Anomalous Carnot cycle

Evaldo M. F. Curado
CBPF & INCT-SC

- Collaborators: Fernando D. Nobre, Andre M. C. Souza, Roberto F. S. Andrade, José Andrade, Veit Schwammle, Andre Moreira, G. F. T. da Silva

- Fokker-Planck equation

Linear Fokker-Planck equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)P(x,t)\}}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2}$$

$$F(x) = -\frac{d\phi}{dx}$$

Linear Fokker-Planck equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)P(x,t)\}}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2}$$

$$F(x) = -\frac{d\phi}{dx}$$

Linear Fokker-Planck equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)P(x,t)\}}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2}$$

$$F(x) = -\frac{d\phi}{dx}$$

H-theorem -> FPE and Boltzmann-Gibbs entropy

BG

$$F = U - TS \; ; \; U = \int_{-\infty}^{\infty} dx \; \phi(x) P(x,t) \; ; \; S = -k_B \int_{-\infty}^{\infty} dx \; P(x,t) \ln P(x,t)$$

Linear Fokker-Planck equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)P(x,t)\}}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2}$$

$$F(x) = -\frac{d\phi}{dx}$$

H-theorem -> FPE and Boltzmann-Gibbs entropy

BG

$$F = U - TS \; ; \; U = \int_{-\infty}^{\infty} dx \; \phi(x)P(x,t) \; ; \; S = -k_B \int_{-\infty}^{\infty} dx \; P(x,t) \ln P(x,t)$$

$$\frac{dF}{dt} \leq 0$$

$$P_{stat}(x) \propto e^{-\phi(x)/D}$$

stationary and stable solution

general solution - Fokker-Planck equation

- time dependence $\longrightarrow$ $F(x) = -kx$

$$P(x,t) = \frac{1}{\sqrt{2\pi D(1-e^{-2t})/k}} e^{-\frac{kx^2}{2D(1-e^{-2t})}}$$

- normal diffusion $(t << 1)$

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt/k}} e^{-\frac{kx^2}{4Dt}} \longrightarrow \langle (x - \langle x \rangle)^2 \rangle = \frac{2Dt}{k}$$

- $t >> 1 \longrightarrow \langle (x - \langle x \rangle)^2 \rangle = \frac{D}{k} \quad \left(P(x) = \frac{1}{\sqrt{2\pi D/k}} e^{-\frac{kx^2}{2D}} \right)$

general solution - Fokker-Planck equation

- time dependence $\longrightarrow$ $F(x) = -kx$

$$P(x,t) = \frac{1}{\sqrt{2\pi D(1-e^{-2t})/k}} e^{-\frac{kx^2}{2D(1-e^{-2t})}}$$

- normal diffusion $(t << 1)$

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt/k}} e^{-\frac{kx^2}{4Dt}} \longrightarrow \langle (x - \langle x \rangle)^2 \rangle = \frac{2Dt}{k}$$

- $t >> 1 \longrightarrow \langle (x - \langle x \rangle)^2 \rangle = \frac{D}{k}$ $\left(P(x) = \frac{1}{\sqrt{2\pi D/k}} e^{-\frac{kx^2}{2D}} \right)$ $\rightarrow$ BG entropy

nonlinear Fokker-Planck Equations

nonlinear Fokker-Planck equation - phenomenological equations

Porous media equation
(M. Muskat - 1937) $\longrightarrow$ $$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P^{\nu}(x,t)}{\partial x^2}$$

$$\langle (x - \langle x \rangle)^2 \rangle \sim t^{\frac{2}{\nu+1}}$$

nonlinear Fokker-Planck equation - phenomenological equations

Porous media equation
(M. Muskat - 1937)

$$\longrightarrow \quad \frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P^{\nu}(x,t)}{\partial x^2}$$

$$\langle (x - \langle x \rangle)^2 \rangle \sim t^{\frac{2}{\nu+1}}$$

- A. R. Plastino and A. Plastino, Physica A 222 (1995) 347;

  C. Tsallis and Bukman D. J., PRE 54 (1996) R2197

nonlinear Fokker-Planck equation - phenomenological equations

Porous media equation (M. Muskat - 1937) $\longrightarrow$ $$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P^{\nu}(x,t)}{\partial x^2}$$

$$\langle (x - \langle x \rangle)^2 \rangle \sim t^{\frac{2}{\nu+1}}$$

- A. R. Plastino and A. Plastino, Physica A 222 (1995) 347;

  C. Tsallis and Bukman D. J., PRE 54 (1996) R2197

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)P(x,t)\}}{\partial x} + D\frac{\partial^2 P^{\nu}(x,t)}{\partial x^2}$$

$$F(x) = -\frac{d\phi}{dx}$$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)P(x,t)\}}{\partial x} + D\frac{\partial^2 P^\nu(x,t)}{\partial x^2}$$

$$F(x) = -\frac{d\phi}{dx}$$

stationary solution $(\nu = 2 - q)$

$$P(x) = C[1 - \beta(1-q)\phi(x)]^{1/(1-q)}$$

$\beta = (1/D)[C^{q-1}/(2-q)]$ (C is a positive constant)

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)P(x,t)\}}{\partial x} + D\frac{\partial^2 P^\nu(x,t)}{\partial x^2}$$

$$F(x) = -\frac{d\phi}{dx}$$

stationary solution $(\nu = 2-q)$

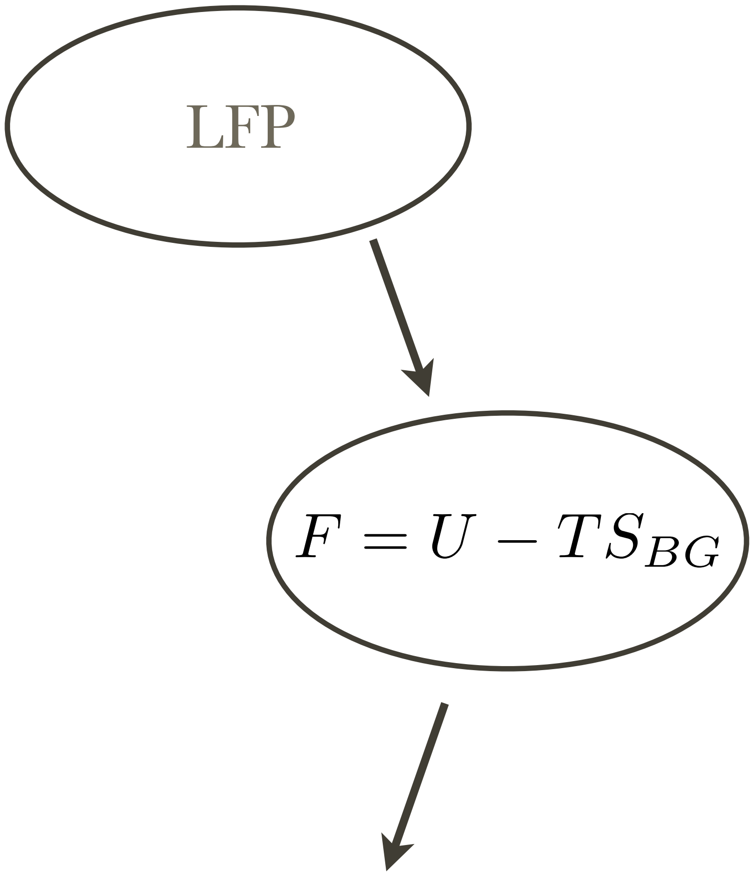
$$P(x) = C[1-\beta(1-q)\phi(x)]^{1/(1-q)}$$

$\beta = (1/D)[C^{q-1}/(2-q)]$ (C is a positive constant)

same distribution that maximizes Tsallis entropy with the external constraint $\phi(x)$!

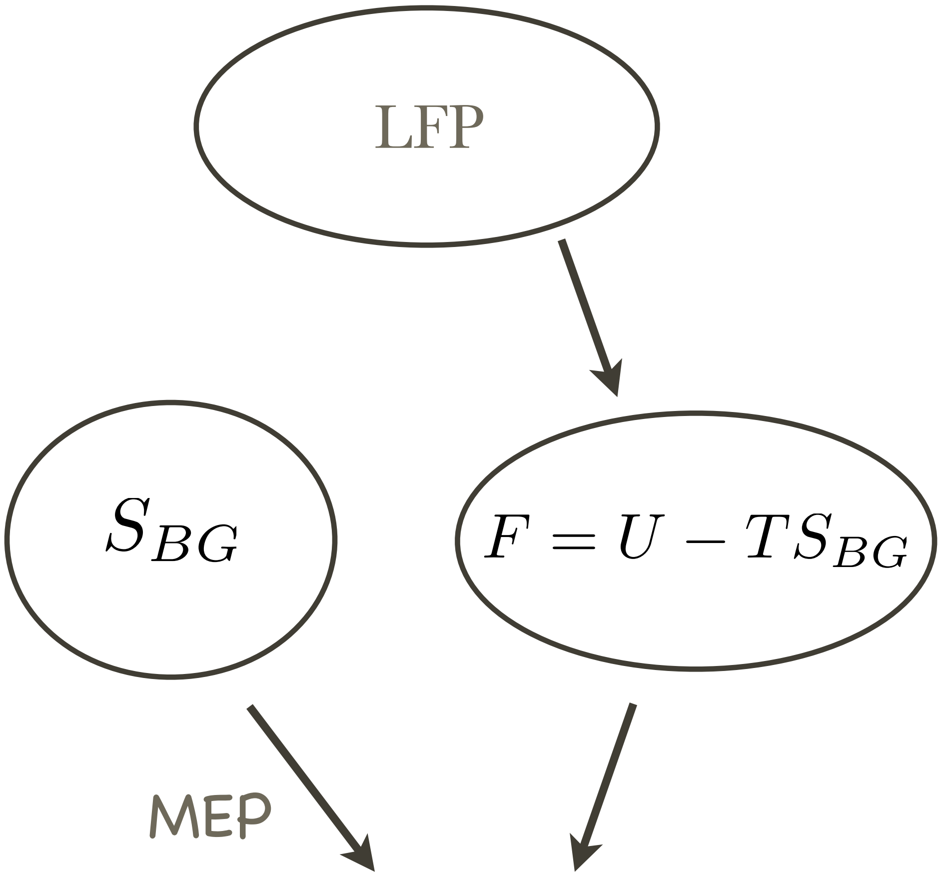
NLFPE <-> Tsallis entropy!

