DIFFUSION: FROM CLASSICAL TO FRACTIONAL

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PART FOUR: THE PERCOLATION MODEL FOR STOCHASTIC DIFFUSION

I. Percolation

Percolation theory deals with the formation of long-range connectivity in random (disordered) systems (like porous rocks, but also turbulent fluids and plasmas) and contains elements of probability theory and of geometry. One might say that it is a branch of statistical geometry. Percolation is not based upon a dynamical law, it is a model describing in simple terms the behavior of very complex dynamical systems. In that sense it resembles the random walk studied in previous chapters.

Percolation theory is an important tool in the study of transport and equilibrium in such diverse fields as the conduction of current in alloys of conducting and non-conducting materials, the flow of liquids in porous media, the diffusion of charged particles in magnetized plasmas etc.. In percolation theory the disorder of the medium is prescribed, is assumed to be given and not created by the processes under consideration.

The standard percolation geometry is a regular lattice of d-dimensions which becomes a random network by assigning to the sites (vertices) or to the edges (bonds) a *probability of occupancy* p. A site may be occupied or empty, in that case nothing is said about bonds. On the other hand, bonds may be active or inactive and sites do not play a role. In this Section we will consider two-dimensional systems ($d = 2$). Each site (or bond) of a very large lattice is occupied independently of its neighbors. No correlations between sites or between bonds are allowed. The lattice should be sufficiently large that boundary effects can be ignored. The probability p that a site is occupied is also called, for obvious reasons, the concentration. Each site has a certain number of neighbors. This number depends on the geometry of the lattice and the number of dimensions. Nearest occupied sites or vertices form *clusters*. A site and its neighbors belong to the same cluster if they are both occupied. A site is isolated if all its neighbors are empty. *Percolation theory deals with the properties of these clusters.* When the probability p is small, only small clusters will exist. With increasing values of p, the number of sites per cluster, i.e. the average size of the clusters will increase. At a critical *percolation threshold* p_c, the long-range connectivity, the infinite cluster will first appear.

As an example, consider the problem of conduction of an electric current through a medium that consists of a mixture of conducting and isolating substances. In this case we call a site occupied if it consists of conducting material and empty or vacant if it consists of non-conducting material. The alloy is bounded by two parallel plates over which a voltage is applied. The ratio between the current and the voltage is the conductivity of the medium. If the probability p that the site is occupied is small, no electrical path between the two plates will exist. This means that the conductivity will be zero. Larger p means that the size of the (conducting) clusters will be larger. When p grows, the clusters will grow until suddenly, when p reaches a critical value p_c, the medium starts to conduct an electric current. This means that at the threshold p_c the long-range connectivity, the infinite cluster, appears. *These geometrical phase transitions described by percolation theory*

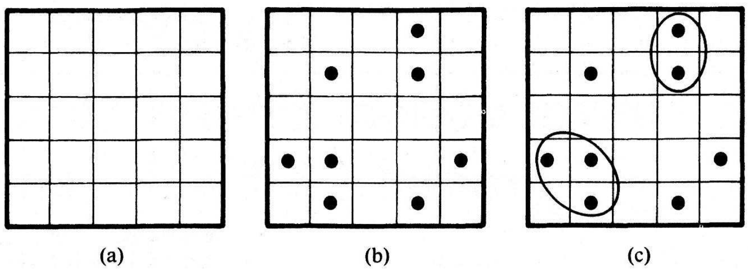

Figure 1: Definition of percolation and its clusters (from [1])

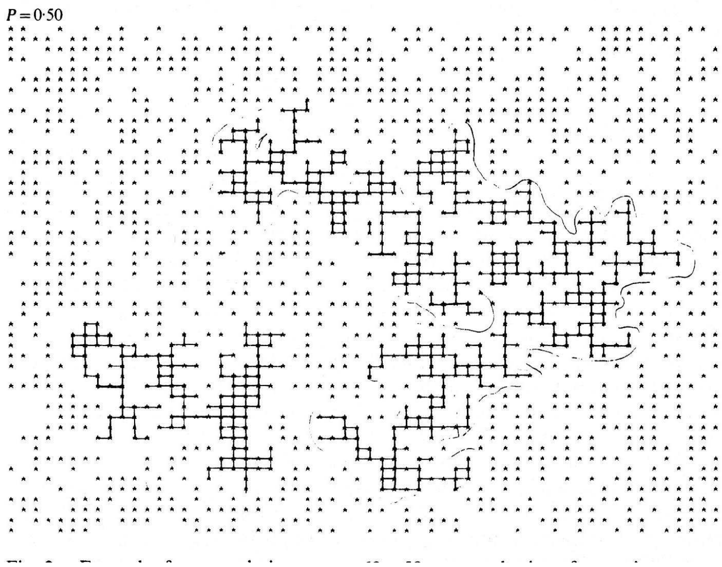

Figure 2: Example of percolation for two values of p (from [1])

exhibit all the characteristic features of physical phase transitions.

The number of s-clusters per site, i.e. the probability that a given site belongs to an s-cluster, is

$$n_s = \sum_t g_{st} p^s (1-p)^t. \tag{1}$$

Here, s is the number of sites of a cluster, also called its mass, and t its perimeter, i.e., the number of neighboring, empty sites of the cluster. Further, g_{st} is a form factor that depends on cluster geometry, but also on the type of lattice that is adopted. It can easily be seen that the number t of neighboring empty sites differs for different cluster geometries. The sum is over all geometric forms that can be formed with s connected sites (bonds).

At $p = p_c$ the *infinite* cluster appears and exists for $p > p_c$. The probability of a site to belong to this infinite cluster is denoted by P_∞.

The probability n_s is a basic concept of percolation theory. Several probability functions are based upon this notion.

- sn_s is the probability that *any* site belongs to an s-cluster,

- sn_s/p is the probability that an *occupied* site belongs to an s-cluster,

- $\sum_s sn_s = p$ is the probability that an arbitrary site belongs to any cluster $(p < p_c)$. For $p > p_c$ there is an infinite cluster and this relation becomes $\sum_s sn_s + P_\infty = p$,

- $w_s = sn_s/\sum sn_s$ is the probability that the cluster to which an arbitrary, occupied site belongs is an s-cluster,

- $S = \sum_s sw_s = \sum_s s^2 n_s/\sum_s sn_s$ is the average cluster size.

Analogously to the radius of gyration in polymer physics, one defines a distance

$$\frac{1}{s}\sum_{i=1}^{s} |\mathbf{r}_i - \mathbf{r}_0|^2 = \frac{1}{2s^2}\sum_{i,j=1}^{s} |\mathbf{r}_i - \mathbf{r}_j|^2,$$

where $\mathbf{r}_0 = \sum_i \mathbf{r}_i/s$ is the center of mass. This squared distance is directly related to the average squared distance between two sites of an s-cluster,

$$R_s^2 = \frac{2}{s(s-1)}\sum_{i,j=1}^{s} |\mathbf{r}_i - \mathbf{r}_j|^2, \tag{2}$$

$s(s-1)/2$ being the number of pairs of sites in an s-cluster.

The correlation function $g(r)$ is defined as the probability that a site at a distance r from an occupied site is also occupied and belongs to the same cluster.

The average number of sites to which an occupied site is connected is $\sum g(r)$. This average number is equal to the average cluster size

$$S = \sum g(r). \tag{3}$$

The sum is over all sites. The correlation or connectivity length is defined in terms of the mean square distance between two sites belonging to the same cluster

$$a^2 = \frac{\sum r^2 g(r)}{\sum g(r)} = \frac{\sum R_s^2 s^2 n_s}{\sum s^2 n_s}, \tag{4}$$

where the sums are over all clusters. Note that in this Section all lengths are normalized to the size of the cell.

