DIFFUSION: FROM CLASSICAL TO FRACTIONAL

Theo J. Schep

Technical University of Eindhoven, P.O.Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: t.schep4@kpnplanet.nl theo.schep@wanadoo.fr

PART TWO: CORRELATION FUNCTIONS and DIFFUSION

I. Diffusion and Autocorrelation Functions

In spite of considerable progress in the understanding of anomalous transport, many questions and issues raised in the early papers in this field of research are still of interest today. In this Section we will briefly discuss some of these issues. In particular the role of the concept of correlation functions in the search for modifications of the conventional diffusion equations and for scaling laws, in order to describe non-classical diffusion.

Turbulent diffusion can also be viewed within the context of correlation functions. A direct relationship between the diffusion coefficient and the autocorrelation function of the velocity was introduced by Taylor [1]. Actually, he introduced a new "tool" for the analysis of diffusion processes. Following the ideas of Einstein and Langevin, Taylor introduced a stochastic equation for the motion of a Lagrangian test particle in a random field.

Consider the equation for the trajectory of a fluid parcel in a turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$,

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t). \tag{1}$$

Since the velocity on the right hand side (rhs) depends on the trajectory $\mathbf{x}(t)$, (1) is highly nonlinear!

Suppose we follow a particle along its Lagrangian trajectory. It is obvious that the Lagrangian velocity $\mathbf{v}(\mathbf{x}_0, t)$ is equal to the Eulerian field value $\mathbf{u}(\mathbf{x}, t)$ if the particle trajectory $\mathbf{x}(t)$ which starts at $\mathbf{x}_0$ passes through the point $\mathbf{x}$ at time t, i.e., $\mathbf{v}(\mathbf{x}_0, t) = \mathbf{u}(\mathbf{x}(t), t)$.

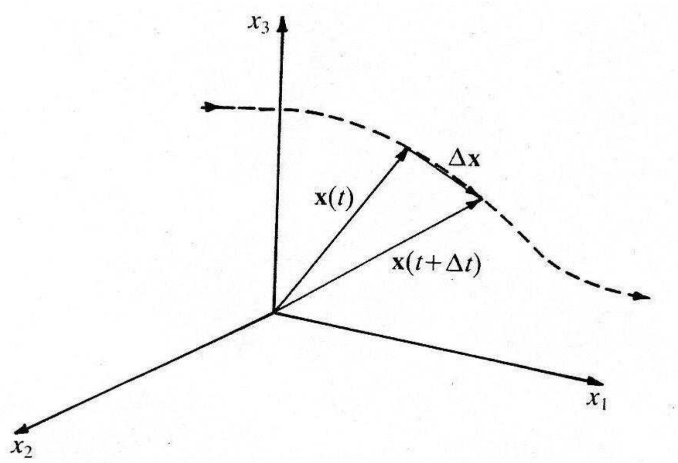

Figure 1: Lagrangian coordinates

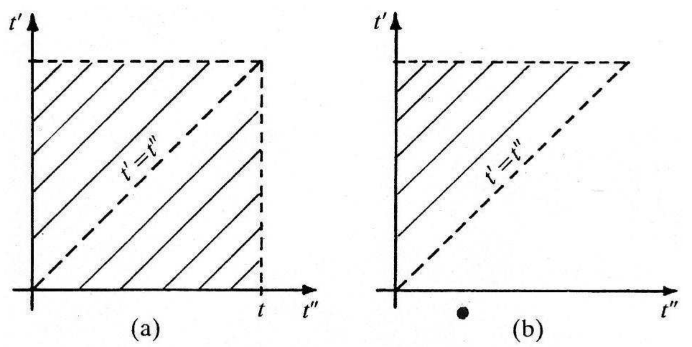

Figure 2: Integration domains

The formal solution of (1) is

$$\Delta \mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}(0) = \int_0^t d\tau \, \mathbf{v}(\mathbf{x}_0, \tau). \tag{2}$$

This solution represents the trajectory of a fluidparcel/particle parametrized by the time t. It is quite general and appears in many circumstances, e.g. it may also describe magnetic field lines. Then, $\mathbf{v}(\mathbf{x}_0, t)$ represents the magnetic field strength with t being an appropriate parameter that indicates the position along a field line. Equation (1) is also called the *V-Langevin equation*, in contrast to the *a-Langevin equation* introduced in Part I.

We assume that the average velocity and, thus, the average displacement of a fluid parcel/particle vanishes, $< \mathbf{v}(\mathbf{x}_0, t) >= 0$ and $< \Delta \mathbf{x}(t) >= 0$.

