JOURNAL OF MATHEMATICAL PHYSICS **51**, 063304 (2010)

New representations of π and Dirac delta using the nonextensive-statistical-mechanics q-exponential function

M. Jauregui[1,a)] and C. Tsallis[1,2]
[1]*Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil*
[2]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*

(Received 25 March 2010; accepted 29 April 2010; published online 29 June 2010)

We present a generalization of the representation in plane waves of Dirac delta, $\delta(x)=(1/2\pi)\int_{-\infty}^{\infty}e^{-ikx}dk$, namely, $\delta(x)=[(2-q)/2\pi]\int_{-\infty}^{\infty}e_q^{-ikx}dk$, using the nonextensive-statistical-mechanics q-exponential function, $e_q^{ix}\equiv[1+(1-q)ix]^{1/(1-q)}$ with $e_1^{ix}\equiv e^{ix}$, x being any real number, for real values of q within the interval $[1,2[$. Concomitantly, with the development of these new representations of Dirac delta, we also present two new families of representations of the transcendental number π. Incidentally, we remark that the q-plane wave form which emerges, namely, e_q^{ikx}, is normalizable for $1<q<3$, in contrast to the standard one, e^{ikx}, which is not.
 [doi:10.1063/1.3431981]

I. INTRODUCTION

Dirac delta is a distribution that is used in almost all branches of physics. Various representations of it have been discovered along the time. For example, it can be represented as a limit of a Gaussian or as a linear combination of plane waves, being the last one strongly related to the Fourier transform (FT), as we will show later.

Dirac delta, $\delta(x)$, obeys the following fundamental property:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0), \tag{1}$$

where $f:\mathbb{R}\to\mathbb{C}$ is a well-behaved function. From the above equation, we can see that if $f(x)=1\ \ \forall x\in\mathbb{R}$, we get the normalization condition

$$\int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{2}$$

Also, choosing $f(x)=f(0)e^{ikx}$ in (1), we obtain

$$\int_{-\infty}^{\infty} \delta(x)e^{ikx}dx = 1, \tag{3}$$

i.e., the FT of $\delta(x)$ equals 1. Therefore, using the expression of the inverse FT, we obtain the following representation of Dirac delta:

[a)]Electronic mail: jauregui@cbpf.br.

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk, \tag{4}$$

which can be interpreted as a linear combination of plane waves. We can rewrite the above expression as

$$\delta(x) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L} e^{-ikx}dk, \tag{5}$$

then, Dirac delta also can be represented as the following improper limit:

$$\delta(x) = \lim_{L\to\infty}\frac{\sin(Lx)}{\pi x}. \tag{6}$$

In 1988, a possible generalization of Boltzmann–Gibbs statistical mechanics was proposed.[1] This new theory, sometimes referred to as *nonextensive-statistical mechanics*,[2] has been satisfactorily applied to handle a large number of physical phenomena (usually, metastable or quasistationary states of systems that are not consistent with the ergodic hypothesis; for example, systems in which long-range interactions or strong correlations exist).[3–17] Furthermore, the elaboration of nonextensive-statistical mechanics required the generalization of some mathematical functions (exponential, logarithm, etc.), operators (sum, product, FT, etc.), and theorems (central limit theorem).[18] Particularly, the generalization of the exponential function, namely, the q-exponential function, is defined by

$$e_q^x \equiv [1+(1-q)x]_+^{1/(1-q)} \quad (e_1^x \equiv e^x) \tag{7}$$

for any $x\in\mathbb{R}$, where the symbol $[y]_+$ means that $[y]_+=y$ if $y\geq 0$ and $[y]_+=0$ if $y<0$. For pure imaginary ix, e_q^{ix} can be defined to be the principal value of

$$e_q^{ix} \equiv [1+(1-q)ix]^{1/(1-q)} \quad (e_1^{ix} \equiv e^{ix}). \tag{8}$$

The main purpose of the present paper is to generalize the representation in plane waves of Dirac delta, introduced in Eq. (4), using the q-exponential function defined above.

