# DIFFUSION: FROM CLASSICAL TO FRACTIONAL 

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## PART FIVE:

CONTINUOUS TIME RANDOM WALK (CTRW)
AND
FRACTIONAL DIFFUSION EQUATIONS

## I. Introduction

In the dynamics of continuous fluids, a tracer is a particle that travels with the local fluid velocity, but that has otherwise no influence on the properties of this fluid. It is a passive particle. The motion of passive tracers in fully developed, isotropic and homogeneous turbulence is well described by Brownian motion. Macroscopically it satisfies a Fick's type, local transport equation, and microscopically it is described by a random walk with Gaussian statistics.
However, in a turbulent system which contains coherent structures like vortices and magnetic islands that may trap particles for long times, and of zonal flows that advect tracers over long distances, this theory breaks down. Trapping in coherent structures and the presence of zonal flows will lead to 'memory' effects, to non-Markovian behavior, and imply that the tracer will undergo Lévy flights that will lead to non-Gaussian statistics.
In this Section we will develop a model for a random walk that incorporates memory effects in time and non-local effects in space. We will abandon the conditions that the walker, i..e. the particle, makes independent steps of fixed length at fixed points in time. We will adopt the model that a particle makes a step x after a time $t$. Then it waits again some time and makes another step. We will treat space and time as continuous variable and we will introduce the distribution $\psi(\mathbf{x}, t)$, which is the pdf that the walker takes a step $\mathbf{x}$ after a time interval $t$.
Transport of passive particles in a system with coherent structures will be characterized as strange diffusion and will lead to diffusion equations that contain fractional operators. In this regime we expect to find that the mean square displacement behaves like

$$
\begin{equation*}
<(\Delta x-<\Delta x>)^{2}>\propto t^{\alpha}, \quad \alpha \neq 1 . \tag{1}
\end{equation*}
$$

In this Section we will frequently make use of Fourier transform and of the Laplace transform. These transforms and their inverses are defined by

$$
\begin{gathered}
\hat{n}(\mathbf{k}, s)=\int_{0}^{\infty} d t \int_{-\infty}^{+\infty} d^{d} x n(\mathbf{x}, t) e^{-s t+i \mathbf{k} \cdot \mathbf{x}} \\
n(\mathbf{x}, t)=\frac{1}{2 \pi i} \frac{1}{(2 \pi)^{d}} \int_{-\infty}^{+\infty} d^{d} k \int_{c-i \infty}^{c+i \infty} d s \hat{n}(\mathbf{k}, s) e^{-i \mathbf{k} \cdot \mathbf{x}+s t}
\end{gathered}
$$

The Bromwich contour in the complex $s$-plane has to be taken to the right of any singularity that might occur in $\hat{n}((\mathbf{k}, s)$.
The transforms of convolution integrals are

$$
\begin{gathered}
F T \int d^{d} x^{\prime} f\left(\mathbf{x}^{\prime}\right) g\left(\mathbf{x}-\mathbf{x}^{\prime}\right)=\hat{f}(\mathbf{k}) \hat{g}(\mathbf{k}) \\
L T \int_{0}^{t} d t^{\prime} f\left(t^{\prime}\right) g\left(t-t^{\prime}\right)=\hat{f}(s) \hat{g}(s)
\end{gathered}
$$

## II. The Continuous Time Random Walk (CTRW)

Suppose that the probability distribution function that the walker is at some arbitrary position $\mathbf{x}$ at some time $t=0$ is $n(\mathbf{x}, 0)=n_{0}(\mathbf{x})$. We want to know the $\operatorname{pdf} n(\mathbf{x}, t)$ that the walker is at some position $\mathbf{x}$ at after a time $t$.

The walker (particle) arrives at the position $\mathrm{x}^{\prime}$ at time $t^{\prime}$. Then it remains immobile for a time interval $t-t^{\prime}$. At time $t$ he jumps to the position $\mathbf{x}$. The pdf of this process is $\psi\left(\mathbf{x}, t ; \mathbf{x}^{\prime}, t^{\prime}\right)$. we will assume that the system is homogeneous in space and in time, so that $\psi\left(\mathbf{x}, t ; \mathbf{x}^{\prime}, t^{\prime}\right)=\psi\left(\mathbf{x}-\mathbf{x}^{\prime}, t-t^{\prime}\right)$. Thus, the probability distribution function (pdf) $\psi(\mathbf{x}, t)$ is the pdf that the walker takes a step $\mathbf{x}$ after a time interval $t$.
The probability that a step of arbitrary length is taken after a time $t$ is

$$
\begin{equation*}
\Psi(t)=\int d^{d} x \psi(\mathbf{x}, t) \tag{2}
\end{equation*}
$$

This is the waiting time distribution.
The pdf that at least one step is taken somewhere in the interval $(0, t)$ is $\int_{0}^{t} \hat{\Psi}(\tau) d \tau$, and the probability that no step is taken during a time $t$ is

$$
\begin{equation*}
\Phi(t)=1-\int_{0}^{t} d t^{\prime} \Psi\left(t^{\prime}\right)=\int_{t}^{\infty} d t^{\prime} \Psi\left(t^{\prime}\right), \quad \Phi(0)=1, \Phi(\infty)=0 \tag{3}
\end{equation*}
$$

which is the pdf that the time interval between steps is greater than $t$. The average waiting time is $T=\int_{0}^{\infty} d t t \Psi(t)$, if this integral exists.

The $\operatorname{pdf} n(\mathbf{x}, t)$ that the walker is at $\mathbf{x}$ at time $t$ is given by a generalized master equation

$$
\begin{equation*}
n(\mathbf{x}, t)=\int d^{d} x^{\prime} \int_{0}^{t} d t^{\prime} \psi\left(\mathbf{x}-\mathbf{x}^{\prime}, t-t^{\prime}\right) n\left(\mathbf{x}^{\prime}, t^{\prime}\right)+\Phi(t) n_{0}(x) \tag{4}
\end{equation*}
$$

The system is assumed to be homogeneous in space and in time. The integrand of the first term on the right gives the pdf that, given that the walker is at $\mathbf{x}^{\prime}$ at $t^{\prime}$, he makes a step $\mathrm{x}-\mathrm{x}^{\prime}$ in a time interval $t-t^{\prime}$. The second term is the probability density that the particle does not make a step at all in the interval $t$, but remains at its initial position.

The Fourier-Laplace transform of (4) is

$$
\begin{equation*}
\hat{n}(\mathbf{k}, s)=\frac{1-\hat{\Psi}(s)}{s} \frac{1}{1-\hat{\psi}(\mathbf{k}, s)} \hat{n}_{0}(\mathbf{k}) . \tag{5}
\end{equation*}
$$

This is the Montroll-Weiss equation [1]. Here,

$$
\begin{equation*}
\hat{\Phi}(s)=\frac{1-\hat{\Psi}(s)}{s} \tag{6}
\end{equation*}
$$

is the LT of $\Phi(t)$ defined in (3). The density is completely determined when the function $\hat{\psi}(\mathbf{k}, s)$ is known.
Rewrite equation (5) as follows.

$$
\begin{equation*}
s \hat{n}(\mathbf{k}, s)-\hat{n}_{0}(\mathbf{k})=\hat{K}(\mathbf{k}, s) \hat{n}(\mathbf{k}, s) \tag{7}
\end{equation*}
$$

with

$$
\begin{equation*}
\hat{K}(\mathbf{k}, s)=\frac{s}{1-\hat{\Psi}(s)}[\hat{\psi}(\mathbf{k}, s)-\hat{\Psi}(s)] \tag{8}
\end{equation*}
$$

Applying the inverse Laplace -Fourier transform yields

$$
\begin{equation*}
\frac{\partial n(\mathbf{x}, t)}{\partial t}=\int d^{d} x^{\prime} \int_{0}^{t} d t^{\prime} K\left(\mathbf{x}-\mathbf{x}^{\prime}, t-t^{\prime}\right) n\left(\mathbf{x}^{\prime}, t^{\prime}\right) \tag{9}
\end{equation*}
$$

where the kernel is given by

$$
\begin{equation*}
K(\mathbf{x}, t)=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} d s \frac{s e^{s t}}{1-\hat{\Psi}(s)}[\hat{\psi}(\mathbf{x}, s)-\hat{\Psi}(s) \delta(\mathbf{x})] . \tag{10}
\end{equation*}
$$

Equation (9) is another version of the generalized, non-Markovian master equation (4). It is nonlocal and contains memory effects.