The mean square displacement follows from (2)

$$< (\Delta x_i)^2(t) >=< \int_0^t dt_1 \int_0^t dt_2 \, v_i(t_1) v_i(t_2) >, \tag{3}$$

$$= 2 \int_0^t dt' \int_0^{t'} d\tau \;\; < v_i(t') v_i(t' - \tau) > \tag{4}$$

Here, it is assumed that we have released at $t = 0$ a large number of particles at different positions in the fluid. The average is taken over the trajectories of all these particles. It clearly requires that the turbulence is homogeneous in space. Since the average is over trajectories, the integrand of (4) is the two-point *Lagrangian* correlation function of the velocity of a single particle.*We will also assume that the turbulence is stationary in time, so that the Lagrangian autocorrelation function of the velocity only depends on the time difference*

$$C_{Lii}(|t_1 - t_2|) =< v_i(t_1) v_i(t_2) > . \tag{5}$$

Note that for *stationary turbulence* the average square of the turbulent velocity does not depend on time, $< v_i^2(t) >=< v_i^2(0) >$.

Upon partial integrating (4), one obtains the following expression for the mean square displacement

$$< (\Delta x_i)^2(t) >= 2 \int_0^t d\tau \; (t-\tau) C_{Lii}(\tau). \tag{6}$$

The 'running' diffusion coefficient is defined as

$$D_{Ti}(t) = \frac{1}{2} \frac{d < (\Delta x_i)^2(t) >}{dt} = \int_0^t d\tau \; C_{Lii}(\tau). \tag{7}$$

This running diffusion coefficient is determined by the *turbulent* velocity field.

The Lagrangian integral time-scale τ_L is defined by

$$\tau_L = \int_0^\infty d\tau \; R_L(\tau) \tag{8}$$

where

$$R_L(\tau) = \frac{< v_i(t) v_i(t-\tau) >}{< v_i^2(t) >} \tag{9}$$

is the normalized correlation function. The MSD can be written as

$$< (\Delta x)_i^2(t) >= 2 < v_i^2(t) > \int_0^t d\tau \; (t-\tau) R_L(\tau). \tag{10}$$

The Lagrangian Taylor micro-scale $t < \tau_L$ corresponds to $\tau \to 0$. In this limit, $R_L(\tau) \to 1$ so that

$$< (\Delta x_i)^2(t) > \approx < v_i^2(t) > t^2. \tag{11}$$

This corresponds to *free streaming*.

It is quite natural to require that events that are widely separated in space and/or time become uncorrelated. This implies that $R_L(\tau) \to 0$ for $\tau \to \infty$. This limit defines the Lagrangian macro scale $t > \tau_L$

$$< (\Delta x_i)^2(t) > \approx 2 < v_i^2(t) > (t \tau_L - constant). \tag{12}$$

This means that for very long times the turbulent diffusion coefficient becomes constant

$$D_{Ti} = \lim_{t \to \infty} D_{Ti}(t) = \int_0^\infty d\tau \; C_{Lii}(\tau) \approx < v_i^2(t) > \tau_L. \tag{13}$$

This expression is appropriate for processes where decorrelation in time is dominant. Since long-term correlations and trapping in structures (e.g. magnetic islands, vortices) are neglected, (13) is only valid for small Kubo numbers (see next section).

In case spatial decorrelation is dominant, one has $< v_i^2 >\approx \lambda_c^2/\tau^2$, where λ_c is the correlation length, set by the average wavelength in the problem at hand, and τ the time it takes for the particle to travel over a distance λ_c. Then, the diffusion coefficient is

$$D_T \approx \frac{\lambda_c^2}{\tau}. \tag{14}$$

In PART ONE it has been shown that the diffusion coefficient of the standard model of the random walk scales like

$$D \approx \frac{\lambda_c^2}{\tau_c}, \tag{15}$$

where λ_c is the correlation length and τ_c the correlation time. In that case $< v^2(t) >\approx \lambda_c^2/\tau_c^2$ with τ_c being the Lagrangian correlation time τ_L, and the coefficients (13), (14), and (15) scale identically.

The classical regime, where (13) is valid, will never be reached in case of turbulence with strong memory effects where long time correlations exist or in case of nonlocal interactions with correlations over long distances. In many cases we will find that the mean square displacement scales algebraically with time according to

$$< (\Delta x_i)^2(t) >\propto t^\alpha, \tag{16}$$

where α is the diffusion exponent. $H = \alpha/2$ is called the Hurst factor. We will encounter the following regimes

$0 < \alpha < 1$ subdifussive regime, *strange* diffusion

$\alpha = 1$ classical regime or *anomalous* diffusion

$1 < \alpha < 2$ superdiffusive regime, *strange* diffusion

$\alpha = 2$ free streaming or *strange* diffusion.