FPE ⟷ entropy



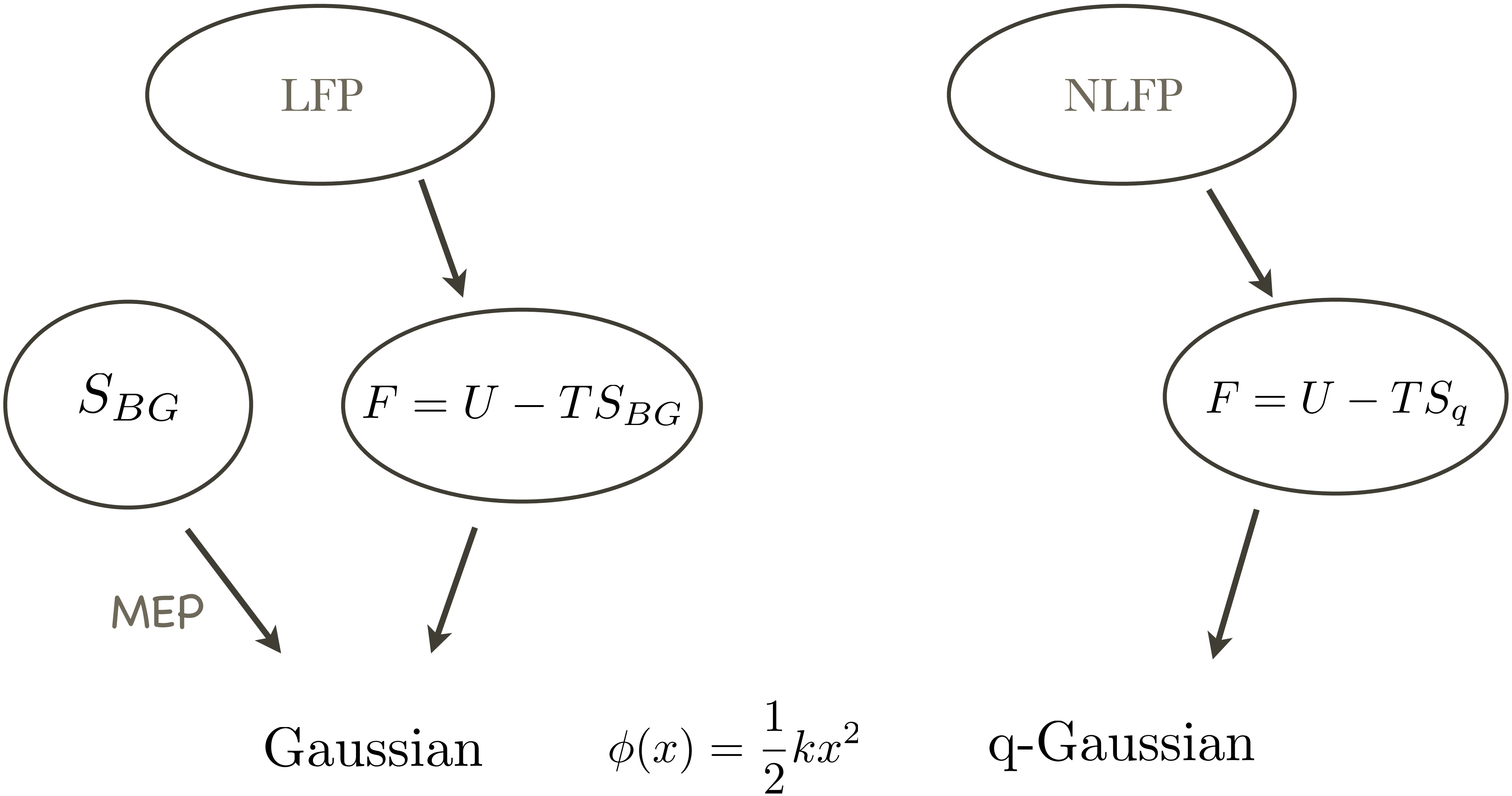
Gaussian $\phi(x) = \frac{1}{2}kx^2$

FPE ⟷ entropy

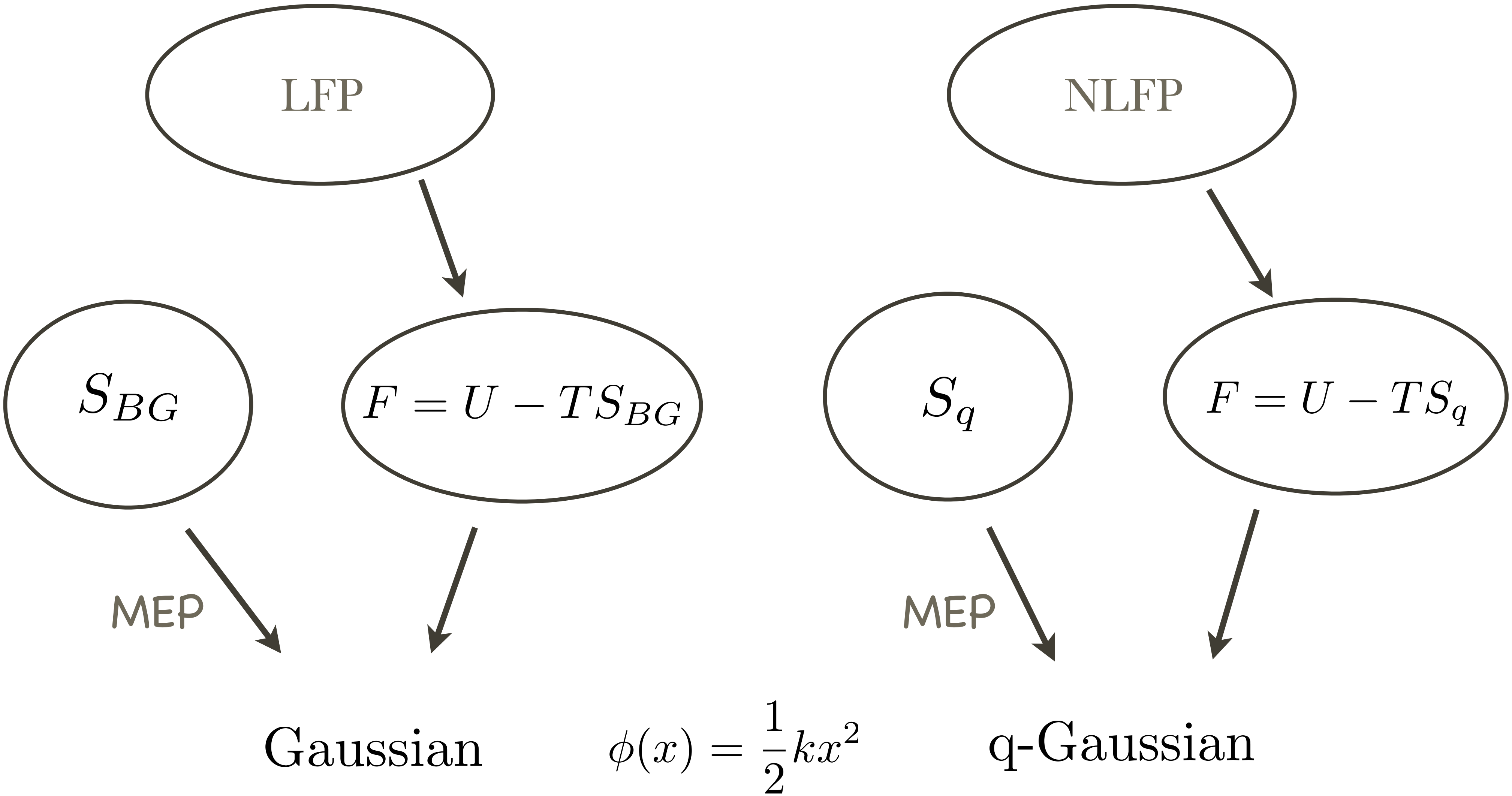


Gaussian $\phi(x) = \frac{1}{2}kx^2$

FPE $\longleftrightarrow$ entropy



FPE ⟷ entropy



LFP
NLFP
S_{BG}
$F = U - TS_{BG}$
S_q
$F = U - TS_q$
MEP
MEP
Gaussian
$\phi(x) = \frac{1}{2}kx^2$
q-Gaussian

how could it be derived?