An important case occurs if the jump lengths in space and the steps in time are independent

$$
\begin{equation*}
\psi(\mathbf{x}, t)=\Psi(t) p(\mathbf{x}) \tag{11}
\end{equation*}
$$

Here, $\Psi(t)$ is the memory kernel that introduces non-Markovian behavior, whereas $p(\mathbf{x})$ is responsible for spatial correlations.
Equation (11) implies $\hat{\psi}(\mathbf{k}, s)=\hat{\Psi}(s) \hat{p}(\mathbf{k})$, so that

$$
\begin{equation*}
\hat{K}(\mathbf{k}, s)=-\hat{\chi}(s)[1-\hat{p}(\mathbf{k}], \tag{12}
\end{equation*}
$$

with

$$
\begin{equation*}
\hat{\chi}(s)=\frac{s \hat{\Psi}(s)}{1-\hat{\Psi}(s)} \tag{13}
\end{equation*}
$$

Then, the master equation (7) can be written as

$$
\begin{equation*}
s \hat{n}(\mathbf{k}, s)-\hat{n}_{0}(\mathbf{k})=-\hat{\chi}(s)[1-\hat{p}(\mathbf{k}] \hat{n}(\mathbf{k}, s) \tag{14}
\end{equation*}
$$

and (9) becomes

$$
\begin{equation*}
\left.\frac{\partial n(\mathbf{x}, t)}{\partial t}=\int_{0}^{t} d t^{\prime} \chi\left(t-t^{\prime}\right)\right)\left\{-n\left(\mathbf{x}, t^{\prime}\right)+\int d^{d} x^{\prime} p\left(\mathbf{x}-\mathbf{x}^{\prime}\right) n\left(\mathbf{x}^{\prime}, t^{\prime}\right)\right\} \tag{15}
\end{equation*}
$$

This is the Montroll-Shlesinger equation [2].

## A. A derivation of the Montroll-Weiss equation (5)

In order to derive equation (5) we introduce two additional pdf's.
The pdf that the walker arrives at the position $\mathbf{x}$ at time $t$ at the jth step is denoted by $n_{j}(\mathbf{x}, t)$. The initial position at $t=0$ is obtained from the $\operatorname{pdf} n_{0}(\mathbf{x})$.
Further, we introduce the function $Q(\mathbf{x}, t)$ which denotes the pdf that the walker arrives at x at time $t$.
Then, it is clear that

$$
Q(\mathbf{x}, t)=\sum_{j=0} n_{j}(\mathbf{x}, t)
$$

and

$$
n_{j+1}(\mathbf{x}, t)=\int d^{d} x^{\prime} \int_{0}^{t} d t^{\prime} \psi\left(\mathbf{x}-\mathbf{x}^{\prime}, t-t^{\prime}\right) n_{j}\left(\mathbf{x}^{\prime}, t^{\prime}\right)
$$

By summing the latter expression over all steps taken during the time $t$, one obtains

$$
Q(\mathbf{x}, t)=\int d^{d} x^{\prime} \int_{0}^{t} d t^{\prime} \psi\left(\mathbf{x}-\mathbf{x}^{\prime}, t-t^{\prime}\right) Q\left(\mathbf{x}^{\prime}, t^{\prime}\right)+n_{0}(\mathbf{x}) \delta(t)
$$

The pdf that the walker is at the position $\mathbf{x}$ at time $t$ is given by the probability that it arrives at $\mathbf{x}$ at an earlier time $\tau$ and stays there until time $t$

$$
n(\mathbf{x}, t)=\int_{0}^{t} d \tau \Phi(t-\tau) Q(\mathbf{x}, \tau)
$$

where $\Phi(t)$, given by (3), is the pdf that no step occurs during the time $t$.
Take the Fourier-Laplace transform of the last two expressions. This yields

$$
\hat{Q}(\mathbf{k}, s)=\hat{\psi}(\mathbf{k}, s) \hat{Q}(\mathbf{k}, s)+\hat{n}_{0}(\mathbf{k})
$$

and

$$
\hat{n}(\mathbf{k}, s)=\hat{\Phi}(s) \hat{Q}(\mathbf{k}, s) .
$$

Upon eliminating $\hat{Q}(\mathbf{k}, s)$ from the last equation and applying (6) one recovers the MontrollWeiss equation (5).

## B. A discrete version of the generalized master equation

A spatially discrete version of the generalized master equation (4) for continuous time and for non-local spatial jumps is [3]

$$
\begin{equation*}
n_{j}(t)=\int_{0}^{t} d \tau\left\{\sum_{n=1}^{\infty} A_{j, n}(t-\tau) n_{j-n}(\tau)+\sum_{n=1}^{\infty} B_{j, n}(t-\tau) n_{j+n}(\tau)\right\}+\Phi(t) \delta_{j, m} \tag{16}
\end{equation*}
$$

The first term represents jumps to the 'right'and the second one jumps to the 'left'. The summations over $n$ mean that jumps from all other sites $j \pm n$ to the site $j$ are possible.

The last term is the probability that the particle does not make a jump but stays at its initial site $m, \delta_{j, m}$ being the Kronecker symbol.
The limit of a continuous distribution of jumps is obtained by taking the limits $j \rightarrow$ $x, j-n \rightarrow x^{\prime}$ in the first term on the right of (16) and $j \rightarrow x, j+n \rightarrow x^{\prime}$ in the second term

$$
\begin{gathered}
n(x, t)=\int_{0}^{t} d \tau\left\{\int_{-\infty}^{x} d x^{\prime} A\left(x^{\prime}, x-x^{\prime}, t-\tau\right) n\left(x^{\prime}, \tau\right)+\int_{x}^{\infty} d x^{\prime} B\left(x^{\prime}, x-x^{\prime}, t-\tau\right) n\left(x^{\prime}, \tau\right)\right\} \\
+\Phi(t) n_{0}(x) .
\end{gathered}
$$

Assume that the functions $A$ and $B$ can be factorized as follows,
$A\left(x^{\prime}, x-x^{\prime}, t-\tau\right)=A\left(x^{\prime}\right) \psi\left(x-x^{\prime}, t-\tau\right), \quad B\left(x^{\prime}, x-x^{\prime}, t-\tau\right)=B\left(x^{\prime}\right) \psi\left(x^{\prime}-x, t-\tau\right)$.
The dependence of $A$ and $B$ on $x^{\prime}$ means that the probability to make a step $\left|x-x^{\prime}\right|$ in a time $t-\tau$ depends on the initial position $x^{\prime}$. The discrete master equation (16) now becomes

$$
\begin{equation*}
n(x, t)=\int_{0}^{t} d \tau \int_{-\infty}^{+\infty} d x^{\prime} \Lambda\left(x, x^{\prime}, t-\tau\right) n\left(x^{\prime}, \tau\right)+\Phi(t) n_{0}(x), \tag{17}
\end{equation*}
$$

with

$$
\Lambda\left(x, x^{\prime}, t-\tau\right)=\psi\left(\left|x-x^{\prime}\right|, t-\tau\right)\left[A\left(x^{\prime}\right) \theta\left(x-x^{\prime}\right)+B\left(x^{\prime}\right) \theta\left(x^{\prime}-x\right)\right]
$$

$\theta(x)$ being the Heaviside function. The normalization requires $A(x)+B(x)=1$. If jumps to the left and to the right have equal probabilities, then $A=B=1 / 2$.