In the classical regime the diffusion process is collision dominated with $\alpha = 1$.

Following [4], we distinguish between anomalous and strange diffusion. *Strange diffusion* is characterized by $\alpha \neq 1$. *Anomalous diffusion* is defined as a diffusive process $(\alpha = 1)$, but with a diffusion coefficient that depends on variables unrelated to collisions, like amplitudes of fluctuating fields, that characterize the disorder and randomness of the medium.

A. Fractional Brownian motion (fBm)

The position $x(t)$ of a Brownian particle is a stochastic variable. This position could also be introduced as follows. Let the displacement of the particle be given by

$$x(t) - x(t_0) \propto \xi |t - t_0|^H, \quad t > t_0, \tag{17}$$

for any two times (t, t_0). Here, ξ is a random variable that may or may not be Gaussian distributed. H is the Hurst factor.
If ξ is Gaussian distributed, then the mean square displacement may be written as

$$< (x(t) - x(t_0))^2 >= 2D\tau(\frac{|t - t_0|}{\tau})^{2H} \tag{18}$$

This is called *fractional Brownian motion*. For $H = 1/2$ classical diffusion is recovered.

Fractinal Brownian motion was introduced by Mandelbrot as a generalization of classical diffusion for $H = 1/2$ to any arbitrary number on the interval $0 < H < 1$. Fractional Brownian motion has *infinitely long time correlations*. The correlation between past and future displacements is

$$< (x(t_0) - x(-t))(x(t) - x(t_0)) > .$$

Take for simplicity $x(t_0) = 0$. Then, using $< (x(t) - x(-t))^2 >= 2 < x^2(t) > -2 < x(t)x(-t) >$, one obtains the normalized correlation

$$C(t) = \frac{- < x(-t)x(t) >}{< x(t)^2 >} = 2^{2H-1} - 1. \tag{19}$$

Note that for $H = 1/2$ the correlation vanishes. This the classical regime. The range $0 < H < 1/2$ corresponds to sub-diffusive behavior, while $1/2 < H < 1$ corresponds to super-diffusive behavior.

B. Velocity shear

In dealing with super-diffusion one must take care to deal with true diffusion processes. The presence of velocity shear could lead to wrong conclusions.

The particle trajectories are given by

$$\frac{dx}{dt} = v_x(t), \qquad \frac{dy}{dt} = v_y(t) + bx(t). \tag{20}$$

Here, $v_{x,y}(t) = v_{x,y}(x(t), y(t), t)$ are the fluctuating Lagrangian velocities and bx is an additional sheared velocity field in the y-direction.
The formal solutions of (20) are

$$x(t) = \int_0^t dt' \ v_x(t'), \tag{21}$$

$$y(t) = \int_0^t dt'\, v_y(t') + b\int_0^t dt' \int_0^{t'} dt''\, v_x(t''). \tag{22}$$

Consider a velocity field with the following statistical properties.

a. The average velocities vanish

$$< v_x >=< v_y >= 0.$$

This implies $< x >= 0, < y >= 0$.

b. The x- and y-velocities are statistically independent

$$< v_x v_y >= 0.$$

c. The turbulence is uniform in time and the velocities are δ-correlated

$$< v_x(t)v_x(t' >) = D_{xx}\delta(t-t'), \quad < v_y(t)v_y(t' >= D_{yy}\delta(t-t'),$$

where $D_{xx} =< v_x^2 > \tau_0, \;\; D_{yy} =< v_y^2 > \tau_0$, τ_0 being the short timescale of the random process. This requires timescales that are large as compared with the time scale set by viscosity (see section on the Langevin approach).

The mean square displacements are obtained from (21) and (22),

$$< x^2(t) >= \int_0^t dt' \int_0^t dt'' < v_x(t')v_x(t'') >, \tag{23}$$

$$< y^2(t) >= \int_0^t dt' \int_0^t dt'' < v_y(t')v_y(t'') > + \tag{24}$$

$$b^2 \int_0^t dt' \int_0^{t'} dt'' \int_0^t ds \int_0^s ds' < v_x(t'')v_x(s') > .$$

