types of stratification.

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