The large clusters and the infinite cluster will not form homogeneous clouds of sites, in particular not near the percolation threshold. They will contain many holes of a wide variety of sizes. This is typical for fractal structures. The concept of fractal dimension will enable us to express the size R_s of a cluster in terms of the number of sites s of the cluster. The mass, i.e., the number of sites of a cluster within a radius $r < a$ in d-dimensions is

$$m(r) \approx r^{d_f}, \tag{5}$$

where d_f is the fractal dimension. From $m(R_s) \approx R_s^{d_f} = s$ follows a scaling for R_s

$$R_s \approx s^{1/d_f}. \tag{6}$$

The probability that an arbitrary site belongs to the infinite cluster is ($p \leq p_c$)

$$P_\infty \approx \frac{a^{d_f}}{a^d}. \tag{7}$$

The central assumption in percolation theory is the following scaling hypothesis. In the remainder of this Section I follow largely Chapters 13 and 14 of the book by Balescu [2].

The probability that a given site belongs to an s-cluster is given by a scaling formula involving only two characteristic exponents

$$n_s(p) = s^{-\tau} f_\pm(|p - p_c|^{1/\sigma} s). \tag{8}$$

This scaling is assumed to hold for all dimensions and all lattice structures and to be valid in the limit $p \to p_c$ and $s \to \infty$. The critical probability p_c depends on the geometry of the lattice. The coefficients σ and τ are universal, they depend only on the dimensionality d, and have the same value above and below the percolation threshold. The function $f_+(x)$ is defined for $x > 0$ i.e. for $p > p_c$ and $f_-(x)$ for $x < 0$ i.e. for $p < p_c$. Their form is not universal but depend on the lattice structure.

There exist a crossover size $s_\xi = |p - p_c|^{-1/\sigma}$

$$f_\pm(x) \;\to\; constant \quad x << 1 \quad i.e. \quad s << s_\xi,$$

$$f_\pm(x) \;\to\; 0\,(very fast) \quad x >> 1 \quad i.e. \quad s >> s_\xi.$$

The first of these statements implies that at the percolation threshold, $n_s(p = p_c)$ has a power law behavior,

$$n_s(p_c) \approx s^{-\tau}. \tag{9}$$

It should be stressed that this scaling law (8) is an assumption. It holds either analytically or numerically for many lattice structures and dimensions, but has not been proven for general structures and dimensions from first principles.

The scaling law (8) implies the scaling of most quantities encountered in this Section, in particular near the percolation threshold. Substitute (8) into the kth moment of n_s,

$$M_k = \sum_s s^k n_s(p) \to \int ds\, s^{k-\tau} f_\pm(|p-p_c|^{1/\sigma} s) = |p-p_c|^{(\tau-k-1)/\sigma} \int dx\, x^{k-\tau} f_\pm(x).$$

Hence, one finds the scaling

$$M_k = \sum_s s^k n_s(p) \propto |p-p_c|^{(\tau-k-1)/\sigma}. \tag{10}$$

The average cluster size is $S = \sum s^2 n_s / \sum s n_s = p^{-1} M_2$, so that we obtain the scaling

$$S \propto |p-p_c|^{-\gamma}, \quad \gamma = \frac{3-\tau}{\sigma}, \qquad (|p-p_c| << p_c). \tag{11}$$

The correlation length (4) is

$$a^2 = \frac{\sum R_s^2 s^2 n_s}{\sum s^2 n_s} = \frac{M_{2+2/d_f}}{M_2} \propto |p-p_c|^{-2/\sigma d_f},$$

where we have used (6) in (10). Thus , the correlation length sales like

$$a \propto |p-p_c|^{-\nu}, \qquad \nu = \frac{1}{\sigma d_f}. \tag{12}$$

The exponent takes the value $\nu = 4/3$ in 2D, $\nu \approx 0.9$ in 3D, and $\nu = 1/2$ for the Bethe network [1]. *For $p < p_c$, (12) is a measure for the size of a cluster. On the other hand for $p > p_c$, (12) measures the size of a hole in the infinite cluster!*

The probability P_∞ of a site to belong to the infinite cluster is

$$\begin{aligned} P_\infty &= -\sum s n_s + p = p - p_c - \sum s n_s + \sum s n_s(p_c) \\ &= p - p_c + \int ds\, s^{1-\tau} f(0) - \int ds\, s^{1-\tau} f_\pm(x). \end{aligned}$$

This yields the scaling

$$P_\infty \approx (p-p_c)^\beta, \quad \beta = \frac{\tau-2}{\sigma} < 1, \qquad p \geq p_c. \tag{13}$$

For sizes below the correlation length, the infinite cluster is self-similar (fractal). According to (7) and (12), $P_\infty \propto r^{d_f-d} \propto a^{d_f-d}$, so that we find the scaling

$$P_\infty \propto |p-p_c|^{-\nu(d_f-d)}. \tag{14}$$

Equations (13) and (14) give the following relation between the exponents

$$d_f = d - \frac{\beta}{\nu}. \tag{15}$$

Another concept that we will need in a later section is the external perimeter or hull of a cluster. The hull consists of sites of a cluster that neighbor vacant sites that are connected to the outside world. One could imagine that a random walker that is hopping on *empty sites* and that starts at the outside of the system, will hit the cluster at one of the sites of this external perimeter (the walker cannot penetrate the cluster). Some of these sites will be easily encountered, other will be hidden in deep fjords. The hull turns out to be a multi-fractal curve

$$L \approx a^{d_h} \tag{16}$$

with dimension

$$d_h = \frac{1+\nu}{\nu}. \tag{17}$$

In 2 dimensions, $\nu = 4/3$ so that $d_h = 7/4$; this is an exact result [6]!

Finally, let us consider the conduction problem introduced in Section I. The configuration is given in figure 3. The alloy consists of a lattice in d dimensions. In each direction the sample has N sites, so that the surface of the plate consists of N^{d-1} sites. The plates are separated by a distance L. The global conductivity is the ratio between the total current I and the voltage V, $I = \hat{\Sigma} V$. Write $\hat{\Sigma}$ in the form

$$\hat{\Sigma} = \frac{N^{d-1}}{L}\Sigma.$$

The conductivity Σ is independent of the size of the plates and their distance, and depends only on the composition of the alloy, $\Sigma = \Sigma(p)$. It is clear that $\Sigma(0) = 0$ and $\Sigma(1) = 1$, where the conductivity is assumed to be normalized to unity if the sample is purely conducting. As long as $p < p_c$ no current will flow through the system. It is only when the infinite cluster appears, $p \leq p_c$, that a current will flow. It turns out that

$$\Sigma(p) \propto (p - p_c)^{\mu}, \qquad p > p_c. \tag{18}$$

The exponent μ is not equal to the exponent β as appears in (13) for the probability P of the infinite cluster. This can be understood from the observation that the infinite cluster contains many dead ends and dangling bonds, so that only paret of it, the 'backbone' participates in the process.

Einstein's relation between conductivity and diffusivity reads ((see Part I)

$$\Sigma = \frac{e^2 n}{T} D_p. \tag{19}$$

This relation will also hold in a percolative medium so that the diffusivity has the same behavior as the conductivity

$$D_p \propto (p - p_c)^{\mu}, \qquad p > p_c. \tag{20}$$

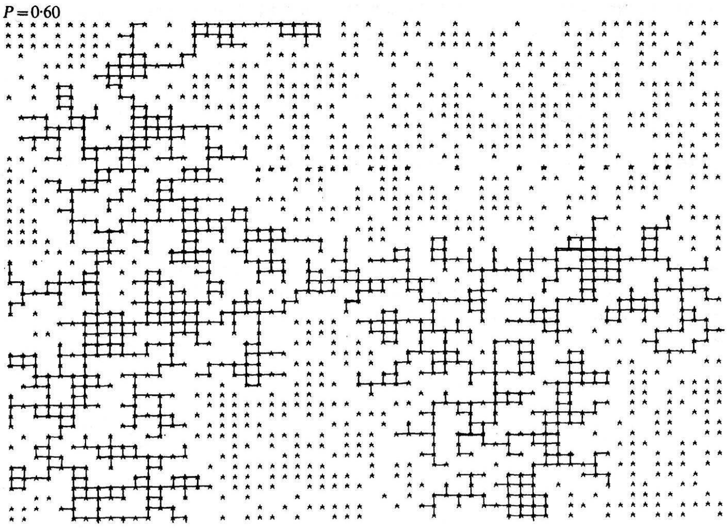

Figure 3: Conduction in a disordered alloy (from [2])

II. Percolation and diffusion

In previous Parts we have explored the relation between diffusion and random walks. In this Section we will continu this subject in a different context. Here, we are interested in diffusion processes in disordered systems that are modelled as percolative structures. In the description of percolation in the previous Section, time did not play a role. The occurrence of random walks on a percolative lattice will introduce time as a new variable in percolation.