This yields

$$< x^2(t) >= 2\int_0^t d\tau\, (t-\tau) < v_x^2 > \tau_0\delta(\tau) = 2D_{xx}t \tag{25}$$

and

$$< y^2(t) >= 2\int_0^t dt' \int_0^{t'} dt'' < v_y(t')v_y(t'') > +$$

$$b^2 \int_0^t dt' \int_0^t ds(t-t')(t-s) < v_x(t')v_x(s) >$$

$$= 2D_{yy}t + \frac{2}{3}b^2 D_{xx}t^3. \tag{26}$$

It looks if the last contribution represents super-diffusion (even faster than free streaming) in the direction of the shear flow. However, this motion in the y-direction is a combined *effect of standard diffusion along x and of shear flow along y. Due to diffusion the particle travels to positions in x with larger $< x^2 >$. At this position the particle undergoes a larger shear velocity along y.*

Independent coordinates can be found as follows. Write $\hat{y} = y - \alpha x$ and determine α from the condition $< x\hat{y} >= 0$. Since $< xy >= bD_{xx}t^2$ one obtains $\alpha = bt/2$. It follows that the independent coordinates are $(x, \hat{y})$ with

$$\hat{y} = y - \frac{1}{2}btx. \tag{27}$$

The mean square displacement is

$$\begin{aligned} < \hat{y}^2 > &=< y^2 > -bt < xy > + \frac{1}{4}b^2t^2 < x^2 > \\ &= 2D_{yy}t + \frac{1}{6}b^2t^3D_{xx}. \end{aligned} \tag{28}$$

On the basis of the Central Limit Theorem, the probability density for the position of a particle (a random walker) starting at $x = 0, y = 0$ at time $t = 0$ is,

$$n(x, y, t) = \frac{1}{\sqrt{4\pi^2 < x^2 >< \hat{y}^2 >}} \exp\left(-\frac{x^2}{2 < x^2 >} - \frac{\hat{y}^2}{2 < \hat{y}^2 >}\right), \tag{29}$$

where $< x^2 >$, $\hat{y}$ and $< \hat{y}^2 >$ are given by (7), (9) and (10), respectively.

Equations (25), (28), and (29) correspond to eqs (18), (19), and to (13) of [3], respectively. It can be shown by substitution that (29) is the solution to the diffusion equation,

$$\frac{\partial n}{\partial t} + bx\frac{\partial n}{\partial x} = D_{xx}\frac{\partial^2 n}{\partial x^2} + D_{yy}\frac{\partial^2 n}{\partial y^2}. \tag{30}$$

II. The Corrsin approximation

The quantity that appears in the Lagrangian correlation function (5) is the product of the velocities at two different times along the same trajectory. The average is taken over all trajectories in the volume. This theory requires a knowledge of Lagrangian trajectories. However, these Lagrangian quantities cannot be determined experimentally. The quantity that is accessible to measurements is the correlation function in *Eulerian coordinates* i.e. at fixed points in space

$$C_{Eij}(\mathbf{x}, t) =< u_i(\mathbf{x}_1, t_1)u_j(\mathbf{x}_1 + \mathbf{x}, t_1 + t) > . \tag{31}$$

This is the average over the product of the velocities at positions that are a distance $\mathbf{x}$ apart in space and t in time. One could also say that the Eulerian average is the result of many fluid particles passing through two measuring points over a period t. These positions are in general not connected by particle trajectories.

This difference between Lagrangian and Eulerian correlation functions is one of the essential difficulties in the theory of turbulence. In order to proceed it seems necessary to find the relationship between Lagrangian and Eulerian coordinates. However, to find such a relationship would practically mean that we are able to solve the general problem of turbulence!

A famous approximation that allows to express the Lagrangian correlation function in terms of the Eulerian one was introduced by Corrsin. The Lagrangian correlation can be expressed as follows

$$C_{Lij}(t) = \int d^d x C^c_{Eij}[\mathbf{x}, t|\mathbf{x}(t) = \mathbf{x}]\rho(\mathbf{x}, t). \tag{32}$$

where E^c_{ij} is the Eulerian velocity correlation under the condition that the trajectory is at $\mathbf{x}$ at time t,

$$C^c_{Eij} =< u_i(0,0)u_j(\mathbf{x}, t) > |_{\mathbf{x}=\mathbf{x}(t)}, \tag{33}$$

and $\rho(\mathbf{x}, t)$ is the probability density that the particle is on the particular trajectory $\mathbf{x} = \mathbf{x}(t)$.