## C. Non-Markovian and Gaussian limits

The time stepping is a Markov process if the probability $\Phi(t)$ that no step occurs in the interval $(t, t+\Delta t)$ is independent of $t$ and equal to $1-\Delta t / T$. Then,

$$
\begin{equation*}
\Phi(t+\Delta t)=\Phi(t)\left(1-\frac{\Delta t}{T}\right) \tag{18}
\end{equation*}
$$

so that

$$
\begin{equation*}
\Phi(t)=\exp -\frac{t}{T}, \quad \Psi(t)=\frac{1}{T} \exp -\frac{t}{T} \tag{19}
\end{equation*}
$$

where $T=\int_{0}^{\infty} d t t \Psi(t)$ is the average waiting time .
The Laplace transforms of these functions are

$$
\begin{equation*}
\hat{\Psi}(s)=\frac{1}{1+s T}, \quad \hat{\Phi}(s)=\frac{T}{1+s T} \tag{20}
\end{equation*}
$$

so that (12) becomes

$$
\begin{equation*}
\hat{K}(\mathbf{k}, s)=T^{-1}[\hat{p}(\mathbf{k})-1] . \tag{21}
\end{equation*}
$$

The Fourier-Laplace transform (14) of the master equation now reads

$$
\begin{equation*}
s \hat{n}(\mathbf{k}, s)-\hat{n}_{0}(\mathbf{k})=\frac{1}{T}[\hat{p}(\mathbf{k})-1] \hat{n}(\mathbf{k}, s) \tag{22}
\end{equation*}
$$

This equation is Markovian.
Let us assume that, in addition, the distribution of step sizes is Gaussian,

$$
p(x)=\frac{1}{\sqrt{2 \pi \sigma}} e^{-x^{2} / 2 \sigma}, \quad \hat{p}(k)=e^{-\sigma k^{2} / 2} \approx 1-\frac{1}{2} \sigma k^{2} .
$$

Upon substitution of this expression into (22) one obtains the Laplace-Fourier transform of the master equation in the form,

$$
\begin{equation*}
\left(s-D k^{2}\right) \hat{n}(k, s)=\hat{n}_{0}(k), \quad D=\frac{\sigma}{2 T} . \tag{23}
\end{equation*}
$$

This is the Laplace-Fourier transform of our classical diffusion (Fick's) equation of Part I.

## III. Distributions with long tails

Since we deal with diffusion problems, we are interested in the behavior of systems on long time- and length-scales, actually we want to find out what happens on macroscopic scales. We do not need to know what happens on small scales. From this point of view, the CRTW master equation contains far too much information for the description of transport on macroscopic scales. We do not require a full kinetic description of the underlying random walk, but we are interested in the continuum (fluid) limit of the master equation. The information we are looking for is contained in the large scales i.e. in the tails of the probability distribution function $\Psi(t)$ and $p(\mathbf{x})$. In Laplace-Fourier space this means that we do not need to know the full pdf's $\hat{\Psi}(s)$ and $\hat{p}(\mathbf{k})$ but basically we need only their asymptotic limits for $s \rightarrow 0, \mathbf{k} \rightarrow 0$. We already applied this point of view at the end of the previous section.

Let us consider the situation where the waiting time distribution $\Psi(t)$ does not decay exponentially, like (18), but decays as a power of $t$ for large $t$,

$$
\begin{equation*}
\Psi(t)=\frac{A}{t^{1+\beta}}, \quad 0<\beta \leq 1, t \rightarrow \infty \tag{24}
\end{equation*}
$$

This term represents the $t \rightarrow \infty$ tail of the distribution. The total probability $\int d t \Psi(t)$ is equal to unity and exists for $\beta>0$ and the average waiting time $\int d t t \Psi(t)$ exists and is finite for $\beta>1$.

Consider the expansion of the Laplace transform $\hat{\Psi}(s)$ for small values of $s$. For $\beta>1$,
the total probability is given by $\hat{\Psi}(s=0)=1$ and the average waiting time by $T=$ $-d \hat{\Psi}(s) /\left.d s\right|_{s=0}$. Hence, the leading order terms in the small $s$ expansion read

$$
\begin{equation*}
\hat{\Psi}(s)=1-T s \tag{25}
\end{equation*}
$$

Upon substitution this expression into (13), it is seen that we recover (22), i.e., for large times we recover the Markovian limit! Therefore, we limit the discussion of (24) to the interval $0<\beta \leq 1$.
The Laplace transform of (24) for $0<\beta \leq 1$ is in the limit of small values of $s$

$$
\begin{equation*}
\hat{\Psi}(s)=1-\tau_{D}^{\beta} s^{\beta}, \tag{26}
\end{equation*}
$$

with $A=\beta \Gamma^{-1}(1-\beta) \tau_{D}^{\beta}$. The constant $\tau_{D}$ is only equal to the average waiting time if $\beta=1$,

$$
<t>=-\left.\frac{d}{d s} \hat{\Psi}\right|_{s=0}=\left.\beta \tau_{D}^{\beta} s^{\beta-1}\right|_{s=0}
$$

Making use of the Tauberian theorem that says that the inverse Laplace transform of a power in $s$ yields a power in $t$,

$$
\begin{equation*}
L^{-1}\left[s^{\gamma}\right]=\frac{-\gamma}{\Gamma(1-\gamma)} t^{-1-\gamma}, \quad s>0 \tag{27}
\end{equation*}
$$

it is seen that (24) is recovered.
The appearance of power laws in representations of pdf's means that these distributions are scale invariant. The algebraic decay of $\Psi(t)$ and /or $p(x)$ implies that there is not a characteristic transport scale. It also means that the distribution of trapping and/or flight events is self-similar.
Further, we assume that the spatial distribution corresponds to a Lévy-type distribution which behaves like

$$
\begin{equation*}
\hat{p}(\mathbf{k})=1-b^{\alpha}|k|^{\alpha}, \quad k \rightarrow 0, \quad 0<\alpha \leq 2 \tag{28}
\end{equation*}
$$

We have seen in Part I that this corresponds to Lévy distributions and to Lévy flights.
The Fourier-Laplace transform of the CTRW master equation (14) (or (5)) reads to leading order

$$
\begin{equation*}
s \hat{n}(\mathbf{k}, s)-\hat{n}_{0}(\mathbf{k})=-\hat{\chi}(s) b^{\alpha} k^{\alpha} \hat{n}(\mathbf{k}, s), \quad 0<\alpha \leq 2, \quad 0<\beta \leq 1, \quad s, k \rightarrow 0 \tag{29}
\end{equation*}
$$

with, to leading order,

$$
\begin{equation*}
\hat{\chi}(s)=s^{1-\beta} . \tag{30}
\end{equation*}
$$

This expression depends

- on the global dimensionality $d$,
- on the two exponents $\alpha$ and $\beta$,
- on the characteristic time $\tau_{D}$ and the characteristic length $b$.