The problem of diffusion in a percolative lattice was coined in [3] as ”the ant in the labyrinth”. The ant (particle) may move from an occupied site to a nearest neighbor that is also occupied. At $t = 0$ the ant starts and looks if there is a nearest site that is occupied. It chooses one at random and moves to that site. If there is no occupied nearest neighbor the ant stays where it is. At $t = 1$ the process is repeated and so forth. After a time t the ant will have travelled a distance $r(t)$ from its initial position. This whole process is repeated by starting from any other initial site. Since we are studying diffusion we are interested in the mean square displacement (or mean square deviation MSD) $< r^2(t) >$. This MSD will depend on the geometry of the lattice and on the value of p. In this Section we will employe percolation theory in order to explore the scaling of the distance over which the ant can travel. Here, we will denote this distance by R.

When the probability p that a site is occupied vanishes $(p = 0)$, the ant has to stay where it is and cannot move so that $< r^2(t) >= 0$. In the opposite limit $p \to 1$, all sites are occupied and, thus, available to the ant, and the ant will perform a classical random walk with

$$R^2 =< r^2(t) >= 2dDt. \tag{21}$$

Next, consider intermediate values of p. Below the percolation threshold, $0 < p < p_c$, all clusters are finite. This means that the ant can diffuse inside the cluster, but can never leave it. This clearly means

$$R^2 \to constant, \qquad t \to \infty, \qquad 0 < p < p_c. \tag{22}$$

This constant will be a function of $|p_c - p|$. In the case of finite clusters, all sites of a cluster will be equally probable for long times. The probability of a site to belong to an s-cluster is sn_s. The size of such a cluster, thus, the distance over which the ant can travel, is R_s. Hence, the mean square distance of the random walk of the ant is

$$R^2 = \Sigma_s sn_s R_s^2 \propto (p_c - p)^{\beta - 2\nu}. \tag{23}$$

Here, I have used (6) and (8) in (10); ν and β are defined by (12) and (13), respectively. When the probability of occupation is larger than the percolation threshold $(p > p_c)$, an infinite cluster exists. This means that the ant may travel arbitrary far from its initial position. However its "diffusive mobility" is limited with respect to the case represented by (21) due to

- the presence of finite clusters in the system which act as traps for any ant that starts from a site inside such a cluster,

- the infinite cluster encloses many holes and contains many dead ends. The infinite cluster really looks as a labyrinth to the ant!

As a result, near the percolation threshold the MSD will not grow linearly with time as in (21) but will increase at a lower rate

$$R^2 = At^\alpha, \qquad \alpha < 1, \qquad p_c \leq p < 1. \tag{24}$$

This represents *sub-diffusion*. The *diffusion exponent* $\alpha(p, d)$ will be a function of the probability p and of the dimensionality d. Numerical simulations have shown that $\alpha(p_c) \approx 2/3$ for $d = 2$, $\alpha \approx 0.4$ for $d = 3$, and that $\alpha \to 1$ for $p \to 1$. Actually, (24) is valid for long but not not too long times. For really large times when R^2 grows beyond the correlation length, the ant does not feel the fractal character anymore, and beyond some cross-over time one will find $\alpha \to 1$ for $t \to \infty$.

For values of p above p_c, we must reobtain Einstein's relation between the conductivity and the diffusion coefficient, $\Sigma \propto D_p$ (see (20)). In a percolative lattice the conductivity will depend on the difference $|p_c - p|$ according to $\Sigma \propto (p - p_c)^\mu$, but otherwise be independent of time. Hence, we must have

$$R^2 \approx D_p t \propto t(p - p_c)^\mu. \tag{25}$$

In order to cover all these limits, we assume that R^2 scales with time and with $p - p_c$ according to

$$R^2 \propto t^\alpha G(|p - p_c| t^x). \tag{26}$$

For $p < p_c$ the validity of both (23) and (26) requires $\alpha + x(\beta - 2\nu) = 0$ and for $p > p_c$ the validity of both (25) and (26) lead to $\alpha + \mu x = 1$. This means that the exponents in the scaling formula (26) are

$$\alpha = \frac{2\nu - \beta}{2\nu + \mu - \beta}, \qquad x = \frac{1}{2\nu + \mu - \beta}. \tag{27}$$

III. Diffusion equation

Now we are ready to consider in more detail a few aspects of a random walk on a fractal, percolative structure. The probability distribution of a particle to be at the position $\mathbf{x}$ at time t might be calculated from a master equation with the specification of all transition probabilities between sites.

Let $P_i(t)$ be the conditional probability that the ant is at position i at time t, given that it starts at $i = 0$ at $t = 0$, i.e., $P_0(0) = 1, P_i(0) = 0$ for $i \neq 0$. This $P_i(t)$ obeys a master equation

$$P_i(t+1) - P_i(t) = \Sigma_j[\sigma_{ji} P_j(t) - \sigma_{ij} P_i(t)], \tag{28}$$

where σ_{ij} is the probability for the ant to hop from site i to the nearest neighbor site j. In the limit of long times, we may write this equation as

$$\frac{dP_i(t)}{dt} = \Sigma_j[\sigma_{ji} P_j(t) - \sigma_{ij} P_i(t)]. \tag{29}$$

This probability P_i will be an extremely complicated and irregular function and will contain singularities on all scales.

Here, we will not continue this approach, but, instead, start from Fick's law for the density $n(\mathbf{x}, t)$ and generalize this equation for the description of diffusion on a fractal substrate [4]. The resulting density $n(\mathbf{x}, t)$ will be the smoothed envelope of the actual, irregular probability density.

Fick's law for the density profile in a d-dimensional space is

$$\frac{\partial n}{\partial t} = -\nabla \cdot \mathbf{\Gamma}, \qquad \mathbf{\Gamma} = -D\nabla^2 n. \tag{30}$$

The Laplacian on the right-hand-side can be expressed in (hyper-)spherical coordinates. For initial conditions that depend only on the radius (e.g. a delta function), we may average over the angles

$$r^{d-1}\frac{\partial n}{\partial t} = -\frac{\partial \Gamma}{\partial r}, \qquad \Gamma = -Dr^{d-1}\frac{\partial n}{\partial r}. \tag{31}$$

This equation can be looked at as a diffusion equation in one dimension with a space dependent diffusion coefficient. The interpretation of the term on the left is that it is the time rate of change of the probability $m(r) \propto r^{d-1}n(r,t)$ of finding a particle in a shell between r and $r + dr$. Here, r^{d-1} is proportional to the number of sites in the shell and $n(r,t)$ is the probability per fractal site of finding a particle at (r,t). This diffusion equation is generalized a follows.

-In a fractal structure embedded in an Euclidean space, not all sites are accessible to a particle like on an regular, Euclidean lattice. Therefore, the number of occupied sites that are accessible to the particle is not proportional to r^{d-1} but to r^{d_f-1}, so that the probability of finding a particle in a shell is $\hat{m}(r) \propto r^{d_f-1}n(r,t)$.

- For the same reason and in a similar way we generalize the flux Γ to

$$\hat{\Gamma} = -\hat{D}r^{d_f-1}\frac{\partial n}{\partial r}.$$

- In a fractal structure it is not obvious that the diffusion coefficient is a constant. Since we deal with sub-diffusion we write

$$\hat{D} = \hat{D}(r) = Kr^{-\theta}$$

in order to account for a slower growth with distance.