II. REPRESENTATION OF DIRAC DELTA IN q-PLANE WAVES

A. Proposition

Let us introduce the following quantity:

$$\delta_q(x) \equiv \frac{1}{c(q)}\int_{-\infty}^{\infty} e_q^{-i\xi x}d\xi \quad \text{with } q\in[1,2[, \tag{9}$$

which can be interpreted as a linear combination of q-plane waves, where $c(q)$ is a constant that may depend on q. We intend to show later that $\delta_q(x)=\delta(x)$ for all $1\leq q<2$.

Analogous to (5), we may write

$$\delta_q(x) = \frac{1}{c(q)}\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda} e_q^{-i\xi x}d\xi \quad \text{with } q\in[1,2[, \tag{10}$$

therefore, by integrating, we can represent $\delta_q(x)$ as the following improper limit:

$$\delta_q(x) = \frac{2}{(2-q)c(q)}\lim_{\Lambda\to\infty}\frac{\sin\left\{\dfrac{2-q}{q-1}\arctan[(q-1)\Lambda x]\right\}}{x[1+(q-1)^2\Lambda^2x^2]^{(2-q)/2(q-1)}} \quad \text{with } q\in]1,2[. \tag{11}$$

.

B. The normalization constant $1/c(q)$ and the transcendental number π

The constant $c(q)$ must be equal to 2π at the $q\to 1^+$ limit. Furthermore, $c(q)$ can be found from the normalization condition (2). Thus, we have

$$c(q)=\frac{2}{(2-q)}\lim_{\Lambda\to\infty}\int_{-\infty}^{\infty}\frac{\sin\left\{\frac{2-q}{q-1}\arctan[(q-1)\Lambda x]\right\}}{x[1+(q-1)^2\Lambda^2x^2]^{(2-q)/2(q-1)}}dx. \tag{12}$$

Using the change of variables $z=(q-1)\Lambda x$, we obtain

$$c(q)=\frac{2}{(2-q)}\lim_{\Lambda\to\infty}\int_{-\infty}^{\infty}\frac{\sin\left(\frac{2-q}{q-1}\arctan z\right)}{z(1+z^2)^{(2-q)/2(q-1)}}dz. \tag{13}$$

As the integral does not depend on Λ, the limit symbol can be omitted. Therefore, we can write

$$c(q)=\frac{2}{(2-q)}\int_{-\infty}^{\infty}\frac{\sin[2\alpha(q)\arctan z]}{z(1+z^2)^{\alpha(q)}}dz, \tag{14}$$

where

$$\alpha(q)\equiv\frac{2-q}{2(q-1)}. \tag{15}$$

We easily verify that $\alpha:]1,2[\subset\mathbb{R}\to\mathbb{R}^+$ is a monotonically decreasing function of q.

In order to solve analytically the integral in (14), let us restrict to integer or half-integer values for $\alpha(q)$, more precisely, $1/2,1,3/2,\ldots$. This implies that q will be allowed to assume just certain rational values within the interval $]1,2[$, namely, $q=3/2,4/3,5/4,\ldots$. Using the change of variables $z=\tan\theta$ in Eq. (14), we obtain

$$c(q)=\frac{4}{2-q}\int_0^{\pi/2}\frac{\sin[2\alpha(q)\theta](\cos\theta)^{2\alpha(q)-1}}{\sin\theta}d\theta. \tag{16}$$

By using now the relation (A4) proved in the Appendix, the above expression yields

$$c(q)=\frac{4}{2-q}\sum_{k=0}^{\lfloor\alpha(q)+1/2\rfloor-1}(-1)^k\binom{2\alpha(q)}{2k+1}\int_0^{\pi/2}d\theta(\cos\theta)^{4\alpha(q)-2k-2}(\sin\theta)^{2k}. \tag{17}$$

We recall that the beta function, $B(x,y)$, is defined by

$$B(x,y)\equiv\int_o^{\pi/2}d\phi 2(\cos\phi)^{2x-1}(\sin\phi)^{2y-1}\quad\text{with}\quad x>0\quad\text{and}\quad y>0, \tag{18}$$

which is related to the gamma function by

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{19}$$

Therefore, using the expressions of beta function shown above, Eq. (17) can be written as

$$c(q)=\frac{4\alpha(q)}{2-q}\sum_{k=0}^{\lfloor\alpha(q)+1/2\rfloor-1}(-1)^k\frac{\Gamma\left(2\alpha(q)-k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(2k+2)\Gamma(2\alpha(q)-2k)}. \tag{20}$$