## A. The density profile

The expression (29) can be written in the form

$$
\begin{equation*}
\hat{n}(\mathbf{k}, s)=\frac{\tau_{D}^{\beta} s^{\beta-1}}{\tau_{D}^{\beta} s^{\beta}+b^{\alpha}|k|^{\alpha}} \hat{n}_{0}(\mathbf{k}) \tag{31}
\end{equation*}
$$

Its inverse Laplace-Fourier transform is obtained from

$$
\begin{equation*}
n(\mathbf{x}, t)=\frac{1}{2 \pi i} \frac{1}{(2 \pi)^{d}} \int_{-\infty}^{+\infty} d^{d} k \int_{c-i \infty}^{c+i \infty} d s \hat{n}(\mathbf{k}, s) e^{-i \mathbf{k} \cdot \mathbf{x}+s t} . \tag{32}
\end{equation*}
$$

Substitute (31) and introduce the variables

$$
\begin{equation*}
\hat{s}=\frac{\tau_{D} s}{\left(b^{\alpha} k^{\alpha}\right)^{1 / \beta}}, \hat{t}=\left(b^{\alpha} k^{\alpha}\right)^{1 / \beta} \frac{t}{\tau_{D}} \tag{33}
\end{equation*}
$$

Then,

$$
\begin{equation*}
n(\mathbf{x}, t)=\int_{-\infty}^{+\infty} \frac{d^{d} k}{(2 \pi)^{d}} \hat{d}_{0}(\mathbf{k}) e^{-i \mathbf{k} \cdot \mathbf{x}} \int_{c-i \infty}^{c+i \infty} \frac{d \hat{s}}{2 \pi i} e^{\hat{s} \hat{t}} \frac{\hat{s}^{\beta-1}}{\hat{s}^{\beta}+1} \tag{34}
\end{equation*}
$$

The function that appears under the integral

$$
\frac{\hat{s}^{\beta-1}}{\hat{s}^{\beta}+1}
$$

is just the Laplace transform of the Mittag-Leffler function $E_{\beta}\left(-\hat{t}^{\beta}\right)$, which is defined as

$$
\begin{equation*}
E_{\beta}(z)=\sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\beta n+1)}, \quad \beta>0, z \in C \tag{35}
\end{equation*}
$$

This is an entire function of $\beta$ and approaches $\exp z$ in the limit $\beta \rightarrow 1$. The behavior of the Mittag-Leffler function for large values of its argument is

$$
E_{\beta}(z) \approx-\frac{\sin \beta}{\beta} \frac{\Gamma(\beta)}{z}
$$

It follows that (32) can be written as

$$
\begin{equation*}
n(\mathbf{x}, t)=\int_{-\infty}^{+\infty} \frac{d^{d} k}{(2 \pi)^{d}} \hat{n}_{0}(\mathbf{k}) e^{-i \mathbf{k} \cdot \mathbf{x}} E_{\beta}\left(-b^{\alpha} k^{\alpha} t^{\beta} / \tau_{d}^{-\beta}\right) \tag{36}
\end{equation*}
$$

Introducing the variables

$$
\begin{equation*}
\hat{\mathbf{k}}=b\left(\frac{t}{\tau_{D}}\right)^{\beta / \alpha} \mathbf{k}, \quad \hat{\mathbf{x}}=\frac{\mathbf{x}}{b}\left(\frac{\tau_{D}}{t}\right)^{\beta / \alpha} \tag{37}
\end{equation*}
$$

one obtains in the one-dimensional case

$$
\begin{equation*}
n(x, t)=b^{-1}\left(\frac{t}{\tau_{D}}\right)^{-\beta / \alpha} \int \frac{d \hat{k}}{2 \pi} e^{-i \hat{k} \hat{x}} E_{\beta}\left(-\hat{k}^{\alpha}\right) \tag{38}
\end{equation*}
$$

where we have taken the initial condition $n_{0}(\mathbf{x})=\delta(\mathbf{x})$ i.e. $\hat{n}(\mathbf{k}, 0)=1$. This expression shows that the solution to (29) has the scaling form

$$
\begin{equation*}
n(x, t)=t^{-\beta / \alpha} G_{\alpha \beta}\left(\frac{x}{t^{\beta / \alpha}}\right) . \tag{39}
\end{equation*}
$$

## B. The mean-square displacement

The form of the function $G$ depends on the parameters $\alpha$ and $\beta$ and on the diffusion regime. Equation (39) leads immediately to the following expression for the mean square displacement

$$
\begin{equation*}
<x^{2}(t)>_{\alpha \beta}=\int d x x^{2} t^{-\beta / \alpha} G_{\alpha \beta}\left(\frac{x}{t^{\beta / \alpha}}\right) . \tag{40}
\end{equation*}
$$

This means that we have obtained the scaling relation

$$
\begin{equation*}
<x^{2}(t)>_{\alpha \beta}=M_{\alpha \beta} t^{\mu}, \quad M_{\alpha \beta}=\int d q q^{2} G_{\alpha \beta}(q) \tag{41}
\end{equation*}
$$

with diffusion exponent

$$
\begin{equation*}
\mu=\frac{2 \beta}{\alpha} . \tag{42}
\end{equation*}
$$

This leads to a criterium for the character of the diffusion process:
$-\beta<\alpha / 2 \leq 1 \quad$ sub-diffusive, strange,
$-\beta=\alpha / 2 \leq 1 \quad$ diffusive, classical, anomalous,
$-\alpha / 2<\beta \leq 1 \quad$ super-diffusive,strange,
$-\beta=\alpha \leq 1 \quad$ free-streaming, strange.

Note that not only the scaling of the second moment, but that the scaling of all moments can be obtained from (39),

$$
\begin{equation*}
<x^{p}(t)>\propto t^{p \beta / \alpha} . \tag{43}
\end{equation*}
$$

## C. The diffusion equation

Let us return to the CTRW master equation in the form (29) and rewrite this expression as follows

$$
\begin{equation*}
\left(s^{\beta}+|k|^{\alpha}\right) \hat{n}(\mathbf{k}, s)=s^{\beta-1} \hat{n}_{0}(\mathbf{k}) \tag{44}
\end{equation*}
$$

where we have taken $\tau_{D} s \rightarrow s$ and $b k \rightarrow k$ (with $t / \tau_{D} \rightarrow t$ and $x / b \rightarrow x$ ). The inverse transform of this equation can be expressed in terms of fractional derivatives. One can write

$$
\begin{equation*}
{ }_{0}^{c} D_{t}^{\beta} n(\mathbf{x}, t)=D_{|\mathbf{x}|}^{\alpha} n(\mathbf{x}, t), \tag{45}
\end{equation*}
$$

where

$$
\begin{equation*}
-|\mathbf{k}|^{\alpha} \rightarrow \frac{\partial^{\alpha}}{\partial|\mathbf{x}|^{\alpha}}=D_{|\mathbf{x}|}^{\alpha} \tag{46}
\end{equation*}
$$

is the Riesz fractional derivative and

$$
\begin{equation*}
s^{\beta} \hat{f}(s)-s^{\beta-1} f(t=0) \rightarrow_{0}^{c} D_{t}^{\beta} f(t) \tag{47}
\end{equation*}
$$

is the Caputo fractional derivative.
There exists an extensive literature on fractional derivatives. The definitions of the fractional derivatives are given in the Section VI together with a few calculational rules.

## IV. The Standard Long Tail CTRW (SLT-CTRW)

An important subclass occurs for $\alpha=2$ when the spatial distribution is Gaussian with $b=\sigma / \sqrt{2 d}$. This case is called the Standard Long Tail CTRW.
The CTRW master equation (31) reads

$$
\begin{equation*}
\hat{n}(\mathbf{k}, s)=\frac{\tau_{D}^{\beta} s^{\beta-1}}{\tau_{D}^{\beta} s^{\beta}+b^{2}|k|^{2}} \hat{n}_{0}(\mathbf{k}), \quad 0<\beta \leq 1, \quad s, k \rightarrow 0 . \tag{48}
\end{equation*}
$$

In this case one can easily derive the mean square displacement.
From (48) one finds

$$
\begin{equation*}
-\left.\frac{\partial^{2} \hat{n}(\mathbf{k}, s)}{\partial \mathbf{k} \cdot \partial \mathbf{k}}\right|_{\mathbf{k}=0}=2 d b^{2} \tau_{D}^{-\beta} s^{-\beta-1} \tag{49}
\end{equation*}
$$

According to a Tauberian theorem, the inverse Laplace transform of a power in $s$ yields a power in $t$, so that we obtain the MSD

$$
\begin{equation*}
<r^{2}(t)>=-\left.\frac{\partial^{2} \hat{n}(\mathbf{k}, t)}{\partial \mathbf{k} \cdot \partial \mathbf{k}}\right|_{\mathbf{k}=0}=\frac{2 d b^{2}}{\Gamma(\beta+1)}\left(\frac{t}{\tau_{D}}\right)^{\beta}, \quad b=\sigma / \sqrt{2 d} \tag{50}
\end{equation*}
$$

Hence, the exponent $\beta$ that occurs in the waiting time distribution $\Psi(t)$, determines the diffusion exponent in the MSD. It is seen that the SLT-CTRW implies strange and subdiffusive transport for $0<\beta<1$.

