DIFFUSION: FROM CLASSICAL TO FRACTIONAL

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PART THREE: TURBULENCE AND DIFFUSION

I. Richardson's model of turbulence

A. Kolmogorov spectrum

Richardson proposed a model of stationary turbulence for fluids with a large Reynold's number

$$R_e = \frac{V l_0}{\nu} >> 1. \tag{1}$$

where V is the typical velocity of the fluid, l_0 its characteristic length, i.e. the length scale at which the fluid is stirred, and ν is the coefficient of viscosity. The Reynold's number R_e measures the ratio of the nonlinear inertial term, $\mathbf{v} \cdot \nabla \mathbf{v}$, and the viscous term, $\nu \nabla^2 \mathbf{v}$, in the Navier-Stokes equation. Thus, large Reynold's numbers refer to fluids that are highly nonlinear in which viscous effects play a subdominant role.

The smallest scale in the system is the viscous scale

$$l_\nu = \frac{\nu}{V_\nu}, \tag{2}$$

V_ν being the characteristic velocity at that scale. In the proposed model the turbulence is composed of 'eddies' of different sizes. The larger eddies are unstable and break up in smaller ones. At each level the turbulence is assumed to be isotropic and the eddies are supposed to be characterized by a single scale length l. The model is sketched in the figure below.

The range of scales where inertia is not important

$$l_\nu << l < l_0, \qquad l_0 < L \tag{3}$$

is called the inertial range. L is the size of the device. The range below l_ν is the dissipation range. The second inequality is required for the assumption that boundary effects do not play a role in what follows.

A convenient definition of the characteristic velocity on a scale l can be given in terms of the Eulerian (spatial) correlation function of the velocity increment within each eddy of scale l

$$V_l^2 =< V^2(\mathbf{r}, l) >=< [\mathbf{v}(\mathbf{r} + \mathbf{l}) - \mathbf{v}(\mathbf{r})]^2 > . \tag{4}$$

It is easily seen that a simple, linear relationship holds between the correlation function $C(l) = V_l^2 =< V^2(\mathbf{r}, l) >$ and the Euclidean spatial correlation function C_E of Part II. The time-scale associated with the velocity V_l is the eddy turn-over time

$$t_l = \frac{l}{V_l}. \tag{5}$$

Kinetic energy is put into the system at scale l_0. The energy of this original eddy will be divided over the smaller ones and transported down the scales in the inertial regime.

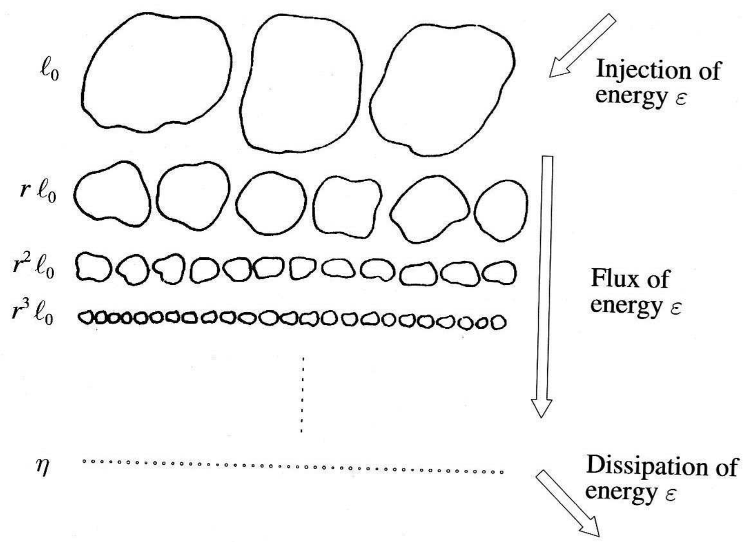

Figure 1: The energy cascade. At each step the eddies are space filling.

The interactions are nonlocal in space, but we assume that they have a 'local' character in the sense that the energy transfer in the inertial range is between nearby scales. This process of energy transfer continues until the energy reaches the viscosity scale where it will be dissipated. This mechanism requires a 3-dimensional system. Due to constraints that are inherent to the Navier-Stokes equations in the inertial regime, in two dimensions the energy flux is actually towards large scales.

Two important assumptions are made:

- the cascade is *self-similar* in the inertial range, i.e., there exist a scaling exponent h such that

$$V(rl) = r^h V(l), \qquad r < 1. \tag{6}$$

An important class of self-similar functions is the one consisting of power laws.