Let us rewrite now the above expression as

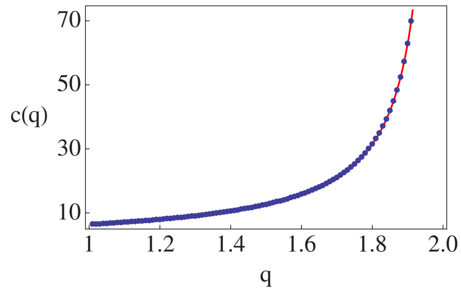

FIG. 1. (Color online) The dots were numerically obtained using expression (14), whereas the continuous curve is the plot of $c(q)$ given by Eq. (24).

$$c(q) = \frac{2}{2-q} S_{n_q}, \tag{21}$$

where

$$S_{n_q} \equiv n_q \sum_{k=0}^{\lfloor (n_q+1)/2 \rfloor -1} (-1)^k \frac{\Gamma\left(n_q - k - \frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(2k+2)\Gamma(n_q-2k)} \quad \text{with} \;\; n_q \equiv 2\alpha(q) \in \mathbb{N}. \tag{22}$$

When $n_q=1$ (which corresponds to $q=3/2$), we obtain straightforwardly that $S_1=\pi$. Also, it is straightforward to verify that S_2, S_3, and S_4 are equal to π. Using a symbolic computation software, we also verified that from $n_q=1$ to $n_q=5000(q=5002/5001)$, $S_{n_q}=\pi$. Hence, we state the following hypothesis:

$$\pi = n \sum_{k=0}^{\lfloor (n+1)/2 \rfloor -1} (-1)^k \frac{\Gamma\left(n - k - \frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(2k+2)\Gamma(n-2k)} \quad \forall\, n \in \mathbb{N}. \tag{23}$$

We thus found a countable infinite family of representations of the transcendental number π (see also Ref. 19).

Using relation (23) in (21), the expression of $c(q)$ becomes

$$c(q) = \frac{2\pi}{2-q}. \tag{24}$$

In addition to the above, this relation has been checked numerically to be correct not only for certain rational values of q within the interval $[1,2[$ but also for all real numbers within that interval (see Fig. 1). Therefore, we conjecture that the integral that appears in (14) equals π for *any* value of q within that interval. Consistently, we obtain another infinite family of representations of the number π, namely,

$$\pi = \int_{-\infty}^{\infty} \frac{\sin(2r \arctan z)}{z(1+z^2)^r} dz \quad \forall\, r \in \mathbb{R}^+. \tag{25}$$

This family is noncountable and contains Eq. (23) as a particular case.

Finally, expressions (9) and (11) of $\delta_q(x)$ become, respectively,

$$\delta_q(x) = \frac{2-q}{2\pi} \int_{-\infty}^{\infty} e_q^{-i\xi x} d\xi \quad \text{with} \;\; q \in [1,2[\tag{26}$$

and

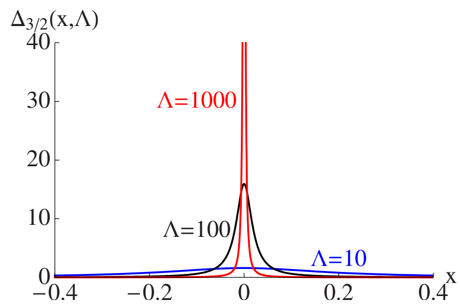

FIG. 2. (Color online) Plot of $\Delta_{3/2}(x,\Lambda)$ for different values of Λ. Similar results are obtained for any value of $q \in]1,2[$.

$$\delta_q(x) = \lim_{\Lambda\to\infty} \frac{\sin\left\{\frac{2-q}{q-1}\arctan[(q-1)\Lambda x]\right\}}{\pi x[1+(q-1)^2\Lambda^2 x^2]^{(2-q)/2(q-1)}} \quad \text{with } q \in]1,2[. \tag{27}$$