- Langevin equation
- master equation

Langevin equation

PHYSICAL REVIEW E VOLUME 57, NUMBER 6 JUNE 1998

Microscopic dynamics of the nonlinear Fokker-Planck equation: A phenomenological model

Lisa Borland*
Centro Brasileiro de Pesquisas Fisicas, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Rio de Janeiro, Brazil
(Received 16 December 1997)

We derive a phenomenological model of the underlying microscopic Langevin equation of the nonlinear Fokker-Planck equation, which is used to describe anomalous correlated diffusion. The resulting distribution-dependent stochastic equation is then analyzed and properties such as long-time scaling and the Hurst exponent are calculated both analytically and from simulations. Results of this microscopic theory are compared with those of fractional Brownian motion. [S1063-651X(98)00206-2]

PACS number(s): 66.10.Cb, 05.20.−y, 05.60.+w, 05.40.+j

PHYSICAL REVIEW E VOLUME 57, NUMBER 6 JUNE 1998

Microscopic dynamics of the nonlinear Fokker-Planck equation: A phenomenological model

Lisa Borland*

Centro Brasileiro de Pesquisas Fisicas, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Rio de Janeiro, Brazil

(Received 16 December 1997)

We derive a phenomenological model of the underlying microscopic Langevin equation of the nonlinear Fokker-Planck equation, which is used to describe anomalous correlated diffusion. The resulting distribution-dependent stochastic equation is then analyzed and properties such as long-time scaling and the Hurst exponent are calculated both analytically and from simulations. Results of this microscopic theory are compared with those of fractional Brownian motion. [S1063-651X(98)00206-2]

PACS number(s): 66.10.Cb, 05.20.−y, 05.60.+w, 05.40.+j

isting theory. The resulting Ito-Langevin equation has the form

$$\frac{dx}{dt}=K(x,t)+\sqrt{Q}f(x,t)^{(\nu-1)/2}\eta(t), \qquad (18)$$

where the evolution of f is given by the Fokker-Planck equation of equation (2). A trajectory of Eq. (18) is determined by both equations simultaneously. It is apparent that there is feedback from the macroscopic level of description of the system in terms of the probability distribution f to the microscopic kinetics.

Available online at www.sciencedirect.com

ScienceDirect

PHYSICS LETTERS A

Physics Letters A 372 (2008) 1236–1239

www.elsevier.com/locate/pla

Computing the non-linear anomalous diffusion equation from first principles

Miguel A. Fuentes [a,b,*], Manuel O. Cáceres [b,c]

[a] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA
[b] Centro Atómico Bariloche, Instituto Balseiro and CONICET, 8400 Bariloche, Argentina
[c] The Abdus Salam ICTP, 34014 Trieste, Italy

Received 8 June 2007; received in revised form 3 September 2007; accepted 10 September 2007

Available online 14 September 2007

Communicated by A.R. Bishop

Abstract

We investigate asymptotically the occurrence of anomalous diffusion and its associated family of statistical evolution equations. Starting from a non-Markovian process *à la* Langevin we show that the mean probability distribution of the displacement of a particle follows a generalized non-linear Fokker–Planck equation. Thus we show that the anomalous behavior can be linked to a fast fluctuation process with memory from a microscopic dynamics level, and slow fluctuations of the dissipative variable. The general results can be applied to a wide range of physical systems that present a departure from the Brownian regime.

PACS: 05.70.Ln; 05.30.Pr

master equation

PHYSICAL REVIEW E **67**, 021107 (2003)

Derivation of nonlinear Fokker-Planck equations by means of approximations to the master equation

Evaldo M. F. Curado[1,*] and Fernando D. Nobre[1,2,†]

[1]*Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro—RJ, Brazil*

[2]*Departamento de Física Teórica e Experimental, Universidade Federal do Rio Grande do Norte, Campus Universitário, Caixa Postal 1641, 59072-970 Natal, Rio Grande do Norte, Brazil*

(Received 25 September 2002; published 21 February 2003)

Nonlinear Fokker-Planck equations (FPEs) are derived as approximations to the master equation, in cases of transitions among both discrete and continuous sets of states. The nonlinear effects, introduced through the transition probabilities, are argued to be relevant for many real phenomena within the class of anomalous-diffusion problems. The nonlinear FPEs obtained appear to be more general than some previously proposed (on a purely phenomenological basis) ones. In spite of this, the same kind of solution applies, i.e., it is shown that the time-dependent Tsallis's probability distribution is a solution of both equations, obtained either from discrete or continuous sets of states, and that the corresponding stationary solution is, in the infinite-time limit, a stable solution.

master equation -> NL Fokker-Planck equation

$$\frac{\partial P(n,t)}{\partial t} = \sum_{m=-\infty}^{\infty} \left[P(m,t) w_{m,n}(t) - P(n,t) w_{n,m}(t) \right]$$

- nonlinear transition rates -> nonlinear Fokker-Planck equations

$$\omega_{m,n}(t) \rightarrow \omega_{m,n}(P,t)$$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial \{F(x)\Psi[P(x,t)]\}}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

master equation -> NL Fokker-Planck equation

$$\frac{\partial P(n,t)}{\partial t} = \sum_{m=-\infty}^{\infty} \left[P(m,t)w_{m,n}(t) - P(n,t)w_{n,m}(t)\right]$$

- nonlinear transition rates -> nonlinear Fokker-Planck equations

$$\omega_{m,n}(t) \longrightarrow \omega_{m,n}(P,t)$$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)\Psi[P(x,t)]\}}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

master equation -> NL Fokker-Planck equation

$$\frac{\partial P(n,t)}{\partial t} = \sum_{m=-\infty}^{\infty} [P(m,t)w_{m,n}(t) - P(n,t)w_{n,m}(t)]$$

- nonlinear transition rates -> nonlinear Fokker-Planck equations

$$\omega_{m,n}(t) \rightarrow \omega_{m,n}(P,t)$$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)\Psi[P(x,t)]\}}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

nonlinear Fokker-Planck equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{F(x)\Psi[P(x,t)]\}}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

$$\Omega[P] > 0 \qquad \Psi[P] > 0 \qquad F(x) = -\frac{d\phi}{dx}$$

$$\int_{-\infty}^{\infty} dx\ P(x,t) = \int_{-\infty}^{\infty} dx\ P(x,t_0) = 1 \qquad (\forall t)$$

$\Psi[P] = P$

$\Omega[P] = \text{const.}$

FPE

$\Psi[P] = P$

$\Omega[P] = DP^{\mu-1}$

NLFPE - Plastino and Plastino, Physica A 1995

master equation -> NLFPE

EMFC & FD Nobre, PRE 2003,
FD Nobre, EMFC & G Rowlands, Physica A 2004

H theorem - general case

$$S[P] = \int_{-\infty}^{\infty} g[P(x,t)]\, dx$$

$$\frac{d^2 g}{dP^2} \leq 0$$

H theorem - general case

$$S[P] = \int_{-\infty}^{\infty} g[P(x,t)]\, dx$$

$$\frac{d^2 g}{dP^2} \leq 0$$

$$-\frac{1}{\beta}\frac{d^2 g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]}$$

↓

$$\frac{dF}{dt} \leq 0$$

H theorem - general case

$$S[P] = \int_{-\infty}^{\infty} g[P(x,t)]\, dx$$

$$\frac{d^2 g}{dP^2} \leq 0$$

$$-\frac{1}{\beta}\frac{d^2 g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]}$$

$$\frac{dF}{dt} \leq 0$$

H theorem - general case

$$S[P] = \int_{-\infty}^{\infty} g[P(x,t)]\, dx$$

$$\frac{d^2 g}{dP^2} \leq 0$$

$$-\frac{1}{\beta}\frac{d^2 g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]}$$

$$\frac{dF}{dt} \leq 0$$

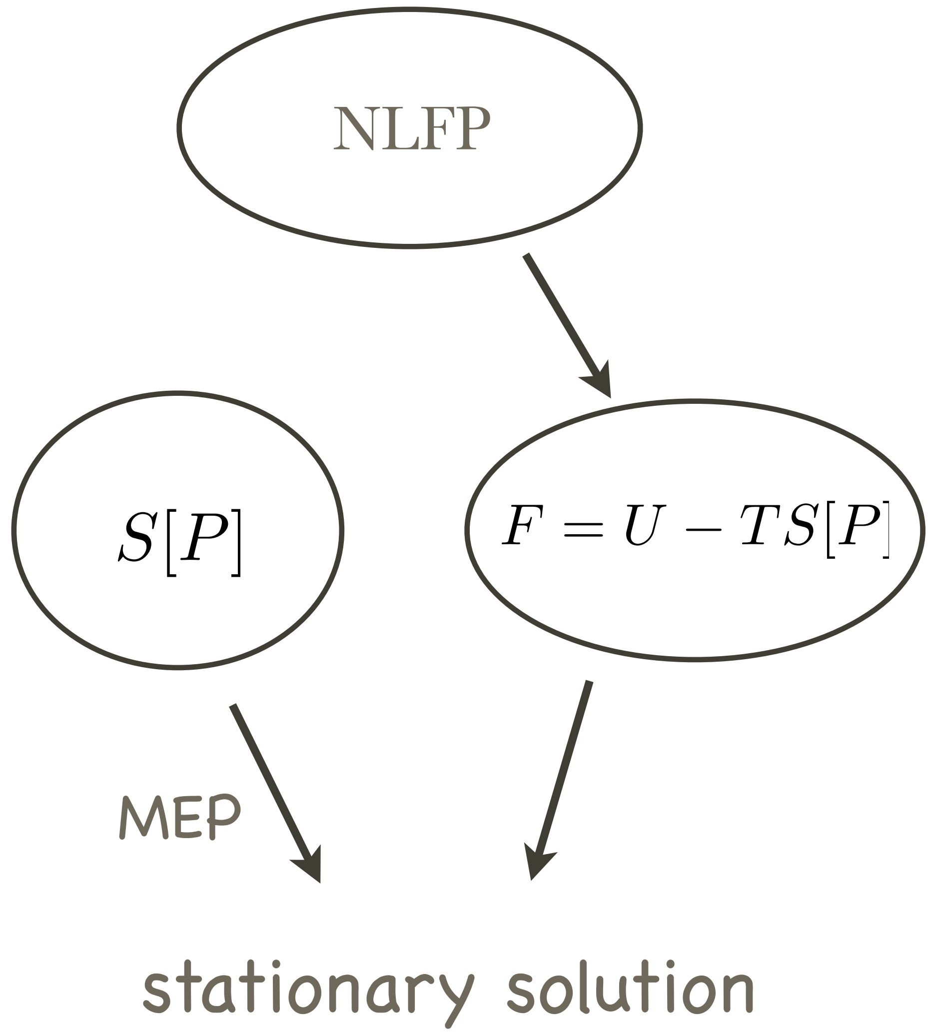
H theorem - general case

$$S[P] = \int_{-\infty}^{\infty} g[P(x,t)]\, dx$$

$$\frac{d^2 g}{dP^2} \leq 0$$

$$-\frac{1}{\beta}\frac{d^2 g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]}$$

$$\frac{dF}{dt} \leq 0$$



- Frank, Chavannis, Nobre, EMFC

- type II disordered supercondutors - overdamped motion of interacting vortices

Flux Front Penetration in Disordered Superconductors

Stefano Zapperi,[1] André A. Moreira,[2] and José S. Andrade, Jr.[2]
[1]*INFM sezione di Roma 1, Dipartimento di Fisica, Università "La Sapienza," P.le A. Moro 2, 00185 Roma, Italy*
[2]*Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil*
(Received 7 November 2000)