With these generalizations, the one-dimensional diffusion equation (31) becomes

$$\frac{\partial n}{\partial t} = \frac{K}{r^{d_f-1}}\frac{\partial}{\partial r}r^{d_f-1-\theta}\frac{\partial n(r,t)}{\partial r}. \tag{32}$$

This is the *generalized diffusion equation* derived in [4].

The diffusion equation (32) has self-similar solutions of the form

$$n(r,t) = t^{-d_s/2}F(\frac{r^{d_w}}{t}). \tag{33}$$

The substitution of this expression into (32) and the requirement that the probability can be normalized, which means that $\int r^{d_f-1}n(r,t)dr = constant$, yield the relations

$$d_s = \frac{2d_f}{2+\theta}, \qquad d_w = 2+\theta. \tag{34}$$

The mean square displacement that follows from (33) is

$$< r^2(t) >= \int drr^2 d_f r^{d_f-1}n(r,t) = C_2 t^{\bar{\alpha}}, \qquad \bar{\alpha} = \frac{2}{d_w} = \frac{2}{2+\theta}, \tag{35}$$

C_2 being a constant independent of space-time. Thus, *diffusion on a fractal substrate is sub-diffusive*. It is seen that the exponent θ in (32) is at the origin of this *strange diffusion*.

The full solution to (32) which satisfies the initial condition $n(r,0) = \delta(r)$ is [4]

$$n(r,t) = \frac{2+\theta}{\Gamma(d_f/(2+\theta))}[\frac{1}{K(2+\theta)^2 t}]^{d_f/(2+\theta)} \exp[-\frac{r^{2+\theta}}{K(2+\theta)^2 t}]. \qquad (36)$$

This can easily be checked by substitution.
A Gaussian distribution is reobtained for $d_f = d = 2, \theta = 0$. Using this expression to find $< r^2(t) >$, gives (17) with

$$C_2 = \frac{\Gamma(\frac{d_f+4}{2+\theta})}{\Gamma(\frac{d_f}{2+\theta})}[(2+\theta)^2 K]^{4/d_w}. \qquad (37)$$

The probability of return to the origin is according to (36)

$$n(0,t) = L^{-2d_f/d_w}(t) = L^{-d_s}(t), \qquad L(t) \propto \sqrt{Kt}, \qquad (38)$$

$L(t)$ being the diffusion length. d_s is called the *fracton* or spectral dimension. Equation (38) is the generalization of the Gaussian $(d_f = 2, \theta = 0)$ result $n(0,t) = L^{-d}(t)$ where $L(t) \propto \sqrt{Dt}$ is the diffusion length.

Again, we recall the Einstein relation (19). According to (32), the diffusion coefficient is space dependent, $D_p = \hat{D} = Kr^{-\theta}$. In addition, we must take into account that the diffusion takes only place on the infinite cluster only, so that $D_p = Kr^{-\theta}P_\infty(|p-p_c|)$. Thus, Einstein's relation takes the form

$$\Sigma \propto \hat{D}(r)P_\infty(p-p_c). \qquad (39)$$

According to (20) and (13) this yields

$$|p-p_c|^\mu \propto r^{-\theta}|p-p_c|^\beta. \qquad (40)$$

This will hold in particular for the correlation length $r \approx \xi \approx |p-p_c|^{-\nu}$, so that

$$\theta = \frac{\mu-\beta}{\nu}. \qquad (41)$$

It follows that the exponent in (35) is

$$\bar{\alpha} = \frac{2\nu}{2\nu+\mu-\beta}. \qquad (42)$$

This exponent differs from the exponent α that is derived in the previous Section and is given in (27). That difference is due to the fact that in this Section the random walk takes place on the infinite cluster only, while in the previous Section the diffusion was considered on the whole lattice.

Using (15) and (41), the spectral dimension (34) can be expressed in terms of the basic exponents

$$d_s = \frac{2d_f}{d_w} = 2\frac{d\nu - \beta}{2\nu + \mu - \beta} \tag{43}$$

It is found numerically that $d_s = 4/3$ is an excellent approximation for $d \geq 2$. For $d \leq 6$ it is even an exact result.

IV. Percolation and continuous fluids

Diffusive transport in turbulent fluids consists of a number of extremely complicated processes. One of the complicating factors is the existence of long-range correlations. In this Section we will apply the concepts of fractality and percolation to continuous fluids in order to catch some aspects of this problem. First we will deal with mono-scale fluids that are characterized by a single scale-length l and a single velocity V. This mono-scale model will be generalized to a multi-scale model for a percolative fluid. An effective way to describe turbulent transport is the use of scaling representations of characteristic parameters to interpret experimental results. Many applications of percolation theory for the description of turbulent diffusion are considered in [8].

Assume that the system under consideration is an incompressible 2D fluid system. Then, we may introduce the representation,

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{e}_z \times \nabla\psi(\mathbf{x}, t). \tag{44}$$

where $\mathbf{v}$ is the fluctuating fluid velocity and ψ the streaming potential. Since $\mathbf{v} \cdot \nabla\psi = 0$, the fluid parcels follow the instantaneous flow lines $\psi = constant$. The level curves of $\psi(\mathbf{x})$ are the fractal stream lines. The characteristic value of ψ on the scale l is $\Psi(l) \approx lV(l)$. The characteristic $V(l)$ represents the ”intensity” of the particular streamline with scale l. The most intense streamlines will contribute most to the transport.

At fixed time the streaming potential $\psi(x)$ can be imagined as a landscape with valleys and mountains connected by passes (saddle points). Suppose that all valleys with $\psi(x) < h$, where h is a constant, are flooded and form lakes. When h is small there are only a few lakes in an area with many mountains and one would not be able to cross the ψ-landscape by boat. For larger values of h the water level rises and some lakes become connected because the water level becomes higher than the pass between the corresponding valleys. For higher and higher values of h more and more lakes will be formed that also become more and more connected. The landscape now resembles more a system of some large lakes with a few rising mountains. Above some critical value h_c, the lakes are so much connected that one would be able to cross the ψ landscape over water. The

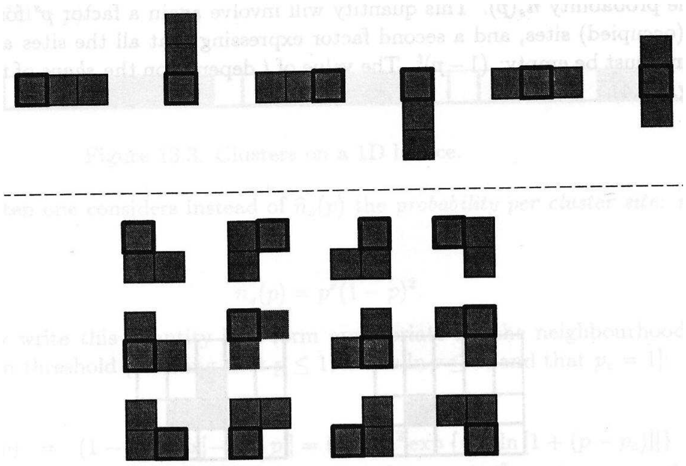

Figure 4: The equivalent percolation lattice of a random function $\psi(x, y)$.

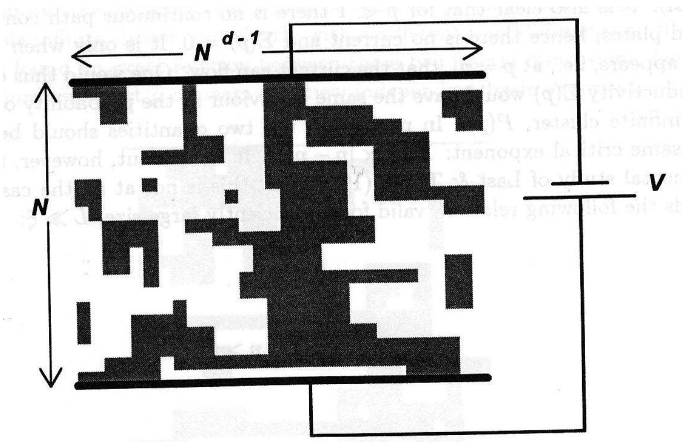

Figure 5:

transition from a landscape dominated by mountains to one dominated by water occurs quite abrupt. One could say that at this critical value h_c there occurs a phase transition. Such transitions and critical phenomena are typical percolation problems.