Diffusion in a stochastic magnetic field would correspond to the exponent $\beta=1 / 2$.
Equation (48) can be rewritten in the form

$$
\begin{equation*}
s \hat{n}(\mathbf{k}, s)-\hat{n}(\mathbf{k}, 0)=-\hat{\chi}(s) k^{2} n(\mathbf{k}, s), \quad \hat{\chi}(s)=s^{1-\beta} \tag{51}
\end{equation*}
$$

The inverse FL transforms yield

$$
\begin{equation*}
\frac{\partial n}{\partial t}=D_{0} \int_{0}^{t} d t_{1} \chi\left(t_{1}\right) \nabla^{2} n\left(\mathbf{x}, t-t_{1}\right) \tag{52}
\end{equation*}
$$

where $\chi(t)$ is the inverse Laplace transform of $\hat{\chi}(s)=s^{1-\beta}$. Equation (52) is the nonMarkovian diffusion equation.
The inverse transform of the long time contribution to $\hat{\chi}(s)$ is

$$
\begin{equation*}
\chi(t)=-\frac{1-\beta}{\Gamma(\beta)} \frac{1}{t^{2-\beta}} \tag{53}
\end{equation*}
$$

However, there will also exists a contribution to the waiting time distribution at shorter times so that

$$
\Psi(t)=f(t)+\frac{A}{t^{1+\beta}} .
$$

Let's assume that this contribution $f(t)$ is Markovian. This means that $f(t) \propto \exp -t / T$. Using (19) and (??) it follows that this bulk contributes a constant to $\hat{\chi}(s)$ for larger values of $s$, so that we find instead of (53)

$$
\begin{equation*}
\chi(t)=A \delta(t)-\frac{1-\beta}{\Gamma(\beta)} \frac{1}{t^{2-\beta}} . \tag{54}
\end{equation*}
$$

It follows that we can write the non-Markovian diffusion equation (52) in the form

$$
\begin{equation*}
\frac{\partial n}{\partial t}-D_{0} \nabla^{2} n(\mathbf{x}, t)=-D_{0} \frac{1-\beta}{\Gamma(\beta)} \int_{t_{\min }}^{t} d t_{1} \frac{1}{t_{1}^{2-\beta}} \nabla^{2} n\left(\mathbf{x}, t-t_{1}\right) \tag{55}
\end{equation*}
$$

where we have introduced a cut-off in the integral on the right in order to avoid the singularity.
The left-hand side of this diffusion equation is just the classical Fick's equation. The right-hand side represents the anomalous part. Note that this contribution vanishes for $\beta=1$, as it should!
The introduction of the cut-off seems to be quite arbitrary. However, it can be justified as follows.
Multiply (55) with $x^{2}$ and integrate over space. It is found that in 1D the MSD satisfies the equation

$$
\frac{\partial}{\partial t}<x^{2}(t)>-2 D_{0}=-2 D_{0} \frac{1-\beta}{\Gamma(\beta)} \int_{t_{\min }}^{t} d t \frac{1}{t^{2-\beta}}
$$

The RHS contains a term proportional to $t^{-1+\beta}$ and a constant term. Since $\beta<1$, this constant would dominate for $t \rightarrow \infty$. This is a spurious effect, so that this term has to cancel against the constant term on the left. The result is

$$
\begin{equation*}
t_{\text {min }}=\Gamma(\beta)^{-1 /(1-\beta)} \tag{56}
\end{equation*}
$$

so that the MSD is

$$
\begin{equation*}
<x^{2}(t)>=2 D_{0} \frac{t^{\beta}}{\Gamma(1+\beta)} . \tag{57}
\end{equation*}
$$

Upon comparing this expression with the general scaling law (41) for $\alpha=2$, we see that (57) has the correct behavior in time. It can be shown that this will also be the case for all higher moments of the distribution function at the same value of $t_{\text {min }}$.

## V. Lévy walks

In a discrete random walk with steps of finite size, the walker can only travel a limited distance in N steps. This idea of a diffusion front beyond which the probability is zero, gets lost in the continuous time limit. In this limit, there exist always a positive probability for the walker to be at some place at any time. A diffusion front can be obtained in CTRW by considering a coupled space-time memory.
Therefore, let us return to equation (11) and consider the spatio-temporal coupled memory kernel

$$
\begin{equation*}
\psi(\mathbf{x}, t)=p(\mathbf{x} \mid t) \Psi(t) \tag{58}
\end{equation*}
$$

with

$$
\begin{equation*}
\Psi(t)=A t^{-1-\beta} \tag{59}
\end{equation*}
$$

for large values of $t$, and

$$
\begin{equation*}
p(\mathbf{x} \mid t)=\delta(r-V(r) t), \quad V(r)=B r^{\alpha} . \tag{60}
\end{equation*}
$$

The $\operatorname{pdf} p(\mathbf{x} \mid t)$ is the conditional probability that the length of the flight is $\mathbf{x}$, given that it took a time $t$ to complete, and $\Psi(t)$ is the probability that the flight time is $t$.
This kernel couples the step size with the waiting time. Steps of arbitrary lengths may be taken, but larger distances take longer times. In a fixed time the particle will reach a finite shell in space. This kernel allows for enhanced, super-diffusion.
An obvious case occurs when the velocity $V$ does not depend on the step length, but is constant. This represents ballistic motion.
In the Richardson model of turbulence, that was discussed in Part III, the energy increases with the size of the eddy. This means that also the velocity increases with size. If the energy flux through the scales is taken to be invariant, one finds $V(r) \propto r^{1 / 3}$, leading to the Kolmogorov spectrum $k^{-5 / 3}$ (see Part III). Values $0<\alpha<1$ correspond to $\nu>1$.
Values $\alpha<0$ i.e. $\nu>1$ would correspond to cases where the velocity decreases for larger scale lengths.
Write (58)-(60) in the equivalent form

$$
\begin{equation*}
\psi(\mathbf{x}, t)=A t^{-1-\beta} \delta\left(r-t^{\nu}\right) \tag{61}
\end{equation*}
$$

with $\nu=1 / 1-\alpha>0$.
An obvious case occurs when the velocity $V$ does not depend on the step length, but is constant. This represents ballistic motion and occurs for $\alpha=0$, i.e. $\nu=1$.
In the Richardson model of turbulence, that was discussed in Part III, the energy increases with the size of the eddy. This means that also the velocity increases with size. If the energy flux through the scales is taken to be invariant, one finds $V(r) \propto r^{1 / 3}$, leading to the Kolmogorov spectrum $k^{-5 / 3}$ (see Part III). Values $0<\alpha<1$ correspond to $\nu>1$.
Values $\alpha<0$, i.e. $\nu<1$, would correspond to cases where the velocity decreases for larger scale lengths.