C. Dirac delta behavior of the distribution $\delta_q(x)$

Let us define the following distribution:

$$\Delta_q(x,\Lambda) \equiv \frac{\sin\left\{\frac{2-q}{q-1}\arctan[(q-1)\Lambda x]\right\}}{\pi x[1+(q-1)^2\Lambda^2 x^2]^{(2-q)/2(q-1)}} \quad \text{with } q \in]1,2[, \tag{28}$$

which is related to $\delta_q(x)$ through

$$\delta_q(x) = \lim_{\Lambda\to\infty} \Delta_q(x,\Lambda). \tag{29}$$

The plot of such a distribution (see Fig. 2) indicates that in the limit $\Lambda\to\infty$, $\Delta_q(x,\Lambda)$ will present a divergence at the origin and will be zero for all $x\neq 0$, i.e., at first glance, $\delta_q(x)$ appears to be a representation of Dirac delta.

Let us now consider an analytic function, $f:\mathrm{dom}\, f \subset \mathbb{R}\to\mathbb{C}$, which can be expanded in Taylor series around the origin such that the expression

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{30}$$

is valid for all $x \in \mathrm{dom}\, f$. Then we have

$$\int_{-\infty}^{\infty} f(x)\delta_q(x)dx = \int_{\mathrm{dom}\, f} f(x)\delta_q(x)dx. \tag{31}$$

Replacing $f(x)$ by its Taylor series, this expression yields

$$\int_{\mathrm{dom}\, f} f(x)\delta_q(x)dx = \lim_{\Lambda\to\infty}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\int_{\mathrm{dom}\, f}\frac{x^{k-1}\sin\left\{\frac{2-q}{q-1}\arctan[(q-1)\Lambda x]\right\}}{\pi[1+(q-1)^2\Lambda^2x^2]^{(2-q)/2(q-1)}}dx, \tag{32}$$

in which we must remark that q belongs to the interval $]1,2[$. If $\mathrm{dom}\, f$ is a bounded interval of $\mathbb{R}$, i.e., $\mathrm{dom}\, f =]a,b[$, with $a<b$, then using the change of variables $z=(q-1)\Lambda x$, we obtain

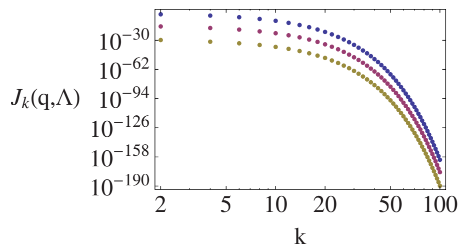

FIG. 3. (Color online) The k-dependence of $J_k(q,\Lambda)$, numerically obtained, considering $b=-a=1$ for $q=1.4$ and different values of Λ. From top to bottom: $\Lambda=10$, $\Lambda=10^{10}$, and $\Lambda=10^{20}$.

$$\int_{\mathrm{dom}\, f} f(x)\,\delta_q(x)dx = \lim_{\Lambda\to\infty}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{(q-1)^k\Lambda^k k!}\int_{(q-1)\Lambda a}^{(q-1)\Lambda b}\frac{z^{k-1}\sin\left(\frac{2-q}{q-1}\arctan z\right)}{\pi(1+z^2)^{(2-q)/2(q-1)}}dz. \tag{33}$$

The first term of the sum that appears above is

$$f(0)\lim_{\Lambda\to\infty}\int_{(q-1)\Lambda a}^{(q-1)\Lambda b}\frac{\sin\left(\frac{2-q}{q-1}\arctan z\right)}{\pi z(1+z^2)^{(2-q)/2(q-1)}}dz. \tag{34}$$

If $0<a<b$ or $a<b<0$, we straightforwardly see that the above expression is equal to zero. If $a<0<b$, then using relation (25) we obtain that expression (34) is equal to $f(0)$. Finally, if we have either $a=0$ or $b=0$ (with $a<b$), then, also using relation (25), we obtain that expression (34) is equal to $f(0)/2$.