Percolation is a physical process that describes transition between two states of a system. It deals with such diverse phenomena as the flow of liquids through semi-porous media, electrical conductivity of alloys of conducting and isolating materials, diffusion of charged particles in a turbulent plasma, the percolation of water through a thin tissue, forest fires, etc.. All these phenomena may be captured under the heading of *diffusion in disordered media*.

Here, we will consider a 2D random flow from the point of view of percolation. We are interested in the behavior of large isolines i.e. lines

$$\psi(\mathbf{x}) = h. \tag{45}$$

The system is illustrated in the figure 3. The mountains are the maxima of the stochastic function $\psi(x, y)$, the valleys are the minima, and the passes are the saddle points of ψ. The steepest descend curves through a saddle points that connects two neighboring minima form the bonds of the system. A bond is 'conducting' if the water level is higher than the elevation of the saddle point, $\psi < h$. The coast lines, i.e., the contours of constant ψ form the perimeters of the percolation clusters. The level line $\psi(\mathbf{x}) = h$ through a randomly chosen point is closed with probability one, i.e. most clusters are finite. Exactly one open line exists at the critical level $h = h_c$. The appearance of the infinite cluster at $h = h_c$ is the phase transition occurring in such a system.

The streaming potential is bounded, statistically sign symmetric, homogeneous and isotropic, and does not contain degeneracies like periodicities or singularities.

The probability of conduction associated with different bonds must be independent. This requires that the correlation function $< \psi(\mathbf{x})\psi(\mathbf{x}') >$ should decay sufficiently fast with the distance $|\mathbf{x} - \mathbf{x}'|$.

Up to now the time did not play a role in our discussion of percolation. This means for the physical system upon which we want to apply this theory, that the Eulerian time on which the global geometry varies is required to be much longer than the Lagrangian time in which a particle circulates around a contour $\psi = constant$. This implies that the model is applicable to systems with long correlation times corresponding to large Kubo numbers.

This picture of continuum percolation was first discussed and applied to diffusion in a magnetized plasma by the authors of [5].

A. Mono-scale flow and percolation

Let us first consider the case of a mono-scale streaming potential with characteristic length l and characteristic value $\Psi \approx Vl$. This model has been introduced in Part III.

In order to apply percolation theory to this physical model, we have to choose a smallness parameter that is equivalent to the distance $|p - p_c|$ to the percolation threshold. On the basis of the discussion in the previous section, we choose this percolation parameter to be the distance to the critical value h_c,

$$|p - p_c| \approx \frac{|h - h_c|}{\Psi} = \epsilon, \qquad \epsilon \to 0. \tag{46}$$

This is the smallness parameter that describes the closeness to the percolation threshold. The maximum size of an contour is the correlation length ξ_h of the mono-scale level lines

$$\frac{\xi_h}{l} \approx |p - p_c|^{-\nu} \approx \epsilon^{-\nu}. \tag{47}$$

Here, ν is the percolation exponent which is $\nu = 4/3$ in 2D geometry. The diameter of the contour is equivalent to the size of the percolation cluster. A contour with a diameter larger than ξ_h is an exponentially improbable event. Therefore, for contours $a \leq \xi_h$, we have

$$\frac{|h - h_c|}{\Psi} = (\frac{\xi_h}{l})^{-1/\nu} \leq \frac{h(a)}{\Psi} = (\frac{a}{l})^{-1/\nu}. \tag{48}$$

This implies the scaling

$$a \approx l\epsilon^{-\nu}, \tag{49}$$

which illustrates that percolation theory applies to systems with long range correlations. In equation (48), $h(a)$ is the distance between two $\psi = constant$ levels

$$h(a) \approx \Delta\psi \approx w(a)|\nabla\psi| \approx w(a)\frac{\Psi}{l}.$$

The length $w(a)$ is taken to be the width of the percolation layer. The stream function (or the magnetic flux function) forms regular curves outside such a layer. This yields the following scaling for the width of the percolation layer

$$w(a) \approx \epsilon l. \tag{50}$$

The contour of diameter a with relative length L/l is a fractal curve with the dimension of a hull

$$d_h = \frac{\ln L/l}{-\ln l/a}, \tag{51}$$

which yields the scaling (2D)

$$\frac{L}{l} = (\frac{a}{l})^{d_h} \approx \epsilon^{-\nu d_h}, \qquad d_h = 1 + \frac{1}{\nu}. \tag{52}$$

This is equivalent to the scaling $L = \epsilon^{-1}a$. Thus, we have derived the scaling that is appropriate for the applicability of the percolation model to a mono-scale 2D flow

$$L \approx \frac{a}{\epsilon} >> a \approx l\epsilon^{-\nu} >> l >> w \approx \epsilon l. \tag{53}$$

The fraction of area occupied by an a-web is

$$\Phi(a) \approx \frac{w(a)L(a)}{a^2} \approx \epsilon^{\nu}. \tag{54}$$

This is the percolation fraction of space. The effective, classical coefficient of diffusion is

$$D_{eff}(\epsilon) \approx \frac{a^2}{\tau}\Phi(a) \approx \frac{a^2}{\tau}\epsilon^{\nu} \approx \frac{l^2}{\tau}\epsilon^{-\nu}, \tag{55}$$

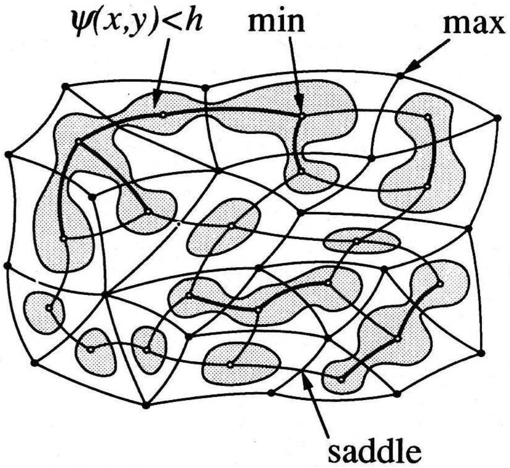

Figure 6: FIGURE SCALE-LENGTHS

a being the correlation length, τ the correlation time, and $\Phi(a)$ the percolation fraction (54).

The minimum lifetime τ of a percolation streamline is determined by the ballistic time L/V it takes for a fluid parcel to complete a flow line. This eddy turn-over time must be small as compared with the global time T during which the flow pattern exists. Thus, we have

$$\frac{L}{V} \leq \tau << T. \tag{56}$$

i. The ratio l/T is a measure for the slow velocity with which a cell changes its shape, so that the minimum correlation time holds when the ballistic time L/V is equal to the time wT/l it takes for a saddle point to cross the percolation width, which would imply an essential change of the cell,

$$\frac{1}{\tau} \approx \frac{V}{L} \approx \frac{l/T}{w}. \tag{57}$$

This is related to the condition that the flow is incompressible. It also means that $\tau \approx \epsilon T$. Collisions do not play a role.