The Laplace transform of the space-averaged waiting time distribution is

$$
\begin{gathered}
\hat{\Psi}(s)=\int_{0}^{\infty} d t \int d^{d} \mathbf{x} \psi(\mathbf{x}, t) e^{-s t}=A S_{d} \int_{a}^{\infty} d t \int d r r^{d-1} t^{-\beta-1} \delta\left(r-t^{\nu}\right) e^{-s t} \\
=A S_{d} \int_{a}^{\infty} d t t^{-\mu-1} e^{-s t}, \quad \mu=\beta-\nu(d-1)
\end{gathered}
$$

We require $\mu>0$, so that the waiting time probability can be normalized, $\hat{\Psi}(s=$ $0)=\int_{0}^{\infty} d t \psi(t)=1$. Further, for $\mu>1$ the average waiting time exists, $\langle t\rangle=$ $-d \hat{\Psi}(s) /\left.d s\right|_{s=0}$. These considerations lead to the following asymptotic expressions for small values of $s$

$$
\begin{equation*}
\hat{\Psi}(s)=1-C s^{\mu}, \quad 0<\mu \leq 1, \tag{62}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\Psi}(s)=1-\tau_{D} s, \quad \mu>1 . \tag{63}
\end{equation*}
$$

Further, considering the two-dimensional case ( $S_{d}=2 \pi$ ), one finds

$$
\hat{\psi}(s, \mathbf{k})-\hat{\Psi}(s)=A \int_{a}^{\infty} d t e^{-s t} \int r d r \int_{-\pi}^{\pi} d \theta\left(e^{i k r \cos \theta}-1\right) t^{-\beta-1} \delta\left(r-t^{\nu}\right) .
$$

This leads to the following result

$$
\begin{equation*}
\hat{\psi}(s, \mathbf{k})-\hat{\Psi}(s)=2 \pi A \int_{a}^{\infty} d t t^{-\mu-1}\left[J_{0}\left(k t^{\nu}\right)-1\right] e^{-s t} \tag{64}
\end{equation*}
$$

where we have used the expression for the Bessel function $J_{0}(z)=(1 / \pi) \int_{0}^{\pi} d \theta e^{i z \cos \theta}$.
In the asymptotic limit

$$
\begin{equation*}
k t^{\nu} \ll s t \tag{65}
\end{equation*}
$$

where the Bessel function does not change very much during the decay of $e^{-s t}$, we may expand this function for small values of its argument, $J_{0}(z) \approx 1-k^{2} t^{2 \nu} / 4$. Then, we obtain

$$
\begin{equation*}
\hat{\psi}(s, \mathbf{k})-\hat{\Psi}(s) \approx-\frac{\pi}{2} A k^{2} \int_{a}^{\infty} d t t^{-\mu-1+2 \nu} e^{-s t}, \tag{66}
\end{equation*}
$$

The integral on the right exists for $-\mu+2 \nu<0$, so that we have to leading order in small quantities

$$
\begin{equation*}
\hat{\psi}(s, \mathbf{k})-\hat{\Psi}(s) \approx C_{1} k^{2}, \quad-\mu+2 \nu<0 \tag{67}
\end{equation*}
$$

For $-\mu+2 \nu>0$, this integral does not exists and one obtains

$$
\begin{equation*}
\hat{\psi}(s, \mathbf{k})-\hat{\Psi}(s) \approx-C_{1} k^{2} s^{\mu-2 \nu}, \quad-\mu+2 \nu>0 \tag{68}
\end{equation*}
$$

From the preceding discussion it is clear that four different cases exist.
I. The first case occurs for $1<\mu<2 \nu$. In this range of values of the exponents we have found

$$
\hat{\psi}(s, \mathbf{k}) \approx 1-\tau_{D} s-C_{1} k^{2} s^{\mu-2 \nu}, \quad \hat{\Psi}(s)=1-\tau_{D} s
$$

The Laplace-Fourier transform of the pdf (5) is

$$
\begin{equation*}
\hat{n}(\mathbf{k}, s)=\frac{\tau_{D} s^{2 \nu-\mu}}{\tau_{D} s^{1+2 \nu-\mu}+C_{1} k^{2}} . \tag{69}
\end{equation*}
$$

This yields the mean square displacement

$$
\begin{equation*}
<\mathbf{x}^{2}(t)>=-\left.\frac{\partial^{2}}{\partial \mathbf{k} \cdot \partial \mathbf{k}} \hat{n}(\mathbf{k}, t)\right|_{\mathbf{k}=0}=\frac{4 C_{1}}{\tau_{D}} \frac{t^{1+2 \nu-\mu}}{\Gamma(2+2 \nu-\mu)} \tag{70}
\end{equation*}
$$

This case describes super-diffusion.
II. The second case occurs when $\mu>1,2 \nu$. In this range we have found

$$
\hat{\psi}(s, \mathbf{k}) \approx 1-\tau_{D} s-C_{1} k^{2}, \quad \hat{\Psi}(s)=1-\tau_{D} s
$$

Upon substituting these expressions into (5) and taking $\hat{n}_{0}(\mathbf{k})=1$, one obtains

$$
\begin{equation*}
\hat{n}(\mathbf{k}, s)=\frac{\tau_{D}}{\tau_{D} s+C_{1} k^{2}} . \tag{71}
\end{equation*}
$$

Here, we recover classical diffusion,

$$
\begin{equation*}
<\mathbf{x}^{2}(t)>=-\left.\frac{\partial^{2}}{\partial \mathbf{k} \cdot \partial \mathbf{k}} \hat{n}(\mathbf{k}, t)\right|_{\mathbf{k}=0}=\frac{4 C_{1}}{\tau_{D}} t . \tag{72}
\end{equation*}
$$

III. The third case is found for $0<\mu<1, \mu<2 \nu$. Then,

$$
\hat{\psi}(s, \mathbf{k}) \approx 1-C s^{\mu}-C_{1} k^{2} s^{\mu-2 \nu}, \quad \hat{\Psi}(s)=1-C s^{\mu}
$$

The Fourier-Laplace transform of the pdf $n(\mathbf{x}, t)$ is

$$
\begin{equation*}
\hat{n}(\mathbf{k}, s)=\frac{C s^{2 \nu-1}}{C s^{2 \nu}+C_{1} k^{2}}, \tag{73}
\end{equation*}
$$

which leads to the MSD

$$
\begin{equation*}
<\mathrm{x}^{2}(t)>=\frac{4 C_{1}}{C} \frac{t^{2 \nu}}{\Gamma(1+2 \nu)} \tag{74}
\end{equation*}
$$

In this interval of values of the exponents, either sub-diffusion $(\nu<1 / 2)$, classical diffusion ( $\nu=1 / 2$ ), or super-diffusion $\nu>1 / 2$ ) occurs.
IV. The fourth case is $0<\mu<1, \mu>2 \nu$. In this regime we have found

$$
\hat{\psi}(s, \mathbf{k}) \approx 1-C s^{\mu}-C_{1} k^{2}, \quad \hat{\Psi}(s)=1-C s^{\mu}
$$

so that

$$
\begin{equation*}
\hat{n}(\mathbf{k}, s)=\frac{C s^{\mu-1}}{C s^{\mu}+C_{1} k^{2}} \tag{75}
\end{equation*}
$$

This gives the MSD

$$
\begin{equation*}
<\mathrm{x}^{2}(t)>=\frac{4 C_{1}}{C} \frac{t^{\mu}}{\Gamma(1+\mu)} \tag{76}
\end{equation*}
$$

This regime has a sub-diffusive character.
Note that for Lévy walks the Fourier-Laplace transform of the pdf $n(\mathbf{x}, t)$ can be written such that it has the same structural form as in the case of decoupled space and time steps that was discussed in Section III (see equation (29)).

In the case of Richardson diffusion we have $\alpha=1 / 3$ in (60), so that $\nu=3 / 2$. The $t^{3}$ behavior of the MSD can be recovered in case III (or, equivalently, in case II for $\mu=1$ ).