In order to analyze the next terms of the sum given in (33), let us first rewrite them as

$$\lim_{\Lambda\to\infty} f^{(k)}(0)J_k(q,\Lambda) \quad \text{with } k\in\mathbb{N}, \tag{35}$$

where

$$J_k(q,\Lambda) \equiv \frac{1}{(q-1)^k\Lambda^k k!}\int_{(q-1)\Lambda a}^{(q-1)\Lambda b}\frac{z^{k-1}\sin\left(\frac{2-q}{q-1}\arctan z\right)}{\pi(1+z^2)^{(2-q)/2(q-1)}}dz \quad \text{with } k\in\mathbb{N}. \tag{36}$$

Here, $J_k(q,\Lambda)$ is a rapidly decreasing function of k (see Fig. 3), which makes the sum given in (33) converge, consistently with the finiteness of the domain of f in integral in Eq. (31). Moreover, from Fig. 3, we can infer that, in the limit $\Lambda\to\infty$, $J_k(q,\Lambda)\to 0$. Therefore, Eq. (33) implies that

$$\int_a^b f(x)\,\delta_q(x)dx = \begin{cases} f(0) & \text{if } a<0<b \\ f(0)/2 & \text{if either } a=0<b \text{ or } a<0=b \\ 0 & \text{if } 0\notin\,]a,b[. \end{cases} \tag{37}$$

In the case when $\mathrm{dom}\, f$ is unbounded, i.e., if $\mathrm{dom}\, f=]a,\infty[$, or $\mathrm{dom}\, f=]-\infty,b[$, or $\mathrm{dom}\, f=\mathbb{R}$, a similar analysis yields once again relation (37). Moreover, we numerically tested the validity of the mentioned relation using some types of functions and distributions (for example, the Gaussian and the Lorentzian). Hence, it seems reasonable to conjecture that, for a wide class of functions, $\delta_q(x)$ indeed is a representation of Dirac delta. Thus, we can finally write

$$\delta(x-x') = \frac{2-q}{2\pi}\int_{-\infty}^{\infty} e_q^{-i\xi(x-x')}d\xi \quad (q\in[1,2[). \tag{38}$$

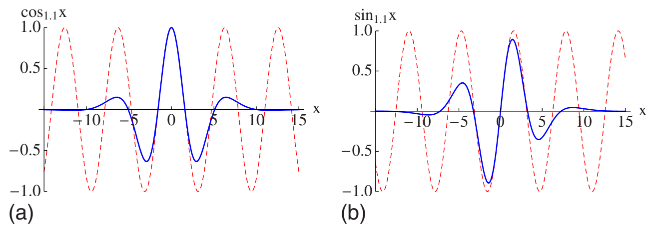

FIG. 4. (Color online) (a) $\cos_{1.1} x$ (continuous curve) and $\cos x$ (dashed curve). (b) $\sin_{1.1} x$ (continuous) and $\sin x$ (dashed). For $1 \leq q < 3$, $\cos_q x$ ($\sin_q x$) is an even (odd) function of x. For $1 < q < 3$, both functions $\cos_q x$ and $\sin_q x$ quickly decay when $|x| \to \infty$, in contrast to $\cos x$ and $\sin x$.

III. SQUARE INTEGRABILITY OF q-PLANE WAVES

Let us consider the following function:

$$\Psi(x) = Ne_q^{i\xi x} = N[\cos_q(\xi x) + i \sin_q(\xi x)] \quad \text{with } q \in \;]1,3[, \tag{39}$$

which can be interpreted as a stationary q-plane wave, where the q-generalized trigonometric functions are defined, for any $x \in \mathbb{R}$, by (see also Ref. 20)

$$\cos_q x \equiv \mathrm{Re}(e_q^{ix}) = \frac{\cos\left\{\dfrac{1}{q-1}\arctan[(q-1)x]\right\}}{[1+(q-1)^2x^2]^{1/2(q-1)}} \tag{40}$$

and

$$\sin_q x \equiv \mathrm{Im}(e_q^{ix}) = \frac{\sin\left\{\dfrac{1}{q-1}\arctan[(q-1)x]\right\}}{[1+(q-1)^2x^2]^{1/2(q-1)}}. \tag{41}$$

We illustrate these functions in Fig. 4.