Equation (57) leads to a relation between the percolation parameter ϵ and the Kubo number $K_u = VT/l$,

$$\epsilon \approx \frac{1}{K_u^{1/1+\nu d_h}}. \tag{58}$$

Then, one obtains from (56) the diffusion coefficient

$$D_{eff} = \frac{l^2}{T} K_u^{\nu d_h/1+\nu d_h} = \frac{l^2}{T} K_u^{0.7}, \qquad K >> 1, \tag{59}$$

where we have used the 2D values $\nu = 4/3$ and $d_h = 7/4$. *This is the percolation limit of* *diffusion at large Kubo numbers.*

ii. Suppose that the correlation time τ in (55) and (56) is set by the slow collisional diffusion through the percolation layer in stead of by the slow change of the configuration as in the previous case. The diffusion time τ_d through the layer is

$$\tau_d = \frac{w^2}{D_0} \approx \frac{(\Delta\psi)^2}{V^2 D_0} \approx \tau_\psi. \tag{60}$$

Here, τ_ψ may be interpreted as the 'field line diffusion time'. The relation $\tau_d \approx l/V$ implies that as many particles flow into the percolation layer as are carried away along the streamlines. This leads to a relation between the smallness parameter ϵ and the Peclet number $P_e = lV/D_0$

$$\epsilon \approx \frac{1}{P_e^{1/(3+\nu)}} = (\frac{D_0}{lV})^{3/13}. \tag{61}$$

Upon substituting (60) and (61) into (56) one obtains for the effective diffusion coefficient

$$D_{eff} \approx lV P_e^{-3/13}. \tag{62}$$

B. Stochastic $\mathrm{E} \times \mathrm{B}$ transport [7]

Let us apply the previous results to transport in electrostatic waves in a plasma that is embedded in a strong magnetic field. At low-frequencies (below the gyration frequency) and for long wavelength (larger than the gyro-radius), the drift motion of a guiding center particle is

$$\frac{d\mathbf{x}}{dt} = v_\parallel \mathbf{e}_z + \frac{c}{B}\mathbf{e}_z \times \nabla\phi(x,y,z,t). \tag{63}$$

The magnetic field is $B\mathbf{e}_z$, ϕ is the random electric potential, and $v_\parallel$ the velocity along the magnetic field. The field is taken to be homogeneous and uniform. Then, $v_\parallel$ is constant, so that the parallel component of (63) can be integrated, $z = z_0 + v_\parallel t$. Hence, (63) can be written in the form (44) with

$$\psi(x,y,t) = \frac{c}{B}\phi(x,y,z_0 + v_\parallel t, t). \tag{64}$$

Typical wavelengths are $(k_\perp, k_\parallel)$, with $k_\parallel \approx 1/qR$ for a tokamak. We will take for the widths of the spectrum $\Delta k_\perp \approx k_\perp$ and $\Delta k_\parallel \approx k_\parallel$. Then, the Eulerian life time T and the fundamental length l in the system are

$$T = \frac{1}{max(\omega_*, k_\parallel v_\parallel)}, \qquad l \approx \frac{1}{k_\perp}, \tag{65}$$

where ω_* is the drift-frequency. Inserting these values into the diffusion coefficient of percolation (59) gives

$$D \approx (\frac{1}{k_\perp^2 T})^{0.3} (\frac{c\Phi}{B})^{0.7}, \tag{66}$$

where Φ is the characteristic value of the electric potential. The effect of collisions has been neglected, which means that the collision time τ_{coll} has to be longer than T and the mean free path λ_{coll} longer than the length L of a flow line.

In case of stochastic magnetic fields in a system with a strong background magnetic field $B_0\mathbf{e}_z$, the field can be represented as

$$\mathbf{B} = B_0(\mathbf{e}_z + \mathbf{e}_z \times \nabla\psi). \tag{67}$$

Here, $\psi(x, y, z)$ is the random magnetic flux function. The stochastic field scales like $\delta\mathbf{b}/B_0 \approx \psi/l$. The field line equations in the transverse plane are

$$\frac{d\mathbf{x}}{dz} = \mathbf{e}_z \times \nabla\psi. \tag{68}$$

This is equivalent to (44). The diffusion coefficient is now $D \propto \Psi_0^{0.7}$.

V. Multi-scale flows

In this Section we will extent the self-similar mono-scale model to a multi-scale-model. We will deal with the scaling properties of transport processes in systems with a hierarchy of superimposed flows. These flows are spatially coupled and characterized by a nested system of scales.

The hierarchy of spatial scales l is

$$l_0 > l_1 > l_2 > ..... > l_m. \tag{69}$$

The total velocity field is the sum of the velocities on each scale

$$v(x) = v_0(x) + v_1(x) + ... + v_m(x). \tag{70}$$

The characteristic velocity on each scale is defined by

$$V_i(l_i) = \sqrt{< (v_i(x + l_i) - v_i(x))^2 >}, \tag{71}$$

and are assumed to obey the self-similar scaling

$$V_i = (\frac{l_i}{l_m})^{M-1} V_m = (\frac{l_i}{l_0})^{M-1} V_0. \tag{72}$$

This is equivalent to equation (6) of PART III with $h = M - 1$. Here, l may denote either the scale-length $l_{\|}$ along or the scale-length $l_{\perp}$ perpendicular to the stream lines or to some other specific direction set by the physics of the problem at hand. In the anisotropic case we will deal with the hierarchy $V_\perp \propto l_\parallel^{M-1}$. This means, e.g., that drift effects in the perpendicular direction depend on longitudinal scales.

Two different situations occur with respect to the value of M. The first case is $M > 1$. This corresponds to

$$V_0 > V_1 > V_2 > ...... > V_m. \tag{73}$$

The characteristic features of transport are determined by the maximum value of l for such a system.

Similar estimates have been considered for the Kolmogorov hierarchy of scales. The Komolgorov scaling of the energy in k-space, $E(k) \propto k^{-5/3}$, implies $\hat{V}_k \propto k^{-1/3}$, which means $V(l) \propto l^{1/3}$. This corresponds to $M = 4/3 > 1$.

The case of greatest interest, however, arises when $M < 1$. Then we have

$$V_0 < V_1 < V_2 < ...... < V_m. \tag{74}$$

This means that the small-scale field V_{i+1} causes small-scale, large amplitude perturbations of the large-scale field V_i. One might also say that the large-scale field V_i is a locally homogeneous, small perturbation of the small-scale field V_{i+1}.

If the random velocity can be represented by a streaming potential, $\mathbf{v} = \mathbf{e}_z \times \nabla\psi$, then we adopt, in agreement with (70),

$$\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \psi_1(\mathbf{x}) + ... + \psi_m(\mathbf{x}). \tag{75}$$

The characteristic value of ψ on the scale l is $\Psi(l) \approx lV(l)$ and is assumed to depend on the spatial scale l as $\Psi_i \approx \Psi_0 \big(l_i/l_0\big)^M$, in agreement with (72).

The physical smallness parameter is

$$\epsilon_0 = \frac{V_i}{V_{i+1}} = (\frac{l_i}{l_{i+1}})^{M-1}, \qquad M < 1. \tag{76}$$

Power spectrum

The Wiener-Khinchin theorem says that the power spectrum of a stationary random process is the Fourier transform of the correlation function,

$$\hat{P}(k) = \int d\rho C(\rho) e^{ik\rho} \tag{77}$$

The velocity correlation function is $C(\rho) =< v(x)v(x+\rho) >$. The inverse transform is

$$C(\rho) =< v(x)v(x+\rho) >= \frac{1}{2\pi} \int dk \hat{P}(k) e^{-ik\rho}, \tag{78}$$

with $\hat{P}(k)\delta(k+k') =< v_k v_{k'} >$. Apply this to the velocity field on the scale l_i. Consider the self-similar case $l_{i+1} = rl_i$ with $r < 1$. The power spectrum $\hat{P}(k)$ is the Fourier transform of the correlation function

$$< v^2 >= \sum_{r<kl_i<1} \hat{P}(k). \tag{79}$$

Assume that the power spectrum is isotropic in d dimensions and is an algebraically decaying function of k,

$$\hat{P}(k) = Ak_L^d k^{-\gamma}, \tag{80}$$

here $k_L = 2\pi/L$ is the global scale. Each interval $r < kl_i < 1$ contains many modes so that we may approximate

$$\sum_{r<kl<1} \rightarrow \frac{1}{k_L^d}\int_{kl=r}^{kl=1} d^d k. \tag{81}$$

This yields the following expression

$$< v^2 >= A\int d^d k\; k^{-\gamma} = AS_d \int_{kl=r}^{kl=1} dk\; k^{-\gamma+d-1} \tag{82}$$

$$= \frac{AS_d}{d-\gamma}(1-r^{d-\gamma})l^{\gamma-d}.$$

According to (72) we also have the scaling $v(l) = A_1 l^{M-1}$. From (82) one obtains the following relations between the exponents and coefficients

$$M = 1 + \frac{\gamma-d}{2}, \;\; A_1 = [\frac{AS_d}{d-\gamma}(1-r^{d-\gamma})]^{1/2}. \tag{83}$$