We investigate flux front penetration in a disordered type-II superconductor by molecular dynamics simulations of interacting vortices and find scaling laws for the front position and the density profile. The scaling can be understood by performing a coarse graining of the system and writing a disordered nonlinear diffusion equation. Integrating numerically the equation, we observe a crossover from flat to fractal front penetration as the system parameters are varied. The value of the fractal dimension indicates that the invasion process is described by gradient percolation.

DOI: 10.1103/PhysRevLett.86.3622 PACS numbers: 74.60.Ge, 05.45.–a, 47.55.Mh

Flux Front Penetration in Disordered Superconductors

Stefano Zapperi,[1] André A. Moreira,[2] and José S. Andrade, Jr.[2]
[1]*INFM sezione di Roma 1, Dipartimento di Fisica, Università "La Sapienza," P.le A. Moro 2, 00185 Roma, Italy*
[2]*Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil*
(Received 7 November 2000)

We investigate flux front penetration in a disordered type-II superconductor by molecular dynamics simulations of interacting vortices and find scaling laws for the front position and the density profile. The scaling can be understood by performing a coarse graining of the system and writing a disordered nonlinear diffusion equation. Integrating numerically the equation, we observe a crossover from flat to fractal front penetration as the system parameters are varied. The value of the fractal dimension indicates that the invasion process is described by gradient percolation.

DOI: 10.1103/PhysRevLett.86.3622 PACS numbers: 74.60.Ge, 05.45.–a, 47.55.Mb

VOLUME 86, NUMBER 16 PHYSICAL REVIEW LETTERS 16 APRIL 2001

for short-range attractive pinning forces such as the one we are investigating. In this case, short wavelength modes yield a macroscopic contribution to pinning that cannot be neglected. Consider, for instance, the flow between two coarse-grained regions: short-range microscopic pinning forces give rise to a macroscopic force that should always oppose the motion, while the random force derived above could, in principle, point in the direction of the flow. In other words, $F_c(\vec{r})$ should be considered as a *friction* force [21] whose direction is always opposed to the driving force $\vec{F}_d$ (in our case $\vec{F}_d = a\vec{\nabla}\rho$) and whose absolute value is given by $|g\vec{\nabla}n|$ for $|\vec{F}_d| > |g\vec{\nabla}n|$ and to $|\vec{F}_d|$ otherwise [22].

Collecting all the terms, we finally obtain a disordered nonlinear diffusion equation for the density of flux lines

$$\Gamma \frac{\partial \rho}{\partial t} = \vec{\nabla}(a\rho\vec{\nabla}\rho - \rho\vec{F}_c) + k_B T \nabla^2 \rho\,. \qquad (5)$$

The boundary conditions representing our MD simulations

crossover length scaling as $\xi_p \sim g_0^{-1/2}$, in agreement with MD simulations (see Fig. 3). In addition, we measure the density profiles and find that they rescale with g_0 in the same way as in MD simulations.

The numerical integration of the diffusion equation allows for a direct analysis of the fluctuations in the front as a function of different internal parameters. Measuring the width W of the fronts as a function of time for different values of g_0, we find that in the initial stage W grows as a power law t^β where $\beta \simeq 0.35$ until it saturates to a value that decreases with g_0. Thus the front crosses over from flat to fractal as it enters into the material. In principle, we can control the strength of the fluctuations and the associated characteristic length ξ^* by tuning g_0, which directly reflects experimentally measurable parameters.

In order to compare the model with experiments, we have to implement appropriate boundary conditions. In Refs. [5,6] the external field was ramped at a constant rate, which corresponds to a constant increase of the boundary

Flux Front Penetration in Disordered Superconductors

Stefano Zapperi,[1] André A. Moreira,[2] and José S. Andrade, Jr.[2]
[1]*INFM sezione di Roma 1, Dipartimento di Fisica, Università "La Sapienza," P.le A. Moro 2, 00185 Roma, Italy*
[2]*Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil*
(Received 7 November 2000)

We investigate flux front penetration in a disordered type-II superconductor by molecular dynamics simulations of interacting vortices and find scaling laws for the front position and the density profile. The scaling can be understood by performing a coarse graining of the system and writing a disordered nonlinear diffusion equation. Integrating numerically the equation, we observe a crossover from flat to fractal front penetration as the system parameters are varied. The value of the fractal dimension indicates that the invasion process is described by gradient percolation.

DOI: 10.1103/PhysRevLett.86.3622 PACS numbers: 74.60.Ge, 05.45.–a, 47.55.Mh

VOLUME 86, NUMBER 16 PHYSICAL REVIEW LETTERS 16 APRIL 2001

for short-range attractive pinning forces such as the one we are investigating. In this case, short wavelength modes yield a macroscopic contribution to pinning that cannot be neglected. Consider, for instance, the flow between two coarse-grained regions: short-range microscopic pinning forces give rise to a macroscopic force that should always oppose the motion, while the random force derived above could, in principle, point in the direction of the flow. In other words, $F_c(\vec{r})$ should be considered as a *friction* force [21] whose direction is always opposed to the driving force $\vec{F}_d$ (in our case $\vec{F}_d = a\vec{\nabla}\rho$) and whose absolute value is given by $|g\vec{\nabla}n|$ for $|\vec{F}_d| > |g\vec{\nabla}n|$ and to $|\vec{F}_d|$ otherwise [22].

Collecting all the terms, we finally obtain a disordered nonlinear diffusion equation for the density of flux lines

$$\Gamma \frac{\partial \rho}{\partial t} = \vec{\nabla}(a\rho\vec{\nabla}\rho - \rho\vec{F}_c) + k_B T \nabla^2 \rho . \qquad (5)$$

The boundary conditions representing our MD simulations

crossover length scaling as $\xi_p \sim g_0^{-1/2}$, in agreement with MD simulations (see Fig. 3). In addition, we measure the density profiles and find that they rescale with g_0 in the same way as in MD simulations.

The numerical integration of the diffusion equation allows for a direct analysis of the fluctuations in the front as a function of different internal parameters. Measuring the width W of the fronts as a function of time for different values of g_0, we find that in the initial stage W grows as a power law t^β where $\beta \simeq 0.35$ until it saturates to a value that decreases with g_0. Thus the front crosses over from flat to fractal as it enters into the material. In principle, we can control the strength of the fluctuations and the associated characteristic length ξ^* by tuning g_0, which directly reflects experimentally measurable parameters.

In order to compare the model with experiments, we have to implement appropriate boundary conditions. In Refs. [5,6] the external field was ramped at a constant rate, which corresponds to a constant increase of the boundary

Flux Front Penetration in Disordered Superconductors

Stefano Zapperi,[1] André A. Moreira,[2] and José S. Andrade, Jr.[2]
[1]*INFM sezione di Roma 1, Dipartimento di Fisica, Università "La Sapienza," P.le A. Moro 2, 00185 Roma, Italy*
[2]*Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil*
(Received 7 November 2000)

We investigate flux front penetration in a disordered type-II superconductor by molecular dynamics simulations of interacting vortices and find scaling laws for the front position and the density profile. The scaling can be understood by performing a coarse graining of the system and writing a disordered nonlinear diffusion equation. Integrating numerically the equation, we observe a crossover from flat to fractal front penetration as the system parameters are varied. The value of the fractal dimension indicates that the invasion process is described by gradient percolation.

DOI: 10.1103/PhysRevLett.86.3622 PACS numbers: 74.60.Ge, 05.45.–a, 47.55.Mh

VOLUME 86, NUMBER 16 PHYSICAL REVIEW LETTERS 16 APRIL 2001

for short-range attractive pinning forces such as the one we are investigating. In this case, short wavelength modes yield a macroscopic contribution to pinning that cannot be neglected. Consider, for instance, the flow between two coarse-grained regions: short-range microscopic pinning forces give rise to a macroscopic force that should always oppose the motion, while the random force derived above could, in principle, point in the direction of the flow. In other words, $F_c(\vec{r})$ should be considered as a *friction* force [21] whose direction is always opposed to the driving force $\vec{F}_d$ (in our case $\vec{F}_d = a\vec{\nabla}\rho$) and whose absolute value is given by $|g\vec{\nabla}n|$ for $|\vec{F}_d| > |g\vec{\nabla}n|$ and to $|\vec{F}_d|$ otherwise [22].