- the flow has a nonvanishing mean rate of dissipation per unit mass. This rate of dissipation ϵ scales as follows

$$\epsilon \propto \frac{V_\nu^3}{l_\nu}. \tag{7}$$

There is no energy input nor energy dissipation in the inertial range. Since the energy is transported locally, it follows that the energy flux is independent of the scale and equal to the energy dissipation rate ϵ at the scale l_ν, so that

$$\epsilon \propto \frac{V_0^3}{l_0} \propto \frac{< V^3(\mathbf{r}, l) >}{l} \propto \frac{V_l^3}{l}. \tag{8}$$

This means that
- the self-similarity exponent in (6) is $h = 1/3$,
- since the odd, 3rd moment does not vanish, the turbulent velocity field is non-Gaussian,
- the characteristic velocities decrease with decreasing scale-length, $V_l = V_0(l/l_0)^{1/3}$.

The invariant energy flux ϵ can be used to express the scale lengths in terms of the quantities that define the system. From (2) and (7) follows the *Kolmogorov dissipation scale*

$$l_\nu = (\frac{\nu^3}{\epsilon})^{1/4}, \tag{9}$$

and from (7) and (8) one obtains a relationship between the global scale and the viscous scale,

$$l_0 \propto l_\nu \, R_e^{3/4}. \tag{10}$$

According to (8) the energy density scales like

$$\frac{1}{2} V_l^2 \propto \epsilon^{2/3} l^{2/3}. \tag{11}$$

The power spectrum is the Fourier transform of the correlation function (Wiener-Khinchin theorem),

$$E(k) = \frac{1}{2} \int dl \; V_l^2 \exp ikl. \tag{12}$$

This means that the spectral energy density in the inertial region behaves like

$$E(k) \propto \epsilon^{2/3} k^{-5/3} \tag{13}$$

at large wave numbers. This is the famous *Kolmogorov spectrum* for stationary fluid turbulence. The Kolmogorov theory predicts the k-spectrum rather well. However, experimentally observed spectra are steeper than $k^{-5/3}$.

It follows from the self-similar model and from $< V^3(\mathbf{r}, l) > \propto \epsilon l$ that the higher order structure functions $S_p = < V^p(\mathbf{r}, l) >$ scale like

$$S_p = c_p \epsilon^{p/3} l^{p/3}.$$

For larger values of p, measured structure functions do, however, deviate from this scaling. See the discussion in [1].

In the preceding pages we have explored the Richardson model in the spatial domain. In the time domain one could conclude from the invariancy of the energy flux that the velocity fluctuations scale like $V_l^2 \propto \epsilon t$. This would imply classical transport (Brownian motion). This is a kind of paradox in this theory of fluid turbulence.

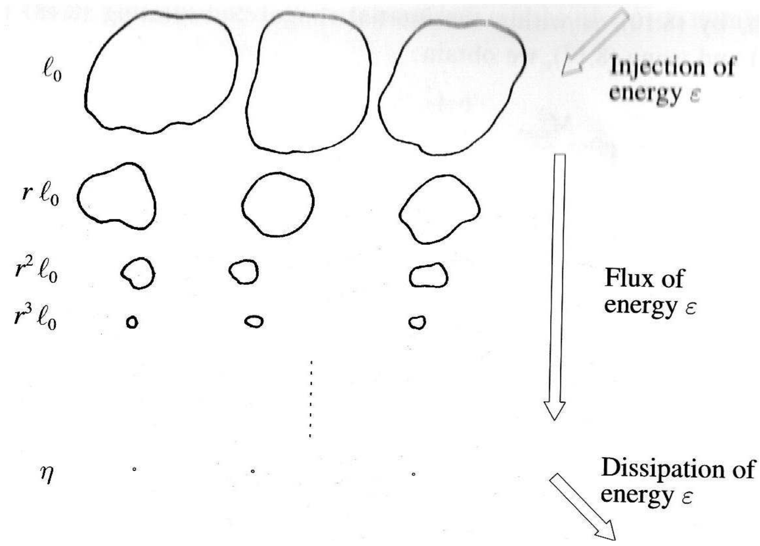

Figure 2: The energy cascade according to the β-model. With each step the eddies become less space filling.