We also have

$$C(\rho) = \Sigma\hat{P}(k)e^{ik\rho} \quad \rightarrow \quad A\int d^d k\; k^{-\gamma}e^{ik\rho}.$$

This yields

$$C(\rho) = \xi_d A\rho^{2M}, \quad l_0 < \rho < l_m, \tag{84}$$

with

$$\xi_d = S_d \int_{k_m\rho}^{k_0\rho} dx\; x^{-1-2M}e^{ix}$$

In the limits $k_0\rho \rightarrow 0$ and $k_m\rho \rightarrow \infty$, the existence of the integral requires $M < 0$.

VI. Multi-scale flows and percolation

Consider the multi-scale streaming potential (75). The associated scales and velocities are given by (69) and (74). The physical smallness parameter is given by (76).

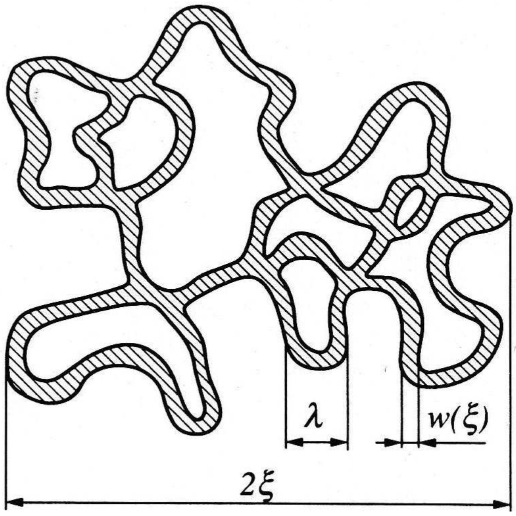

Figure 7: conservation

The scaling (72) with $M < 1$ implies that the large scale field V_{i-1} is a locally homogeneous, small perturbation of the small-scale field V_i. This means that the field ψ_{i-1} causes a change in the critical level $h_c(i)$ of ψ_i,

$$h_c = h_c(i) + w_{i-1,i}|\nabla\psi_i| \tag{85}$$

so that with $\Psi_i = V_i/l_i$, the smallness parameter of the percolation model is

$$\epsilon = \frac{|h - h_c(i)|}{\Psi_i} \approx \frac{w_{i-1,i}}{l_i}. \tag{86}$$

The multi-scale correlation lengths scales like

$$a_{i-1,i} \approx l_i \epsilon^{-\nu}. \tag{87}$$

In agreement with (52), the length of a contour on the scale of l_i is

$$L_i \approx l_i (\frac{a_{i-1,i}}{l_i})^{d_h}, \qquad d_h = 1 + 1/\nu. \tag{88}$$

The smallness parameter ϵ of percolation theory is not a physically observable quantity. Therefore, we introduce a renormalization by relating ϵ to the physical smallness parameter ϵ_0. Let us assume that

$$\frac{V_0}{V_1} \approx (\frac{l_1}{l_0})^{1-M} = \epsilon_0 = \epsilon^{\alpha+\nu} \tag{89}$$

and choose $\alpha = 1$. This choice implies

$$V_{i-1} a_{i-1,i} \approx V_i w_{i-1,i} \tag{90}$$

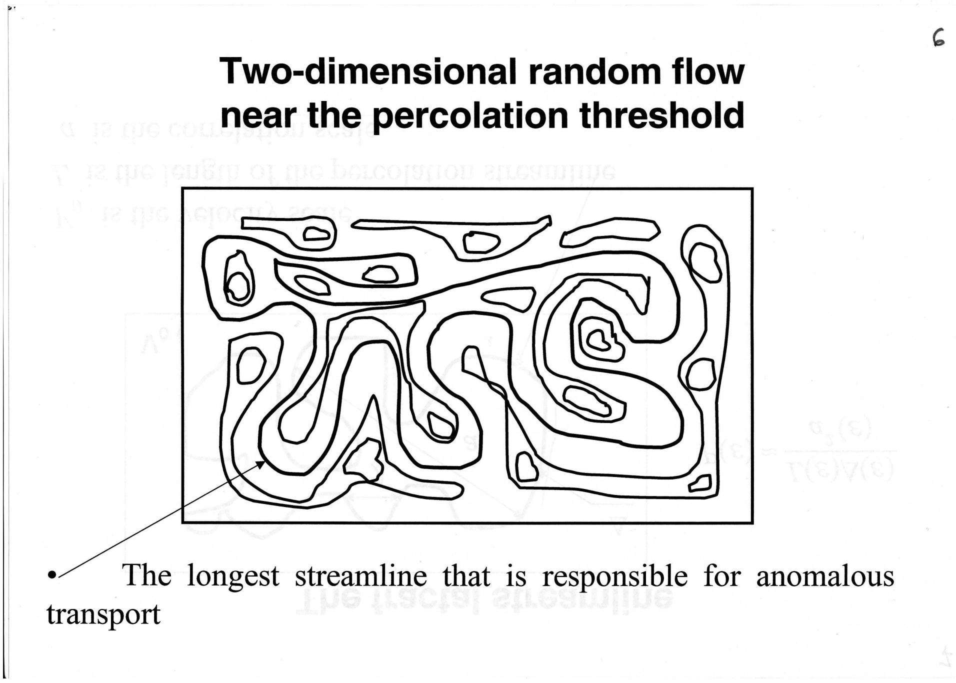

Figure 8: NOTES// ISICHENCO

or

$$\Phi_{i-1,i} V_{di} \approx V_{i-1}, \tag{91}$$

where $V_{di} = a_{i-1,i} V_i / L_i$ is the slow drift of the particle across the correlation length, and

$$\Phi_{i-1,i} \approx \frac{w_{i-1,i} L_i}{a_{i-1,i}^2} \approx \epsilon_0^{\frac{\nu}{1+\nu}} \tag{92}$$

is the area fraction covered by a streamline that extends over a correlation length $a_{i-1,i}$. Thus, the slow drift on the i'th scale has the same scaling as the velocity on the $(i-1)$'th scale.

The lengths $w_{i-1,i}$ and $a_{i-1,i}$ can now be expressed in terms of the physical smallness parameter or in terms of the ratio of the scale lengths

$$\frac{w_{i-1,i}}{l_i} = \epsilon_0^{\frac{1}{1+\nu}} = (\frac{l_i}{l_{i-1}})^{\frac{1-M}{1+\nu}}, \qquad \frac{a_{i-1,i}}{l_i} = \epsilon_0^{\frac{-\nu}{1+\nu}} = (\frac{l_i}{l_{i-1}})^{\frac{-\nu(1-M)}{1+\nu}}. \tag{93}$$

The ordering

$$...l_{i+1} < a_{i,i+1} < w_{i-1,i} < l_i < a_{i-1,i} < ....., \tag{94}$$

requires

$$-\frac{1}{\nu} < M < 1. \tag{95}$$

These inequalities guarantee that the perturbation ψ_{i-1} is weak and quasi-homogeneous with respect to ψ_i.