Collecting all the terms, we finally obtain a disordered nonlinear diffusion equation for the density of flux lines

$$\Gamma \frac{\partial \rho}{\partial t} = \vec{\nabla}(a\rho\vec{\nabla}\rho - \rho\vec{F}_c) + k_B T \nabla^2 \rho \,. \qquad (5)$$

The boundary conditions representing our MD simulations

crossover length scaling as $\xi_p \sim g_0^{-1/2}$, in agreement with MD simulations (see Fig. 3). In addition, we measure the density profiles and find that they rescale with g_0 in the same way as in MD simulations.

The numerical integration of the diffusion equation allows for a direct analysis of the fluctuations in the front as a function of different internal parameters. Measuring the width W of the fronts as a function of time for different values of g_0, we find that in the initial stage W grows as a power law t^β where $\beta \simeq 0.35$ until it saturates to a value that decreases with g_0. Thus the front crosses over from flat to fractal as it enters into the material. In principle, we can control the strength of the fluctuations and the associated characteristic length ξ^* by tuning g_0, which directly reflects experimentally measurable parameters.

In order to compare the model with experiments, we have to implement appropriate boundary conditions. In Refs. [5,6] the external field was ramped at a constant rate, which corresponds to a constant increase of the boundary

Flux Front Penetration in Disordered Superconductors

Stefano Zapperi,[1] André A. Moreira,[2] and José S. Andrade, Jr.[2]
[1]*INFM sezione di Roma 1, Dipartimento di Fisica, Università "La Sapienza," P.le A. Moro 2, 00185 Roma, Italy*
[2]*Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil*
(Received 7 November 2000)

We investigate flux front penetration in a disordered type-II superconductor by molecular dynamics simulations of interacting vortices and find scaling laws for the front position and the density profile. The scaling can be understood by performing a coarse graining of the system and writing a disordered nonlinear diffusion equation. Integrating numerically the equation, we observe a crossover from flat to fractal front penetration as the system parameters are varied. The value of the fractal dimension indicates that the invasion process is described by gradient percolation.

DOI: 10.1103/PhysRevLett.86.3622 PACS numbers: 74.60.Ge, 05.45.–a, 47.55.Mh

VOLUME 86, NUMBER 16 PHYSICAL REVIEW LETTERS 16 APRIL 2001

for short-range attractive pinning forces such as the one we are investigating. In this case, short wavelength modes yield a macroscopic contribution to pinning that cannot be neglected. Consider, for instance, the flow between two coarse-grained regions: short-range microscopic pinning forces give rise to a macroscopic force that should always oppose the motion, while the random force derived above could, in principle, point in the direction of the flow. In other words, $F_c(\vec{r})$ should be considered as a *friction* force [21] whose direction is always opposed to the driving force $\vec{F}_d$ (in our case $\vec{F}_d = a\vec{\nabla}\rho$) and whose absolute value is given by $|g\vec{\nabla}n|$ for $|\vec{F}_d| > |g\vec{\nabla}n|$ and to $|\vec{F}_d|$ otherwise [22].

Collecting all the terms, we finally obtain a disordered nonlinear diffusion equation for the density of flux lines

$$\Gamma \frac{\partial \rho}{\partial t} = \vec{\nabla}(a\rho\vec{\nabla}\rho - \rho\vec{F}_c) + k_B T \nabla^2 \rho . \qquad (5)$$

The boundary conditions representing our MD simulations

crossover length scaling as $\xi_p \sim g_0^{-1/2}$, in agreement with MD simulations (see Fig. 3). In addition, we measure the density profiles and find that they rescale with g_0 in the same way as in MD simulations.

The numerical integration of the diffusion equation allows for a direct analysis of the fluctuations in the front as a function of different internal parameters. Measuring the width W of the fronts as a function of time for different values of g_0, we find that in the initial stage W grows as a power law t^β where $\beta \simeq 0.35$ until it saturates to a value that decreases with g_0. Thus the front crosses over from flat to fractal as it enters into the material. In principle, we can control the strength of the fluctuations and the associated characteristic length ξ^* by tuning g_0, which directly reflects experimentally measurable parameters.

In order to compare the model with experiments, we have to implement appropriate boundary conditions. In Refs. [5,6] the external field was ramped at a constant rate, which corresponds to a constant increase of the boundary

- overdamped motion

$$\mu \vec{\mathbf{v}}_i = \sum_{j \neq i} \vec{J}(\vec{r}_i - \vec{r}_j) + \vec{F}^e(\vec{r}_i) + \eta(\vec{r}_i, t)$$

$$\langle \eta \rangle = 0$$

$\vec{J}(\vec{r}) \equiv f_0 \, K_1(|\vec{r}|/\lambda)\hat{r}$ (repulsive interaction)

$$\langle \eta^2 \rangle = \frac{kT_h}{\mu}$$

- overdamped motion

$$\mu \vec{\mathbf{v}}_i = \sum_{j \neq i} \vec{J}(\vec{r}_i - \vec{r}_j) + \vec{F}^e(\vec{r}_i) + \eta(\vec{r}_i, t)$$

$\vec{J}(\vec{r}) \equiv f_0 \, K_1(|\vec{r}|/\lambda)\hat{r}$ (repulsive interaction)

$$\langle \eta \rangle = 0$$

$$\langle \eta^2 \rangle = \frac{kT_h}{\mu}$$

- continuous description

$$\mathcal{P}(\vec{r_1}, \vec{r_2}, \cdots, \vec{r_N}, t) \longrightarrow \mu \frac{\partial \mathcal{P}}{\partial t} = \sum_i \vec{\nabla}_i(-\vec{f}_i \mathcal{P} + k_B T_h \vec{\nabla}_i \mathcal{P})$$

- overdamped motion

$$\mu \vec{\mathbf{v}}_i = \sum_{j \neq i} \vec{J}(\vec{r}_i - \vec{r}_j) + \vec{F}^e(\vec{r}_i) + \eta(\vec{r}_i, t)$$

$$\langle \eta \rangle = 0$$

$\vec{J}(\vec{r}) \equiv f_0\, K_1(|\vec{r}|/\lambda)\hat{r}$ (repulsive interaction)

$$\langle \eta^2 \rangle = \frac{kT_h}{\mu}$$

- continuous description

$$\mathcal{P}(\vec{r_1}, \vec{r_2}, \cdots, \vec{r_N}, t) \longrightarrow \mu \frac{\partial \mathcal{P}}{\partial t} = \sum_i \vec{\nabla}_i(-\vec{f}_i \mathcal{P} + k_B T_h \vec{\nabla}_i \mathcal{P})$$

$\mathcal{P}(\vec{r_1}, \vec{r_2}, \cdots, \vec{r_N}, t) \to \rho(\vec{r}, t)$ (single particle density)

$$\mu \frac{\partial \rho}{\partial t} = -\vec{\nabla} \left[\int d^2 r' \vec{J}(\vec{r} - \vec{r'}) \rho^{(2)}(\vec{r}, \vec{r'}, t) + \vec{F}^e(\vec{r}) \rho(\vec{r}, t) \right] + kT_h \nabla^2 \rho(\vec{r}, t)$$

- overdamped motion

$$\mu \vec{\mathbf{v}}_i = \sum_{j \neq i} \vec{J}(\vec{r}_i - \vec{r}_j) + \vec{F}^e(\vec{r}_i) + \eta(\vec{r}_i, t)$$

$\vec{J}(\vec{r}) \equiv f_0\, K_1(|\vec{r}|/\lambda)\hat{r}$ (repulsive interaction)

$$\langle \eta \rangle = 0$$

$$\langle \eta^2 \rangle = \frac{kT_h}{\mu}$$

- continuous description

$$\mathcal{P}(\vec{r_1}, \vec{r_2}, \cdots, \vec{r_N}, t) \longrightarrow \mu \frac{\partial \mathcal{P}}{\partial t} = \sum_i \vec{\nabla}_i(-\vec{f}_i \mathcal{P} + k_B T_h \vec{\nabla}_i \mathcal{P})$$

$\mathcal{P}(\vec{r_1}, \vec{r_2}, \cdots, \vec{r_N}, t) \to \rho(\vec{r}, t)$ (single particle density)

$$\mu \frac{\partial \rho}{\partial t} = -\vec{\nabla} \left[\int d^2 r' \vec{J}(\vec{r} - \vec{r'}) \rho^{(2)}(\vec{r}, \vec{r'}, t) + \vec{F}^e(\vec{r}) \rho(\vec{r}, t) \right] + kT_h \nabla^2 \rho(\vec{r}, t)$$

$$\rho^{(2)}(\vec{r}, \vec{r}', t) \simeq \rho(\vec{r}, t) \rho(\vec{r}', t)$$

$$\int d^2 r' \vec{J}(\vec{r} - \vec{r'}) \rho(\vec{r'}, t) \approx -a \vec{\nabla} \rho(\vec{r}, t)$$

$$a \equiv \int d^2 r\, \vec{r} \cdot \vec{J}(\vec{r})/2 = 2\pi f_0 \lambda^3$$

$$\mu \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left\{ \rho \left[a \frac{\partial \rho}{\partial x} - A(x) \right] \right\} + kT_h \frac{\partial^2 \rho}{\partial x^2}$$

$$\vec{F}^e = -A(x)\hat{x}$$

- for T_h = 0 and $A(x) = -\alpha x \ (\alpha > 0)$

$$\rho_{stat}(x) = \frac{\alpha}{2a}(x_e^2 - x^2), \quad |x| < x_e$$

$$\mu \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left\{ \rho \left[a \frac{\partial \rho}{\partial x} - A(x) \right] \right\} + kT_h \frac{\partial^2 \rho}{\partial x^2}$$