B. β-model of fluid turbulence

The self-similarity of the system can also be introduced by requiring that at each step in the cascade the scale of the substructure is a factor r smaller than its parent structure, $l_n = r l_{n-1}$, so that

$$l_n = r^n l_0, \qquad n = 0, 1, 2...., \quad 0 < r < 1. \tag{14}$$

In mathematical terms one would say that *the set S of eddies is self-similar if S is the union of N non-overlapping subsets each of which is scaled down by a factor r from the original and is identical in all statistical aspects to $r(S)$.* The 'local' energy flux at scale l is the energy transferred from the scale $r^{-1}l$ to the scale rl.

The eddies are space filling if the number N of sub-eddies per parent eddy is

$$N = \frac{l_{n-1}^3}{l_n^3} = r^{-3}. \tag{15}$$

However, if the volume is filled with structures with a *fractal* geometry, then the effective volume, i.e., the volume filled with active elements, is smaller than the nominal volume. This is illustrated in Figure 2.

In the fractal case, the number of substructures per parent structure can be expressed as $N l_n^D = l_{n-1}^D$, i.e. $N r^D = 1$, so that

$$D = \frac{\ln N}{\ln 1/r}. \tag{16}$$

D is the *fractal dimension*. This dimension is only defined for self-similar fractals.

Suppose that at each step in the cascade the effective volume is a fraction β of the volume of the parent structure

$$Nl_n^3 = \beta l_{n-1}^3. \tag{17}$$

It follows that

$$\beta = (\frac{l_n}{l_{n-1}})^{3-D}. \tag{18}$$

The fraction of the space that is active within an eddy of size $l = r^n l_0$ is

$$p_l = \beta^n = (\frac{l}{l_0})^{3-D}, \qquad l \to 0. \tag{19}$$

This is also the probability that two points a distance l apart belong to the same fractal set. $3 - D$, or in general $d - D$, is called the codimension.

The energy per unit mass associated with scale l is

$$E_l \propto p_l V_l^2 = V_l^2 (\frac{l}{l_0})^{3-D}. \tag{20}$$

and the energy flux $\epsilon \propto E_l/\tau_l$ through the scales is

$$\epsilon \propto \frac{V_0^3}{l_0} \propto \frac{V_l^3}{l} (\frac{l}{l_0})^{3-D}. \tag{21}$$

It is seen that the scaling of the characteristic velocities is

$$V_l \propto \epsilon^{1/3} l^{\frac{1}{3} - \frac{3-D}{3}}. \tag{22}$$

Thus, the self-similarity exponent h in (6) is

$$h = \frac{1}{3} - \frac{3-D}{3}. \tag{23}$$

Note that for $D < 2$, the self-similarity exponent becomes negative, so that the characteristic velocity V_l increases with decreasing scale-length l.

Following the same reasoning as before one finds for the spectral energy density at large wave numbers

$$E_k = \frac{1}{2} \int dl V_l^2 \exp ikl \propto \int dl \epsilon^{2/3} l^{2[1/3-(3-D)/3]} l^{3-D}$$

so that

$$E_k \propto \epsilon^{2/3} k^{-[5/3+(3-D)/3]}. \tag{24}$$

This spectrum is steeper than the Kolmogorov spectrum (16), which is recovered for $D = 3$.

REMARK

Other definitions of a fractal dimension that often play a role in theories of turbulence are:

- box-counting dimension. The number of boxes of size b that are needed to cover a structure (per unit volume) is $Nb^{D_B} = 1$, so that

$$D_B = \frac{\ln N}{\ln 1/b}. \tag{25}$$

- divider dimension. obtained by measuring a curve with a yardstick of length δ. For a self-similar curve like a coastline, one has $N\delta_d^D = 1$ so that

$$D_d = \frac{\ln N}{\ln \delta^{-1}}. \tag{26}$$

II. Non-locality: Richardson's law of relative diffusion

Relative diffusion was first analyzed by Richardson in 1926. Two particles placed in a turbulent fluid and initially close together, have a separation R at time t,

$$\mathbf{R}(t) = \mathbf{x}_2(t) - \mathbf{x}_1(t). \tag{27}$$

In the spirit of the theory of single particle diffusion Richardson introduced the mean square relative displacement $< R^2(t) >$.