The Corrsin approximation consists of two elements.
1. At long diffusion times the pdf of the particle displacements and the one of he Eulerian velocity field become independent of each other. The particle trajectories are statistically independent of the stochastic velocity field. This means that the Lagrangian character of E^c_{ij} is neglected and that C^c_{Eij} may be replaced by the Eulerian correlation C_{Eij}.

$$C_{Lij}(t) = \int d^d x C_{Eij}(\mathbf{x}, t)\rho(\mathbf{x}, t). \tag{34}$$

Note that $\mathbf{x}$ in $\rho(\mathbf{x}, t)$ is the difference between two positions along a trajectory, while in $C_{Eij}(\mathbf{x}, t)$ it is the difference between the positions of two arbitrary points.
2. The Lagrangian orbits have a diffusive character, i.e., the displacements have a Gaussian distribution

$$\rho(\mathbf{x}, t) = \frac{1}{(2\pi < x^2(t) >)^{d/2}} \exp -\frac{x^2}{2 < x^2(t) >}. \tag{35}$$

If the process is diffusive one has $< x^2(t) >= 2dDt$.

III. Anisotropy and double diffusion

In this Section we will consider diffusion processes in which the anisotropy of the medium plays an important role. We consider systems in which the particles undergo a classical

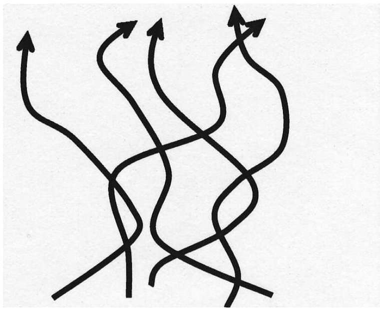

Figure 3: A layered medium with random jets.

diffusive motion in, let's say, the longitudinal direction and an additional stochastic motion in the perpendicular direction. Transport in such anisotropic media is either characterized by super-diffusive or by sub-diffusive processes. The transverse displacement (i.e. the root mean square displacement) is described by the scaling law,

$$\lambda_\perp \propto t^H. \tag{36}$$

Here, H is the Hurst factor. In the case of classical diffusive behavior we find $H = 1/2$.

A. A model with super-diffusion

A physical model of particle behavior under the influence of strongly anisotropic diffusion, was considered by Dreizin and Dykhnes [2]. The basic idea is the following.

A conducting fluid (plasma) is embedded in a magnetic field. The particles experience a "seed" diffusion with coefficient $D_\parallel$ in the direction of the field. During its diffusive motion along magnetic field lines, the particle travels through a set of perpendicular layers of width a. The time it takes to diffuse through such a layer is

$$\tau = \frac{a^2}{2D_\parallel} \tag{37}$$

In these layers, random jets with velocity $\pm V_0$ translate these particles in the perpendicular plane creating narrow convective flows of width a. During this diffusion time τ the particle will take a step $V_0\tau$ either to the right or to the left. The diffusion coefficient in the transverse direction $D_\perp$ will depend on the number of times a particle returns to the same layer during its motion along the magnetic field.

The transverse diffusion is described by

$$D_\perp \approx \frac{\lambda_\perp^2}{t}, \quad \lambda_\perp^2 \approx V_0^2 t^2 P, \quad P = \frac{N_r}{\hat{N}}. \tag{38}$$

Here, $\lambda_\perp = \sqrt{< \mathbf{x}_\perp^2 >}$ is the transverse displacement during the time t and P is the relative number of ”non-compensated” fluctuations. The number of *different* shear flows intersected by the particle during its longitudinal motion is

$$N \approx \frac{\sqrt{2D_\| t}}{a}, \tag{39}$$

while the *total* number of shear flows crossed in time t is

$$\hat{N} \approx \frac{t}{\tau} \approx \frac{2D_\| t}{a^2}. \tag{40}$$

The number of times a particle visits the same layer along its diffusive trajectory is denoted by N_r.

The particle undergoes a classical random walk in the z-direction along the field lines. The probability density $\rho(z,t)$ to find the walker at a distance z at time t is

$$\rho(z,t) = \frac{1}{(4\pi D_\| t)^{1/2}} \exp\left(\frac{-z^2}{4D_\| t}\right). \tag{41}$$

The limit $z \to 0$ corresponds to the probability of return to the initial layer

$$\rho(0,t)a = \frac{a}{\sqrt{4\pi D_\| t}}. \tag{42}$$

This yields

$$N_r \approx \hat{N}\frac{a}{\sqrt{4\pi D_\| t}} \propto \sqrt{\frac{t}{\tau}} \approx \sqrt{\hat{N}}. \tag{43}$$

Thus, one obtains from (38)

$$\lambda_\perp^2 \approx V_0^2 t^2 \frac{N_r}{\hat{N}} \approx \frac{V_0^2 a}{\sqrt{4\pi D_\|}} t^{3/2} >> t \tag{44}$$

and

$$D_\perp \approx V_0^2 a \sqrt{\frac{t}{4\pi D_\|}}. \tag{45}$$

This is the *super-diffusive regime with a Hurst factor* $H = 3/4$.