$$\vec{F}^e = -A(x)\hat{x}$$

- for T_h = 0 and $A(x) = -\alpha x \ (\alpha > 0)$

$$\rho_{stat}(x) = \frac{\alpha}{2a}(x_e^2 - x^2), \quad |x| < x_e$$

$$\Psi[\rho] = \rho \qquad \Omega[\rho] = \frac{a}{\mu}\rho \quad \longrightarrow \quad -\frac{1}{\beta}\frac{d^2 g[\rho]}{d\rho^2} = \frac{\Omega[\rho]}{\Psi[\rho]}$$

$$\mu \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left\{ \rho \left[a \frac{\partial \rho}{\partial x} - A(x) \right] \right\} + kT_h \frac{\partial^2 \rho}{\partial x^2}$$

$$\vec{F}^e = -A(x)\hat{x}$$

- for T_h = 0 and $A(x) = -\alpha x \;\; (\alpha > 0)$

$$\rho_{stat}(x) = \frac{\alpha}{2a}(x_e^2 - x^2), \;\; |x| < x_e$$

$$\Psi[\rho] = \rho \qquad \Omega[\rho] = \frac{a}{\mu}\rho \longrightarrow -\frac{1}{\beta}\frac{d^2 g[\rho]}{d\rho^2} = \frac{\Omega[\rho]}{\Psi[\rho]}$$

$$S[\rho] = k\left(1 - \int_{-\infty}^{\infty} dx\, \rho(x)^2\right)$$

$$\mu\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left\{\rho\left[a\frac{\partial\rho}{\partial x} - A(x)\right]\right\} + kT_h\frac{\partial^2\rho}{\partial x^2}$$

$$\vec{F}^e = -A(x)\hat{x}$$

- for T_h = 0 and $A(x) = -\alpha x \;(\alpha > 0)$

$$\rho_{stat}(x) = \frac{\alpha}{2a}(x_e^2 - x^2), \quad |x| < x_e$$

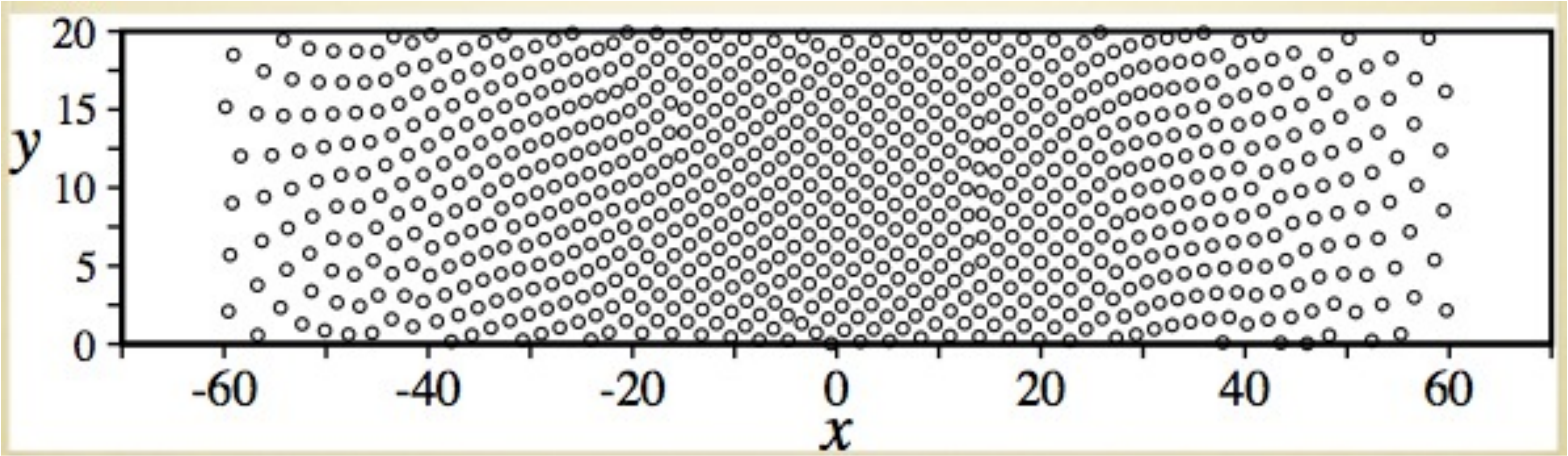
$$\Psi[\rho] = \rho \qquad \Omega[\rho] = \frac{a}{\mu}\rho \quad \longrightarrow \quad -\frac{1}{\beta}\frac{d^2 g[\rho]}{d\rho^2} = \frac{\Omega[\rho]}{\Psi[\rho]}$$

$$S[\rho] = k\left(1 - \int_{-\infty}^{\infty} dx\,\rho(x)^2\right) \qquad U = \int \phi(x)\rho(x)dx$$

molecular dynamics simulations (2d)

$L_x = 100\lambda$ $\qquad$ $L_y = 20\lambda \; (pbc)$

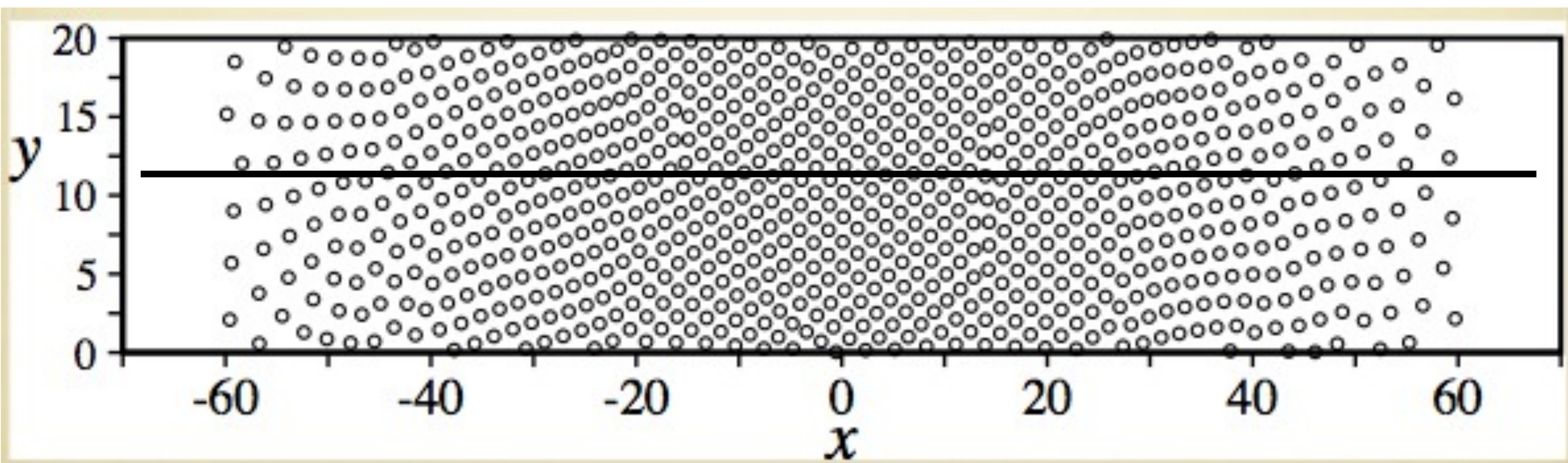
$N = 800$ $\qquad$ $\alpha = 10^{-3} f_0 \lambda$