A. Application to a tokamak [9]

Consider a toroïdal plasma with magnetic field

$$\mathbf{B} = RB_\phi \nabla\phi + rB_\theta \nabla\theta + \tilde{\mathbf{B}} = B_\phi(\boldsymbol{e}_\phi + \frac{r}{qR}\mathbf{e}_\theta) + \nabla \times \tilde{\psi}\nabla\phi, \tag{96}$$

where

- ϕ and θ are the toroïdal and poloïdal angles,

- r is the minor radius,

- $R = R(1 + \epsilon \cos\theta)$ is the major radius,

- $\epsilon = r/R << 1$ is the ratio of the minor to the major axis (not to be confused with the percolation parameter),

- $RB_\phi \approx constant$ in a low-β plasma,

- $q = rB_\phi/RB_\theta$ is the inverse rotational transform,

- $\tilde{\mathbf{B}} = \nabla\tilde{\psi} \times \nabla\phi$ is the fluctuating magnetic field.

Consider turbulence with frequencies of the order of the drift frequency

$$\omega \approx \omega_* = k_\perp cT/eBl_n, \qquad l_n = n/|\nabla n|,$$

l_n being the density scale-length. These waves have long wavelengths parallel to the main field and short perpendicular wavelengths

$$k_\parallel^{-1} \approx qR, \qquad k_\perp^{-1} \approx \rho_i.$$

The width $\Delta\omega$ of the spectrum satisfies

$$\Delta\omega < \omega_* < \omega_{ti}, \qquad \omega_{ti} = v_{ti}/qR_0,$$

ω_{ti} being the thermal ion transit frequency.

We adopt the standard mixing length rule $\nabla\tilde{n} \approx \nabla n_0$ to get an estimate for the random electric potential $\tilde{\Phi}$

$$\frac{\tilde{n}}{n_0} \approx \frac{e\tilde{\Phi}}{T} \approx \frac{1}{k_\perp L_n}. \tag{97}$$

The toroidal electric field has to be small in a high temperature plasma. This means that contribution from the vector potential is of the same order as the one from the electric potential,

$$\frac{\tilde{\Phi}}{qR_0} \approx \frac{\omega}{cR_0}\tilde{\psi}.$$

This implies the following scaling for $\tilde{\mathbf{B}}$

$$\frac{\tilde{B}}{B} \approx \frac{k_\perp \tilde{\psi}}{R_0} \approx \frac{1}{k_\perp q R_0}. \tag{98}$$

The motion of the guiding center along the magnetic field is determined by the parallel electric field and the magnetic mirror force $-\mu(\mathbf{B}/B) \cdot \nabla B$,

$$\frac{dv_\parallel}{dt} = \frac{e}{m}\tilde{E}_\parallel - \mu B \frac{\epsilon}{qR}\sin\theta. \tag{99}$$

The guiding center motion is given by

$$\mathbf{u}_g = v_\parallel \frac{\mathbf{B}}{B} + \mathbf{u}_E + \mathbf{u}_B, \tag{100}$$

where

$$\mathbf{u}_E = \frac{c}{B}\nabla\phi \times \nabla\tilde{\Phi} \tag{101}$$

is the $E \times B$ velocity and $\mathbf{u}_B$ is the magnetic drift velocity of the electrons

$$\begin{aligned} \mathbf{u}_B &= -\frac{v_\perp^2 + 2v_\parallel^2}{2\omega_{ce}} \frac{\mathbf{B}}{B} \times \frac{\nabla B}{B} \\ &= \frac{v_\perp^2 + 2v_\parallel^2}{2\omega_{ce}} \nabla\phi \times \nabla R. \end{aligned} \tag{102}$$

The first term on the right of (99) is small as compared with the second term,

$$\left(\frac{e}{m}\tilde{E}_\parallel\right)\left(\mu B \frac{\epsilon}{qR}\right)^{-1} \approx \frac{\rho_i}{\epsilon l_n} << 1. \tag{103}$$

Hence, the fluctuating field has hardly any influence on the motion along the magnetic field. This means that the number of trapped and circulating particles is not affected by the fluctuations.

In principle, (99) can be solved for the parallel motion along the background field. This solution can be used to eliminate the dependence on the parallel coordinate in (100). The fluctuating part of the contribution $v_\parallel(t)\tilde{B}/B$ can be incorporated in a generalized potential. The guiding center motion can then be written as

$$\frac{d\mathbf{r}_\perp}{dt} = \nabla\phi \times \nabla\left(\frac{c}{B}\tilde{\Phi} - \frac{v_\parallel(t)}{B}\tilde{\psi}\right) + \mathbf{u}_B. \tag{104}$$

where the fluctuating potentials are functions of (r, θ, t). The time dependence consists of a contribution with frequency $\omega \approx \omega_*$ and of a part with 'frequency' $k_\parallel v_\parallel \approx v_\parallel / qR_0$, which is introduced as a result of the elimination of the parallel coordinate.

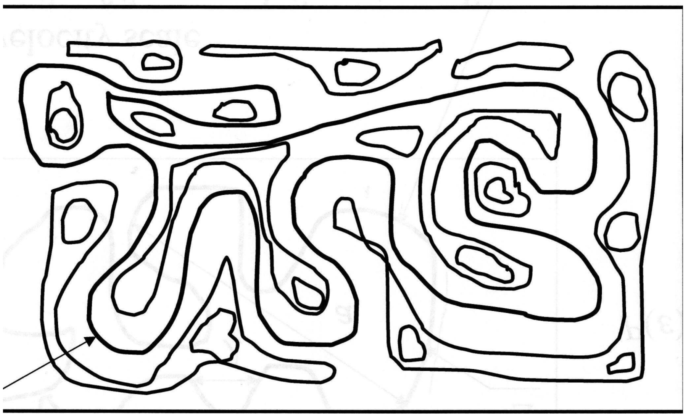

Figure 9: YUSMANOV

The ratio of the fluctuating part of the guiding center velocity to the magnetic drift velocity is

$$\frac{\tilde{u}_g}{u_B} = max\ \frac{R}{v_{ti}k_\perp \rho_i}(\omega_*,\ \frac{v_\|}{qR_0}) = max\ (\frac{R}{l_n},\ \frac{v_\|}{v_{ti}}). \tag{105}$$

Thus, the typical levels of the random velocity exceed the magnetic drift, which means that the deviation of the particle from the magnetic surface is determined by the fluctuating fields, not by the magnetic drift. The magnetic drift velocity can be considered as a large scale perturbation of the fluctuating part of the velocity. This is the essential point of this treatment.

Now we are in the position to apply the multi-scale percolation model of the proceeding section. We take $u_B = V_0$ and $\tilde{u}_g = V_1$. The physical smallness parameter is

$$\epsilon_0 = \frac{V_0}{V_1} = \frac{u_B}{\tilde{u}_g}. \tag{106}$$

According to (87), (88), and (89), the correlation length is $a_{01} = l_1 \epsilon_0^{-\nu/1+\nu}$ and the length of a streamline on the scale l_1 is $L = l_1 \epsilon_0^{-\nu d_h/1+\nu}$. The particle runs around a flux line in the time $\tau = L/V_1$. During this time the particle drifts over a distance of the correlation length a_{01}. The average drift velocity of such a particle is (see 91) and (92))

$$V_d \approx \frac{a_{01}}{\tau} \approx \frac{a_{01}}{L} V_1 \approx \epsilon_0^{-\nu/1+\nu} V_1 = (\frac{\tilde{u}_g}{u_B})^{\nu/1+\nu} u_B >> u_B. \tag{107}$$

This can be written as $\Phi_{01} V_d \approx u_B$. Thus, in the percolation region the drift velocity is enhanced over u_B by the area fraction.

The effective diffusion coefficient is

$$D_{eff} \approx \Phi_{01} \frac{a_{01}^2}{\tau_c} \approx \epsilon_0^{\nu/1+\nu} V_d^2 \tau_c \approx u_B^2 \tau_c (\frac{\tilde{u}_g}{u_B})^{4/7}, \tag{108}$$

where the correlation time is set by the dependence on the 'frequency' $\tau_c^{-1} \approx \omega_t = v_\| / qR$ of the streaming potential.

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