Consider the correlation function in the form,

$$C(t_1,t_2) = \int_{-\infty}^{\infty} < V(0)V(z) > \rho(z, t_2 - t_1)dz, \tag{46}$$

where $\rho(z, t_2 - t_1)$ is given by (41) Here, $V(z)$ is the velocity of the flow at the point z. This representation corresponds to the Corrsin conjecture of the diffusive nature of decorrelations.

The probability to return to the initial point is given by the limit $z \to 0$ in (41). In this limit, one obtains the expression,

$$C(t_1, t_2) = C(\tau) \approx \frac{V_0^2 a}{\sqrt{4\pi D_{\|}\tau}}, \qquad \tau = t_1 - t_2. \tag{47}$$

It is seen that $C(\tau) \approx V_0^2/N(\tau)$.

The correlation function (47) leads to the diffusion coefficient,

$$D_{\perp} = \frac{d}{dt}\lambda_{\perp}^2 = \int_0^t C(\tau)d\tau, \tag{48}$$

so that

$$\lambda_{\perp}^2 \approx \frac{V_0^2 a}{\sqrt{4\pi D_{\|}}} \int_0^t \int_0^t \frac{dt_1 dt_2}{\sqrt{t_1 - t_2}} \approx \frac{V_0^2 a}{\sqrt{4\pi D_{\|}}} t^{3/2}. \tag{49}$$

This is identical to (45).

It is concluded that even a small number of uncompensated flows $P = N_r/\hat{N} \approx 1/\sqrt{\hat{N}}$, leads to a considerable deviation of transport from the standard diffusive behavior.

B. Diffusion in a stochastic magnetic field: sub-diffusion

Next we analyze the ”double diffusion” scaling law, which is one of the first models of anisotropic diffusion in a magnetic field.
Consider a plasma embedded in a magnetic field. This field consists of a strong, homogeneous and uniform axial field $B_0\mathbf{e}_z$ and a stochastic field $\mathbf{B}_{\perp} = B_0\mathbf{b}(\mathbf{x}_{\perp}, z)$ in the transverse plane,

$$\mathbf{B}(\mathbf{x}_{\perp}, z) = B_0(\mathbf{e}_z + \mathbf{b}(\mathbf{x}_{\perp}, z)), \tag{50}$$

where $\mathbf{x}_{\perp} = (x, y, 0)$. The field line equation is given by

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_0}, \tag{51}$$

which can be written as

$$\frac{d\mathbf{x}_{\perp}}{dz} = \mathbf{b}(\mathbf{x}_{\perp}, z). \tag{52}$$

This equation is equivalent to equation (1) for the trajectory of a fluid parcel, the normalized magnetic field strength plays the role of the velocity and the position z along the field line the role of time.

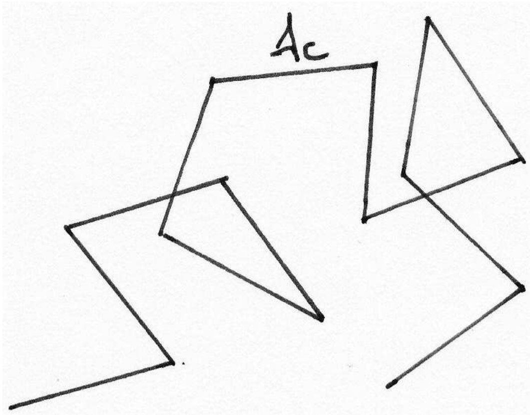

Figure 4: Stochastic field lines.

The magnetic field lines execute a random motion in the transverse plane. Analogously to the treatment in the previous sections we will now find a magnetic diffusion coefficient D_m. For stochastic variables b_x and b_y that are independent, have zero averages, and equal variances, we have

$$D_m = \frac{1}{4}\int_{-\infty}^{+\infty} dz \ < b_x(\mathbf{x}_\perp(z), z) b_x(0,0) + b_y(\mathbf{x}_\perp(z), z) b_y(0,0) > . \tag{53}$$

Assume that the $\mathbf{x}_\perp$*-dependence in the right hand side of (52) is weak and may be neglected. This is the quasi-linear approximation.* Then, the magnetic analogue of (13) is

$$D_m = \frac{1}{2}\int_{-\infty}^{+\infty} dz \ < b_x(0,z) b_x(0,0) > \approx b_0^2 \lambda_\parallel, \tag{54}$$

where $< b_x^2 > = < b_y^2 > = b_0^2$, and $\lambda_\parallel$ is the correlation length along the main field. The displacement of the magnetic field line in the transverse plane over a distance $l_\parallel$ in the longitudinal direction [4] is

$$\lambda_\perp^2 \approx 2 D_m l_\parallel . \tag{55}$$

The approximation (54) clearly requires $b_0 \lambda_\parallel << l_\perp$, where $l_\perp$ is the characteristic scale of the magnetic field in the transverse directions. Equation (54) corresponds to the quasi-linear approximation and is only valid for small magnetic Kubo number, $K_m = b_0 \lambda_\parallel / l_\perp << 1$.