## VI. Fractional integration and differentiation

Until 20 years ago fractional integration and differentiation were unknown in physics. Since then, physicists have discovered that fractional calculus may be very useful in the description of anomalous and 'strange' diffusion problems. Fractional diffusion equations generalize Fick's and Fokker-Planck equations and represent memory and non-local aspects.

A way to introduce fractional calculus is to start from $n$-fold repeated integration,

$$
\begin{equation*}
{ }_{a} I_{x}^{n} f(x)=\int_{a}^{x} \int_{a}^{y_{1}} \ldots \ldots . \int_{a}^{y_{n}} d y_{1} \ldots . d y_{n} f\left(y_{n}\right)=\frac{1}{\Gamma(n)} \int_{a}^{x} d y(x-y)^{n-1} f(y) \tag{77}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma(z)=\int_{0}^{\infty} t^{z-1} e^{-t} d t, \quad \operatorname{Re} z>0 \tag{78}
\end{equation*}
$$

is the Gamma function. The Gamma function diverges for negative integer values of its argument. Further we have used Dirichlet's formula

$$
\int_{a}^{x} d y_{1} \int_{a}^{y_{1}} d y_{2} f\left(y_{1}, y_{2}\right)=\int_{a}^{x} d y_{2} \int_{y_{2}}^{x} d y_{1} f\left(y_{1}, y_{2}\right)
$$

The n-tuple integral (77) can be generalized to a fractional integral of arbitrary order $\alpha$

$$
\begin{equation*}
{ }_{a} I_{x}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)} \int_{a}^{x} d y(x-y)^{\alpha-1} f(y), \quad x \geq a \tag{79}
\end{equation*}
$$

where $\alpha$ is positive real. This is the Riemann-Liouville fractional integral.
A. On the basis of (79) we can define the fractional derivative of order $\alpha$

$$
\begin{equation*}
{ }_{a} D_{x}^{\alpha} f(x)=\frac{d^{n}}{d x^{n}} a I_{x}^{n-\alpha} f(x)=\frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{d x^{n}} \int_{a}^{x} d y(x-y)^{n-\alpha-1} f(y), \quad x \geq a . \tag{80}
\end{equation*}
$$

where n is a positive integer such that $0<n-1<\operatorname{Re} \alpha<n$. More precisely, this is the left Riemann-Liouville fractional derivative. The right Riemann-Liouville fractional derivative is

$$
\begin{equation*}
{ }_{x} D_{b}^{\alpha} f(x)=\frac{(-1)^{n}}{\Gamma(n-\alpha)} \frac{d^{n}}{d x^{n}} \int_{x}^{b} d y(x-y)^{n-\alpha-1} f(y), \quad x \leq b . \tag{81}
\end{equation*}
$$

It is easily seen that ${ }_{a} I_{x}^{\alpha}={ }_{a} D_{x}^{-\alpha}$ and ${ }_{x} I_{b}^{\alpha}={ }_{x} D_{b}^{-\alpha}$.
B. Note that the definitions (80) and (81) are not unique. It is clear that the following definitions are also valid,

$$
\begin{equation*}
{ }_{a}^{c} D_{x}^{\alpha} f(x)={ }_{a} I_{x}^{n-\alpha} \frac{d^{n}}{d x^{n}} f(x)=\frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} d y(x-y)^{n-\alpha-1} \frac{d^{n}}{d y^{n}} f(y), \quad x \geq a, \tag{82}
\end{equation*}
$$

and

$$
\begin{equation*}
{ }_{x}^{c} D_{b}^{\alpha} f(x)=\frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} d y(x-y)^{n-\alpha-1} \frac{d^{n}}{d y^{n}} f(y), \quad x \leq b . \tag{83}
\end{equation*}
$$

These equations form the Caputo definition of the fractional derivatives. The RiemannLiouville and Caputo fractional derivatives differ by boundary terms of the function $f(x)$ and its derivatives.

C A third derivative that plays a role in fractional diffusion is the symmetric derivative

$$
\begin{equation*}
D_{|x|}^{\alpha}=\frac{1}{2 \cos \pi \alpha / 2}\left[-\infty D_{x}^{\alpha}+{ }_{x} D_{\infty}^{\alpha}\right] . \tag{84}
\end{equation*}
$$

This is the Riesz fractional derivative.
Below we list a number of properties of fractional integrals and derivatives.
I. The fractional integrals satisfy the group property

$$
\begin{equation*}
{ }_{a} I_{x}^{\alpha} I_{x}^{\beta}={ }_{a} I_{x}^{\alpha+\beta} . \tag{85}
\end{equation*}
$$

The proof is as follows.

$$
\begin{gathered}
{ }_{a} I_{x}^{\alpha}{ }_{a} I_{x}^{\beta} f(x)=\frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{a}^{x} d \xi \frac{1}{(x-\xi)^{1-\alpha}} \int_{a}^{\xi} d \tau \frac{f(\tau)}{(\xi-\tau)^{1-\beta}} \\
\quad=\frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{a}^{x} d \tau \frac{f(\tau)}{(\xi-\tau)^{1-\beta}} \int_{\tau}^{x} d \xi \frac{1}{(x-\xi)^{1-\alpha}}
\end{gathered}
$$

Apply the transformation of variables $\xi \rightarrow s$ with $\xi=\tau+s(x-\tau)$. Then, it follows that

$$
\begin{aligned}
{ }_{a} I_{x a}^{\alpha} I_{x}^{\beta} f(x) & =\frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{a}^{x} d \tau \frac{f(\tau)}{(x-\tau)^{1-\alpha-\beta}} \int_{0}^{1} d s \frac{1}{s^{1-\beta}(1-s)^{1-\alpha}} \\
= & \frac{B(\alpha, \beta)}{\Gamma(\alpha) \Gamma(\beta)} \int_{a}^{x} d \tau \frac{f(\tau)}{(x-\tau)^{1-\alpha-\beta}}={ }_{a} I_{x}^{\alpha+\beta} f(x) .
\end{aligned}
$$

Here, $B(\alpha, \beta)$ is the Beta-function

$$
\begin{equation*}
B(\alpha, \beta)=\int_{0}^{1} s^{\alpha-1}(1-s)^{\beta-1} d s=\frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{86}
\end{equation*}
$$

II. It is seen from (80) and (81) that

$$
\begin{equation*}
{ }_{a} D_{x}^{\alpha} f(x)=\frac{d}{d x}{ }^{a} D_{x}^{\alpha-1} f(x), \quad{ }_{x} D_{b}^{\alpha} f(x)=-\frac{d}{d x}{ }_{x} D_{b}^{\alpha-1} f(x) . \tag{87}
\end{equation*}
$$

III. From the definition of the fractional derivative it follows that

$$
\begin{equation*}
{ }_{0} D_{x}^{\alpha} x^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha} . \tag{88}
\end{equation*}
$$

This is equivalent to

$$
\begin{equation*}
{ }_{a} D_{x}^{\alpha}(x-a)^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-a)^{\mu-\alpha} . \tag{89}
\end{equation*}
$$

This result can be obtained as follows. According to (80) we have

$$
\begin{gathered}
{ }_{0} D_{x}^{\alpha} x^{\mu}=\frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{d x^{n}} \int_{0}^{x} d y(x-y)^{n-\alpha-1} y^{\mu} \\
=\frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{d x^{n}} x^{n-\alpha+\mu} \int_{0}^{1} d y y^{\mu}(1-y)^{n-\alpha-1} \\
=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha},
\end{gathered}
$$

where we have used definition (86) of the Beta function.
The Riemann-Liouville fractional derivative of a function that may be Taylor expanded around $x=a$,

$$
f(x)=\sum_{p} \frac{(x-a)^{p}}{\Gamma(1+p)} f^{(p)}(a),
$$

is according to (89)

$$
\begin{equation*}
{ }_{a} D_{x}^{\alpha} f(x)=\frac{1}{\Gamma(1-\alpha)} \frac{f(a)}{(x-a)^{\alpha}}+\frac{1}{\Gamma(2-\alpha)} \frac{f^{\prime}(a)}{(x-a)^{-1+\alpha}}+\sum_{p=0}^{\infty} \frac{f^{(p+2)}(a)}{\Gamma(p-\alpha+3)}(x-a)^{p-\alpha+2} . \tag{90}
\end{equation*}
$$