Initially the particles move apart under the action of the smallest eddies. When the particle distance becomes larger, more and more eddies influence the motion, although the eddy of the size of the inter-particle distance will be the most important one. When the particles separation further increases the eddy with the largest energy will become dominant. Finally the distance between the particles will be so large that their separation will be governed by the random walk of each individual particle.

The analysis of experimental data of atmospheric diffusion led Richardson to the conclusion that atmospheric diffusion differs in a essential way from the standard, classical proces. His investigations led to the following expression for the time dependence of the separation between two particles

$$\frac{1}{2}\frac{d < R^2(t) >}{dt} = A < R^2(t) >^{2/3} . \tag{28}$$

This yields the scaling

$$< R^2(t) >\propto t^3. \qquad (29)$$

This result differs essentially from the classical diffusive scaling, which would yield $< R^2(t) >\approx 4D_T t$. Richardson's law (28) implies an accelerated growth of the relative distance, which can not be explained on the basis of a standard diffusion equation with a constant coefficient. From the point of view of a spatial scaling law we deal with

$$D_R \propto< R^2(t) >^{2/3} . \qquad (30)$$

In fact his expression mirrors the non-local character of transport under conditions of atmosphere turbulence, where the separation between diffusing particles significantly changes under the influence of vortex motions. Richardson [2] attributed the non-local character to the increasing scale of eddies that take part in the process. In his approach the diffusion coefficient D_R is a function of the characteristic inter-particle distance which is the characteristic scale l of an eddy (See section III.1).

Richardson suggested to use a diffusion equation for the description of the evolution of the probability density $P(l, t)$ to find two particles, which are initially close together, on a distance l from one another at time t

$$\frac{\partial P(l,t)}{\partial t} = \frac{\partial}{\partial l} D_R \frac{\partial P(l,t)}{\partial l}, \qquad (31)$$

with

$$D_R(l) \propto l^{4/3}. \qquad (32)$$

Later, this idea was exploited by Kolmogorov and Obuchov within the framework of the theory of stationary, isotropic turbulence in fluids. Their assumption that the rate of energy dissipation rate ϵ is scale independent in the inertial range, yields the dimensional estimate $D_R(l) \approx \epsilon^{1/3} l^{4/3}$.

An alternative interpretation was discussed by Batchelor, who considered the diffusion coefficient D_R to be the result of statistical averaging and used the temporal scaling for D_R. Within this approach, dimensional considerations lead to the expression

$$D_R(t) \propto \frac{< R^2(t) >}{t} \propto [l^2(t)]^{2/3} \propto t^2. \qquad (33)$$

The diffusion equation (31) can be solved for both models. Assuming a point-source $\delta(l)$ of particles, we obtain for the Richardson model (32)

$$P(l,t) = \frac{8}{315\pi^{8/2}} \left(\frac{9}{4t}\right)^{3/2} \exp\left(-\frac{9l^{2/3}}{4t}\right), \qquad (34)$$

whereas for the Batchelor model (33) we obtain:

$$P(l,t) = \left(\frac{1}{2\pi < l^2(t) >}\right)^{3/2} \exp\left(-\frac{l^2}{2 < l^2(t) >}\right). \qquad (35)$$

Thus the models (32) and (33) into (31) give widely different results in spite of the same underlying law (28).

The arguments in favor of either model have a qualitative character and it cannot be decided which one is correct. Moreover, the combination of both approaches, where D_R depends algebraically on both the distance l and the time t,

$$D_R \propto l^{(2-\alpha)2/3} t^{\alpha}, \tag{36}$$

might also be relevant. The solution to (31) with D_R given by (36) for a $\delta(l)$ source at $t = 0$ is

$$G(l,t) = C_1 t^{-3/2} \exp -C_2 \frac{l^{(1+\alpha)2/3}}{t^{(1+\alpha)}}. \tag{37}$$

The nonlocal character, either in space or in time, inherent to a turbulent process that yields (28), can not be accounted for by the local approximation (31) to the diffusion process. In order to deal with non-localities, the conventional description of diffusion in terms of a transport equation that is second order in space and first order in time, has to be abandoned.

Many models have been put forward to improve the diffusion equation. Besides the different phenomenological methods, non-local effects can be also described in terms of a random walk model.

References

[1] U. Frisch, Turbulence, the Lgacy of A.N. Kolmogorov, Cambridge University Press, 1995.

[2] L.F. Richardson, Proc. Roy. Soc. London, Ser.A **110** 709 ((1926).