The relationship between particle diffusion and the stochastic motion of the field lines is in general quite complex. Let us assume that the particles are tied to the magnetic field lines, so that while moving along field lines, they wander stochastically in the transverse plane. Since the particles are tied to the field lines, the perpendicular particle motion is also stochastic with the same standard deviation $\lambda_\perp^2$. Further, assume that the particles undergo a classical diffusion process along the field lines with diffusion coefficient

$$l_\parallel = \sqrt{2 D_\parallel t}, \qquad D_\parallel = \frac{\lambda_{coll}^2}{\tau_{coll}} . \tag{56}$$

Hence, the particles undergo a double diffusion process: a stochastic motion in the transverse plane and a classical diffusion process along the field lines. From (53) and (56) one obtains the following estimate for the particle diffusion

$$\lambda_\perp^2 \approx 2D_m l_\| \approx 2D_m \sqrt{2D_\| t}. \tag{57}$$

This is much smaller than t for large t. Thus, the scaling law for the transverse displacement of the particles has a *sub-diffusive* form with Hurst factor $H = 1/4$. This sub-diffusive character is absent if the motion along the magnetic field is ”ballistic”, $l_\| \approx Vt$. Thus, the character of transverse diffusion is determined by the actual longitudinal transport mechanism.

IV. Kubo and Péclet numbers

A. Kubo number

The Kubo number is the ratio of the distance covered by a particle in the correlation time τ_c to the correlation length λ_c of the stochastic field

$$K = \frac{V\tau_c}{\lambda_c}, \tag{58}$$

where
- V is the characteristic amplitude of the *fluctuating* velocity,
- λ_c the average wavelength in the Eulerian velocity correlation function $C_E(\Delta, t)$,
- τ_c correlation time of $C_E(\Delta, t)$.

These are all Eulerian quantities. However, the Kubo number can also be interpreted as the ratio of the Eulerian time τ_c to the Lagrangian time λ_c/V.
In case of standard diffusion processes, $V \approx \lambda_c/\tau_c$ so that $K \approx 1$.
Analogous to equation (58), we may define on the basis of (52) a magnetic Kubo number

$$K_m = \frac{b_0 l_\|}{l_\perp}, \tag{59}$$

where b_0 is the characteristic value of the fluctuating field, $l_\|$ the correlation length in the direction along the main field (the coordinate along this field plays the role of time), and $l_\perp$ the correlation length in the transverse plane.

A.1 Small K

The particle covers only a small distance as compared with the correlation length, $\lambda \propto V\tau_c < \lambda_c$, before it is decorrelated. The particle cannot explore the spatial structure of the

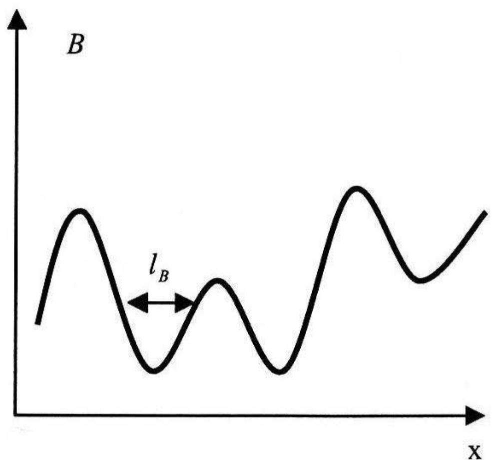

Figure 5: Schematic trajectory of a particle for $K >> 1$.

field. This corresponds to *decorrelation in time*. The scaling of the diffusion coefficient with K is

$$D \approx \frac{\lambda^2}{2\tau_c} \propto V^2\tau_c = K^{2}\frac{\lambda_c^2}{\tau_c}. \qquad (60)$$

This the *quasi-linear regime.* Examples were discussed in section II.