The first two terms are written separately because they are singular for $1<\alpha<2$. The Caputo derivative takes care of these singularities,

$$
\begin{equation*}
{ }_{a}^{c} D_{x}^{\alpha} f(x)=\sum_{p=0}^{\infty} \frac{f^{(p+2)}(a)}{\Gamma(p-\alpha+3)}(x-a)^{p-\alpha+2} . \tag{91}
\end{equation*}
$$

IV. The rule (88) implies that the derivative of a constant does not vanish unless the order of integration is an integer,

$$
\begin{equation*}
{ }_{0} D_{x}^{\alpha} C=C \frac{x^{-\alpha}}{\Gamma(1-\alpha)} . \tag{92}
\end{equation*}
$$

On the other hand, the Caputo derivative of a constant vanishes

$$
\begin{equation*}
{ }_{x}^{c} D_{b}^{\alpha} C=0 . \tag{93}
\end{equation*}
$$

$\mathbf{V}$. The derivative of the exponential function is

$$
{ }_{0} D_{x}^{\alpha} e^{x}=\frac{d^{n}}{d x^{n}} \frac{1}{\Gamma(n-\alpha)} \int_{0}^{x} d y(x-y)^{n-\alpha-1} e^{-y}
$$

$$
\begin{equation*}
=e^{x} \frac{\gamma(-\alpha, x)}{\Gamma(-\alpha)}, \tag{94}
\end{equation*}
$$

where

$$
\gamma(\alpha, x)=\frac{1}{\Gamma(\alpha)} \int_{0}^{x} e^{-t} t^{\alpha-1}, \text { Re } \alpha>0
$$

is the incomplete gamma function.
VI. From

$$
\begin{equation*}
{ }_{-\infty}^{c} D_{x}^{\alpha} e^{i k x}=(-i k)^{\alpha} e^{i k x}, \quad{ }_{x}^{c} D_{\infty}^{\alpha} e^{i k x}=(i k)^{\alpha} e^{i k x}, \tag{95}
\end{equation*}
$$

follows hat the Fourier transform of the Riemann-Liouville fractional derivatives are

$$
\begin{equation*}
F T_{-\infty}^{c} D_{x}^{\alpha} f(x)=(-i k)^{\alpha} \hat{f}(k), \quad{ }_{x}^{c} D_{\infty}^{\alpha} e^{i k x}=(i k)^{\alpha} \hat{f}(k) \tag{96}
\end{equation*}
$$

The Fourier transform of a fractional integral is

$$
\begin{equation*}
F T_{-\infty} I_{x}^{\alpha} f(x)=(-i k)^{-\alpha} \hat{f}(k) \tag{97}
\end{equation*}
$$

This can be shown as follows

$$
\begin{aligned}
& F T_{-\infty} I_{x}^{\alpha} f(x)=\int_{-\infty}^{\infty} d x e^{i k x} \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} d x^{\prime} \frac{f\left(x^{\prime}\right)}{\left(x-x^{\prime}\right)^{1-\alpha}} \\
& =\int_{-\infty}^{\infty} d x e^{i k x} \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} d s \frac{f(x-s)}{s^{1-\alpha}}=\frac{\hat{f}(k)}{\Gamma(\alpha)} \int_{0}^{\infty} d s \frac{e^{i k s}}{s^{1-\alpha}} \\
& =(-i k)^{-\alpha} \hat{f}(k) \frac{1}{\Gamma(\alpha)} \int_{0}^{-i \infty} d z z^{\alpha-1} e^{-z}=(-i k)^{-\alpha} \hat{f}(k) .
\end{aligned}
$$

In the same way one can also find the FT of the fractional derivatives.
VII. The Laplace transform of the left Riemann-Liouville fractional derivative for $0<$ $\alpha<1$ is

$$
\begin{gather*}
L T{ }_{0} D_{t}^{\alpha} f(t)=\int_{0}^{\infty} d t \frac{1}{\Gamma(1-\alpha)} e^{-s t} \frac{d}{d t} \int_{0}^{t} d y(t-y)^{-\alpha} f(y) . \\
=-\left.{ }_{0} D_{t}^{\alpha-1} f\right|_{t=0}+s \int_{0}^{\infty} d t \frac{e^{-s t}}{\Gamma(1-\alpha)} \int_{0}^{t} d y(t-y)^{-\alpha} f(y) \\
=s^{\alpha} \hat{f}(s)-\left.{ }_{0} D_{t}^{\alpha-1} f\right|_{t=0} . \tag{98}
\end{gather*}
$$

This Laplace transform depends on the initial value of the fractional derivative. This is not very convenient since the initial value of the function $f(t)$ is usually given.
IX. The Laplace transform of the Caputo fractional derivative (82) is

$$
L T_{0}^{c} D_{t}^{\alpha} f(t)=\int_{0}^{\infty} d t e^{-s t} \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} d y \frac{d f / d y}{(t-y)^{\alpha}}
$$

$$
=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{\infty} d y \frac{d f}{d y} \int_{y}^{\infty} d t e^{-s t}(t-y)^{-\alpha}
$$

which gives

$$
\begin{equation*}
L T{ }_{0}^{c} D_{x}^{\alpha}=s^{\alpha} \hat{f}(s)-s^{\alpha-1} f(0) . \tag{99}
\end{equation*}
$$

The Laplace transform of the Caputo fractional derivative depends on the initial value of the function itself.

## References

[1] E.W. Montroll, E.W. Weiss, J. Math. Pys. 6 (1965) 167
[2] E.W. Montroll, M.F Shlesinger, in: Studes of Statistical Mechanics, vol. 11, p. 5 (J.L. Lebowitz and E.W. Montroll, eds.) North Holland, Amsterdam
[3] R. Metzler, Non-homogeneous random walks, generalised master equation, fractional Fokker-Planck equations, and the generalised Kramers-Moyal equation, Eur.Phys.J.B 19 (2001) 249-2589
[4] Igor M. Sokolov, Joseph Klafter, Alexander Blumen, Fractional Kinetics, Physics Today, Nov. 2002, 48
[5] R. Metzler, E. Barkai, J. Klafter, Europhysics Letters 46 (1999) 431-436
[6] George M. Zaslavsky, Hamiltonian Chaos and Fractional Dynamics, Oxford UP, 2005
[7] Radu Balescu, Statistical Dynamics, Imperial College, 1997
[8] R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, J.Phys.A:Math.Gen. 37 (2004) 161-208
[9] Radu Balescu, V-Langevin equations, continuous time random walks and fractional diffusion, Chaos, Sol.Fract. 34 (2007) 62
[10] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Gordon and Breach Science Publishers, 1993
[11] A.V. Chekhin, R. Metzler, J. Klafter, V.Yu. Gonchar, Introduction to the Theory of Lévy Flights,,, in Anomalous Transport, eds. R. Klages, G.Radons, I.M. Sokolov, Wiley-VCH 2008
[12] Diego del-Castillo-Negrete, Fractional Diffusion Models in Anomalous Transport, in Anomalous Transport, eds. R. Klages, G.Radons, I.M. Sokolov, Wiley-VCH 2008
[13] J. Klafter, A. Blumen, M.F. Shlesinger, Stochastic pathway to anomalous diffusion, Phys.Rev. A 35 (1987) 3081
[14] A. Blumen, G. Zumofen, J. Klafter, Transport aspects in anomalous diffusion: Lévy walks, Phys.Rev. A 40 (1989) 3964