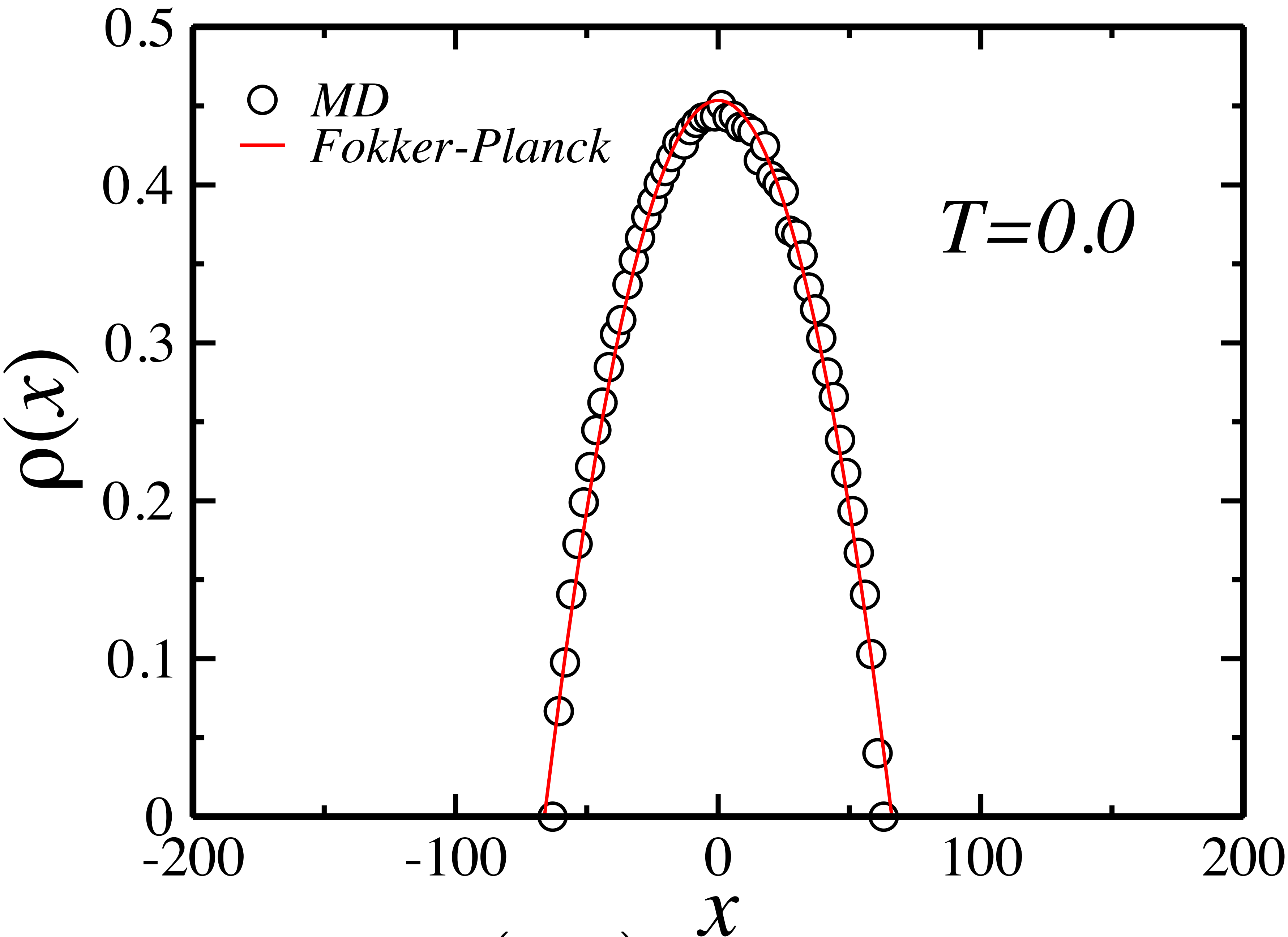


molecular dynamics simulations (2d)

$L_x = 100\lambda$ $\qquad L_y = 20\lambda \ (pbc)$

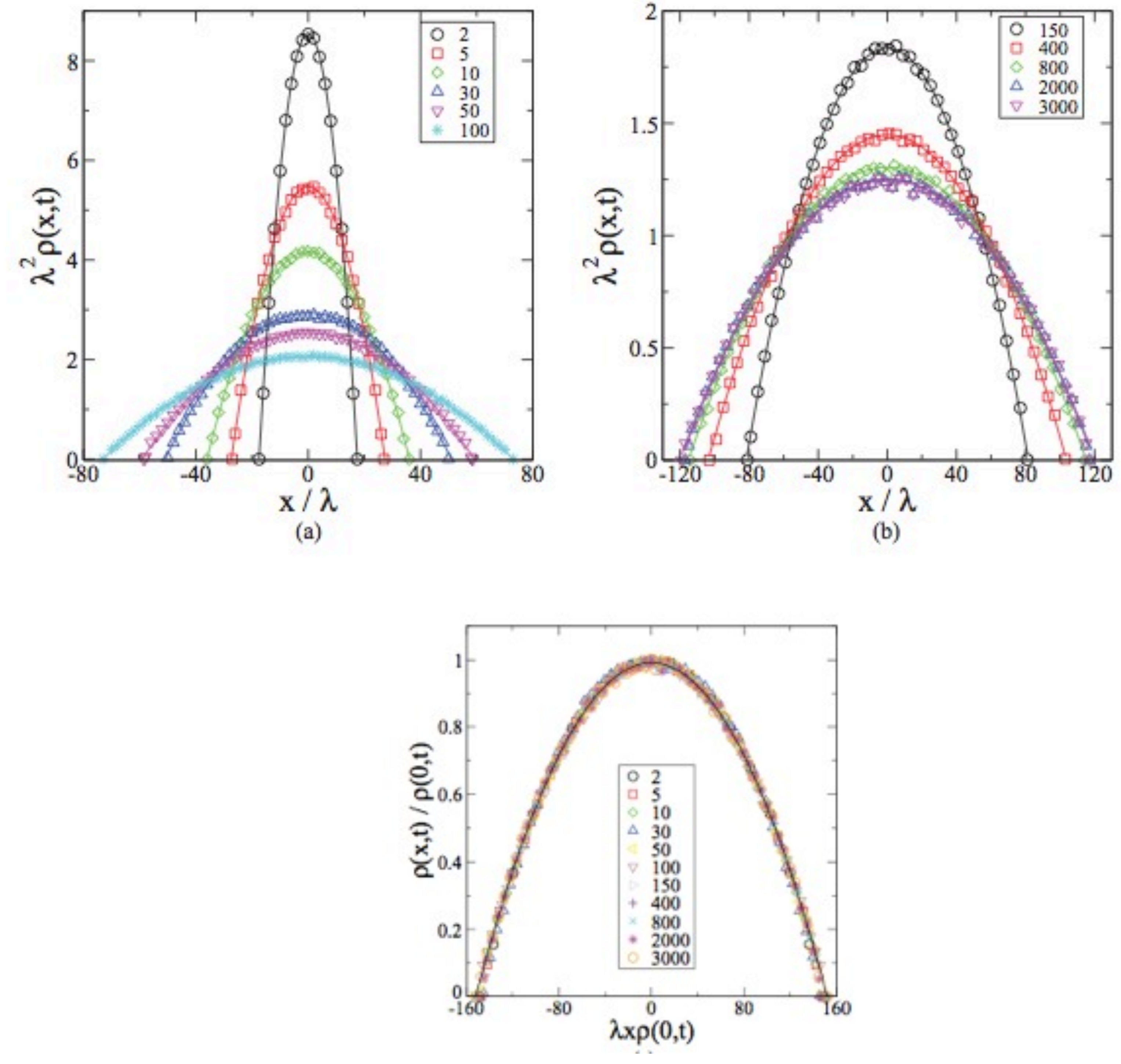
$N = 800$ $\qquad \alpha = 10^{-3} f_0 \lambda$





$\langle x^2 \rangle \propto t^{2/3} \left(t^{\frac{2}{\nu+1}} \right)$

$P_{stat} \propto x_e^2 - x^2$

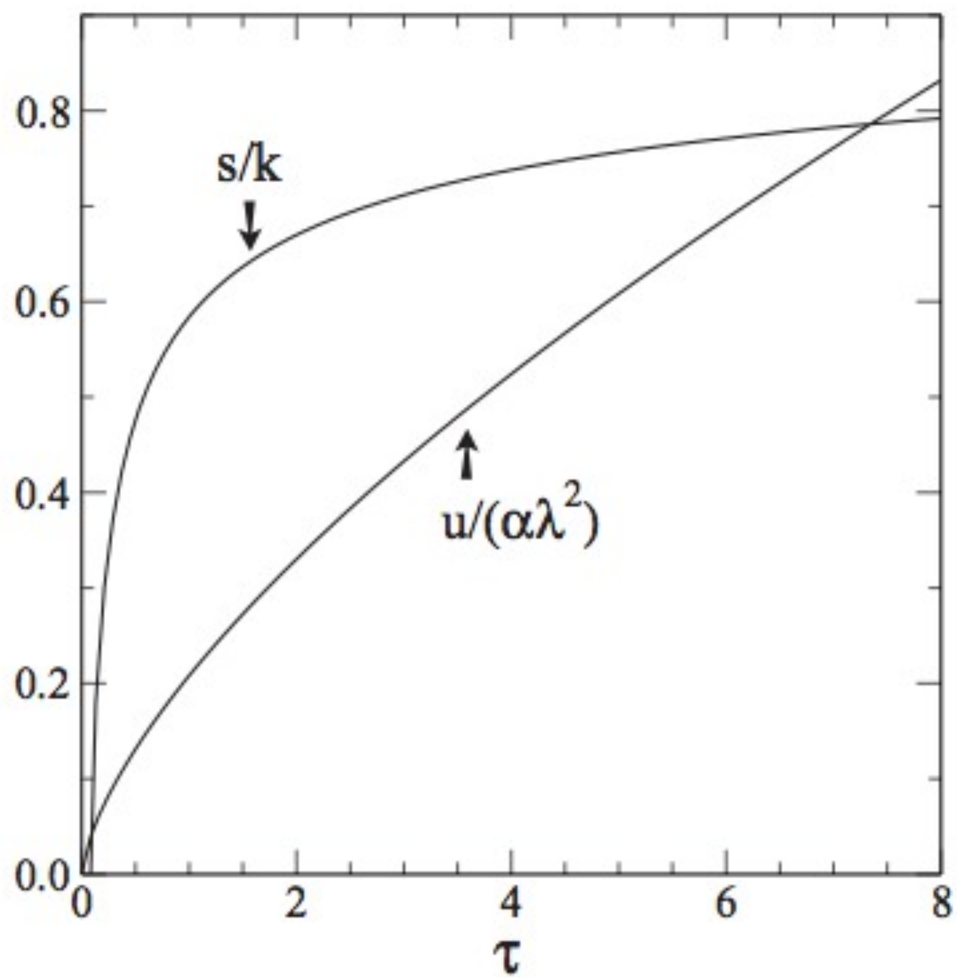

λ²ρ(x,t)
x / λ
2
5
10
30
50
100
-80 -40 0 40 80
0 2 4 6 8
(a)
λ²ρ(x,t)
x / λ
150
400
800
2000
3000
-120 -80 -40 0 40 80 120
0 0.5 1 1.5 2
(b)
ρ(x,t) / ρ(0,t)
λxρ(0,t)
2
5
10
30
50
100
150
400
800
2000
3000
-160 -80 0 80 160
0 0.2 0.4 0.6 0.8 1

effective temperature

$$k\theta \equiv D = \frac{N\pi f_0 \lambda^2}{L_y} = n\pi f_0 \lambda^2$$

effective temperature

$$k\theta \equiv D = \frac{N\pi f_0 \lambda^2}{L_y} = n\pi f_0 \lambda^2$$



portanto...