We will determine now the value of the constant N using the normalization condition given by

$$\int_{-\infty}^{\infty} \Psi^*(x)\Psi(x)dx = 1. \tag{42}$$

Thus, we have

$$\frac{1}{N^2} = \int_{-\infty}^{\infty} e_q^{-i\xi x} e_q^{i\xi x} dx, \tag{43}$$

which, using the definition of the q-exponential function given in (8), can be written as

$$\frac{1}{N^2} = \int_{-\infty}^{\infty} \frac{1}{[1+(q-1)^2\xi^2x^2]^{1/(q-1)}} dx. \tag{44}$$

Using the change of variables $\tan\theta = (q-1)|\xi|x$, this relation yields

$$\frac{1}{N^2} = \frac{1}{(q-1)|\xi|}\int_{-\pi/2}^{\pi/2} (\cos\theta)^{(4-2q)/(q-1)} d\theta. \tag{45}$$

Therefore, we obtain that the normalization constant is given by

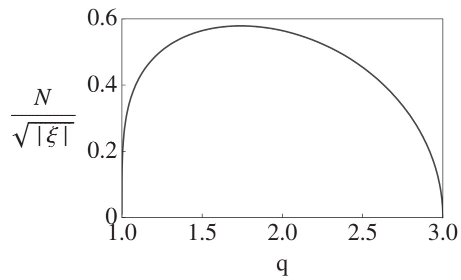

FIG. 5. The normalization constant N as a function of q. N goes to zero in the $q \to 1$ limit, thus recovering the well known non-normalizability of the plane waves.

$$N = \left[\frac{(q-1)|\xi|\Gamma\left(\frac{1}{q-1}\right)}{\sqrt{\pi}\Gamma\left(\frac{3-q}{2(q-1)}\right)} \right]^{1/2}. \tag{46}$$

Let us emphasize that the function $\psi(x)=e_1^{ikx}$ (plane wave) cannot be normalized, whereas q-plane waves, with $q \in]1,3[$, have a finite norm (see Fig. 5).

IV. CONCLUSIONS

From the analytical and numerical results shown in Sec. II, we conjecture Eq. (38), i.e., that $\delta_q(x)$ is indeed a generalization of the standard representation of Dirac delta in plane waves. Further research is welcome in order to establish which precise class of functions satisfies relation (37).

Concomitantly, we found two new families of representations, namely, expressions (23) and (25), of the transcendental number π. We tested the validity of such expressions for a set of values $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. A demonstration is still required in order to formally establish these new families of representations of π.

A generalization of FT, namely, the so-called q-Fourier transform (q-FT) was developed in order to generalize the central limit theorem. The possible analytic expression of the inverse q-FT remains to be found. It is known that, using the representation in plane waves of Dirac delta together with the expression of the direct FT, it is possible to find the expression of the inverse FT. Consequently, we suppose that the present q-generalization of the representation in plane waves of Dirac delta might be helpful in searching for an analytic expression of the inverse q-FT. Moreover, the present new representations of Dirac delta could be useful to handle some integrals that may appear in the analysis of certain physical phenomena.

Finally, we prove a physically appealing property, namely, that the q-plane wave form e_q^{ikx} is square integrable (in other words, normalizable) for $1<q<3$, in contrast to the standard form, e^{ikx}, which is not.

ACKNOWLEDGMENTS

We acknowledge fruitful discussions with E. M. F. Curado, R. S. Mendes, and F. D. Nobre, useful remarks by L. T. Cardoso, Y. Stein, and C. Vignat, as well as partial support by Faperj and CNPq (Brazilian agencies).