A.2 Large K

The spatial step is the correlation length of the stochastic field, $\lambda \approx \lambda_c << V\tau_c$. This is *decorrelation in space*. This regime is valid in case of long-term correlations and trapping in field structures where particles execute semi-periodic motions before they escape again. The time step

$$\tau = \frac{\lambda_c}{V} = \frac{\tau_c}{K} << \tau_c. \qquad (61)$$

is much smaller than the correlation time. If trapping does not occur, the diffusion coefficient is

$$D \approx \frac{\lambda_c^2}{\tau} \approx V^2\tau = K\frac{\lambda_c^2}{\tau_c}. \qquad (62)$$

The total length of the diffusive path covered in time τ_c contains many correlation lengths

$$L \approx V\tau_c \approx N\lambda_c \qquad (N \approx K >> 1). \qquad (63)$$

The diffusion length covered in time $t = N\tau$ is much smaller

$$l_{diff} \approx \sqrt{2Dt} = \sqrt{2DN\tau} = \lambda_c\sqrt{N}. \qquad (64)$$

In case of trapping in a field structure, one might expect to obtain an approximate expression for the decorrelation time like $\tau = \tau_t + \lambda_c/V$, where τ_t represents the trapping time.

Then, one would find

$$D \approx \frac{\lambda_c^2}{\lambda_c/V + \tau_t} = \frac{\lambda_c^2}{\tau_c}\frac{K}{1 + K\tau_t/\tau_c}. \tag{65}$$

B. Peclet number

The Peclet number is a measure for the strength of the convective transport with respect to diffusive transport

$$P_e = \frac{Vl}{D}, \tag{66}$$

where V is the characteristic velocity of the fluid, l its characteristic length (e.g. $l^{-1} \approx \nabla V/V$), and D its diffusivity.

For large values of the Kubo number K, one finds with (58) and (62)

$$P_e = \frac{Vl}{D} \approx \frac{l}{\lambda_c}. \tag{67}$$

Thus, $P_e \approx 1$ for large values of K if the scale-length l and the correlation length λ_c are of the same order of magnitude.

For small Kubo numbers one finds, using (58) and (60)

$$P_e \approx \frac{l}{K\lambda_c}. \tag{68}$$

If the scale-length l is smaller than or of the order of the correlation length λ_c, small Kubo numbers imply large values of the Péclet number.

This case can be illustrated with an example of 2D streaming , the convective cell (see figure). Here, the cell size l is of the same order as the correlation length. The width of the layer is the diffusive displacement across the streamlines in a time of the order of the rotation around the cell l/V

$$D \approx \frac{\delta^2}{l/V} \tag{69}$$

this leads to

$$\delta \approx (\frac{Dl}{V})^{1/2} = lP_e^{-1/2}. \tag{70}$$

Since the model requires $\delta << l$, the convective cell only exists at large P_e values and, thus, for small Kubo numbers.

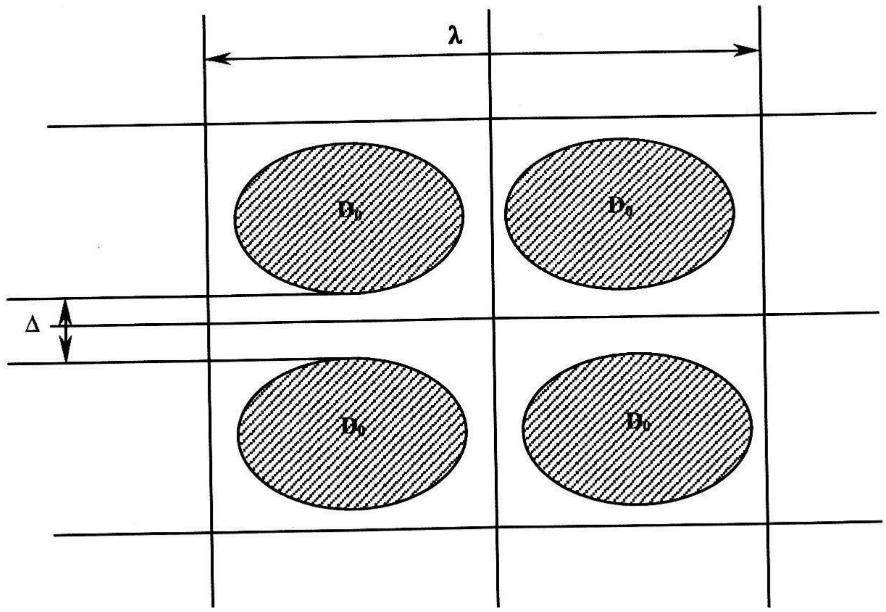

Figure 6: A convective cell

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