APPENDIX: TRIGONOMETRIC IDENTITY

We establish here an expression for $\sin[2\alpha(q)\theta]$, with $2\alpha(q) \in \mathbb{N}$ and $\theta \in \mathbb{R}$, written in terms of $\sin\theta$ and $\cos\theta$. First, we have

$$\sin[2\alpha(q)\theta] = \mathrm{Im}[(\cos\,\theta + i\,\sin\,\theta)^{2\alpha(q)}], \tag{A1}$$

then, using binomial expansion, we have

$$\sin[2\alpha(q)\theta] = \mathrm{Im}\left[\sum_{k=0}^{2\alpha(q)} \binom{2\alpha(q)}{k} (\cos\,\theta)^{2\alpha(q)-k}(\sin\,\theta)^k (i)^k\right] \tag{A2}$$

$$= -\sum_{\substack{k=1\\ (\mathrm{odd})}}^{2\alpha(q)} (i)^{k+1}\binom{2\alpha(q)}{k}(\cos\,\theta)^{2\alpha(q)-k}(\sin\,\theta)^k. \tag{A3}$$

Finally, this expression can be rewritten as follows:

$$\sin[2\alpha(q)\theta] = \sum_{k=0}^{\lfloor\alpha(q)+1/2\rfloor-1} (-1)^k \binom{2\alpha(q)}{2k+1}(\cos\,\theta)^{2\alpha(q)-2k-1}(\sin\,\theta)^{2k+1}, \tag{A4}$$

where we have used the *floor function* $\lfloor\ \rfloor$, defined, for any real number x, by $\lfloor x \rfloor = n$ such that $n \le x < n+1$, with $n \in \mathbb{Z}$.

[1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[2] *Nonextensive Entropy—Interdisciplinary Applications*, edited by M. Gell-Mann and C. Tsallis (Oxford University Press, New York, 2004); J. P. Boon, and C. Tsallis, Europhys. News **36**, 185 (2005); C. Tsallis, *Introduction to Nonextensive Statistical Mechanics* (Springer, New York, 2009).
[3] A. Upadhyaya, J.-P. Rieu, J. A. Glazier, and Y. Sawada, Physica A **293**, 549 (2001).
[4] K. E. Daniels, C. Beck, and E. Bodenschatz, Physica D **193**, 208 (2004).
[5] R. Arévalo, A. Garcimartín, and D. Maza, Eur. Phys. J. E **23**, 191 (2007).
[6] P. Douglas, S. Bergamini, and F. Renzoni, Phys. Rev. Lett. **96**, 110601 (2006); G. B. Bağci and U. Tirnakli, Chaos **19**, 033113 (2009).
[7] B. Liu and J. Goree, Phys. Rev. Lett. **100**, 055003 (2008).
[8] R. G. DeVoe, Phys. Rev. Lett. **102**, 063001 (2009).
[9] L. Borland, Phys. Rev. Lett. **89**, 098701 (2002).
[10] S. M. D. Queirós, Quant. Finance **5**, 475 (2005).
[11] L. F. Burlaga and A. F. Viñas, Physica A **356**, 375 (2005).
[12] L. F. Burlaga and N. F. Ness, Astrophys. J. **703**, 311 (2009).
[13] B. Bakar and U. Tirnakli, Phys. Rev. E **79**, 040103(R) (2009).
[14] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra, and A. Rapisarda, Phys. Rev. E **75**, 055101(R) (2007).
[15] J. C. Carvalho, R. Silva, J. D. do Nascimento, and J. R. de Medeiros, Europhys. Lett. **84**, 59001 (2008).
[16] R. M. Pickup, R. Cywinski, C. Pappas, B. Farago, and P. Fouquet, Phys. Rev. Lett. **102**, 097202 (2009).
[17] CMS Collaboration, J. High Energy Phys.02, (2010) 041.
[18] S. Umarov, C. Tsallis, and S. Steinberg, Milan J. Math. **76**, 307 (2008); S. Umarov, C. Tsallis, M. Gell-Mann, and S. Steinberg, J. Math. Phys. **51**, 033502 (2010).
[19] G. Gasper and M. Rahman, *Basic Hypergeometric Series* (Cambridge University Press, Cambridge, 2004); G. E. Andrews, *The Theory of Partitions* (Addison-Wesley, Reading, MA, 1976).
[20] E. P. Borges, J. Phys. A **31**, 5281 (1998).