$$\mu \frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + 2D\frac{\partial}{\partial x}\left\{[\lambda P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

$$P_{\text{st}}(x) = \frac{\alpha}{4k\theta\lambda}(x_e^2 - x^2) = \frac{\alpha\lambda}{4k\theta}\left[\left(\frac{x_e}{\lambda}\right)^2 - \left(\frac{x}{\lambda}\right)^2\right]$$

$$(x_e = (3k\theta\lambda/\alpha)^{1/3})$$

portanto...

$$\mu \frac{\partial P(x,t)}{\partial t} = -\frac{\partial [A(x)P(x,t)]}{\partial x} + 2D\frac{\partial}{\partial x}\left\{[\lambda P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

$$P_{\rm st}(x) = \frac{\alpha}{4k\theta\lambda}(x_e^2 - x^2) = \frac{\alpha\lambda}{4k\theta}\left[\left(\frac{x_e}{\lambda}\right)^2 - \left(\frac{x}{\lambda}\right)^2\right]$$

$$(x_e = (3k\theta\lambda/\alpha)^{1/3})$$

$$\frac{s}{k} = 1 - \lambda\int_{-x_e}^{x_e} dx\ \ [P_{\rm st}(x)]^2 = 1 - \frac{3^{2/3}}{5}\left(\frac{\alpha\lambda^2}{k\theta}\right)^{1/3},$$

$$u = \int_{-x_e}^{x_e} dx\ \frac{\alpha x^2}{2}\ P_{\rm st}(x) = \frac{3^{2/3}}{10}(\alpha\lambda^2)^{1/3}(k\theta)^{2/3}$$

portanto...

$$\mu \frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + 2D\frac{\partial}{\partial x}\left\{[\lambda P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

$$P_{\rm st}(x) = \frac{\alpha}{4k\theta\lambda}(x_e^2 - x^2) = \frac{\alpha\lambda}{4k\theta}\left[\left(\frac{x_e}{\lambda}\right)^2 - \left(\frac{x}{\lambda}\right)^2\right]$$

$$(x_e = (3k\theta\lambda/\alpha)^{1/3})$$

$$\frac{s}{k} = 1 - \lambda\int_{-x_e}^{x_e} dx \; [P_{\rm st}(x)]^2 = 1 - \frac{3^{2/3}}{5}\left(\frac{\alpha\lambda^2}{k\theta}\right)^{1/3},$$

$$u = \int_{-x_e}^{x_e} dx \; \frac{\alpha x^2}{2} \; P_{\rm st}(x) = \frac{3^{2/3}}{10}(\alpha\lambda^2)^{1/3}(k\theta)^{2/3}$$

$$s(u,\alpha) = k\left[1 - \frac{3}{5}\left(\frac{\alpha\lambda^2}{10u}\right)^{1/2}\right]$$

portanto...

$$\mu \frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + 2D\frac{\partial}{\partial x}\left\{[\lambda P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

$$P_{\rm st}(x) = \frac{\alpha}{4k\theta\lambda}(x_e^2 - x^2) = \frac{\alpha\lambda}{4k\theta}\left[\left(\frac{x_e}{\lambda}\right)^2 - \left(\frac{x}{\lambda}\right)^2\right]$$

$$(x_e = (3k\theta\lambda/\alpha)^{1/3})$$

$$\frac{s}{k} = 1 - \lambda\int_{-x_e}^{x_e} dx \;\; [P_{\rm st}(x)]^2 = 1 - \frac{3^{2/3}}{5}\left(\frac{\alpha\lambda^2}{k\theta}\right)^{1/3},$$

$$u = \int_{-x_e}^{x_e} dx \; \frac{\alpha x^2}{2} \; P_{\rm st}(x) = \frac{3^{2/3}}{10}(\alpha\lambda^2)^{1/3}(k\theta)^{2/3}$$

$$s(u,\alpha) = k\left[1 - \frac{3}{5}\left(\frac{\alpha\lambda^2}{10u}\right)^{1/2}\right] \qquad \left(\frac{\partial s}{\partial u}\right)_\alpha = \frac{1}{\theta}$$

Legendre

- free energy

$$u(s, \alpha) \to f(\theta, \alpha)$$

Legendre

- free energy

$$u(s, \alpha) \to f(\theta, \alpha)$$

$$f(\theta, \alpha) = u - \theta s \; ; \quad \Rightarrow \quad df = -s d\theta + \sigma d\alpha$$

Legendre

- free energy

$$u(s,\alpha) \to f(\theta,\alpha)$$

$$f(\theta,\alpha) = u - \theta s \ ; \quad \Rightarrow \quad df = -s d\theta + \sigma d\alpha$$

$$f(\theta,\alpha) = \frac{3^{5/3}}{10}\ (\alpha\lambda^2)^{1/3}(k\theta)^{2/3} - k\theta$$

$$\left(\frac{\partial f}{\partial \theta}\right)_\alpha = -s \ ; \qquad \left(\frac{\partial f}{\partial \alpha}\right)_\theta = \sigma$$

and so on...

- calor e trabalho

$$du = \delta Q + \delta W = \theta ds + \sigma d\alpha$$

$$\delta Q = \theta ds \qquad\qquad \delta W = \sigma d\alpha$$

$$\sigma = \frac{3^{2/3}}{10}\ \lambda^2 \left(\frac{k\theta}{\alpha\lambda^2}\right)^{2/3}$$

- isothermal process

$$Q = \int_{s_i}^{s_f} \theta ds = \frac{3^{2/3}}{5} (k\theta\lambda)^{2/3} \left({\alpha_i}^{1/3} - {\alpha_f}^{1/3} \right)$$

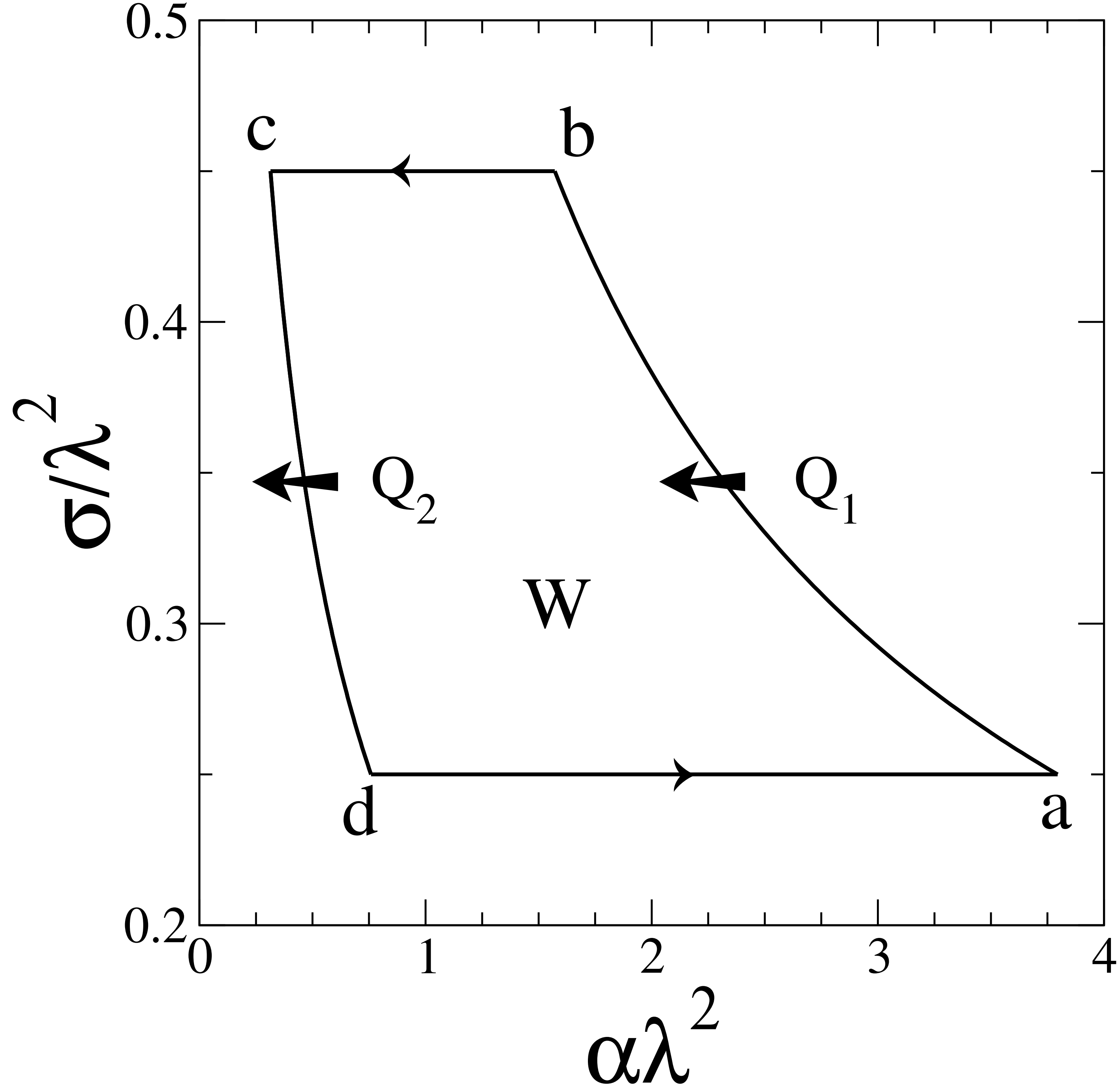
$$W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \frac{3^{5/3}}{10} (k\theta\lambda)^{2/3} \left({\alpha_f}^{1/3} - {\alpha_i}^{1/3} \right)$$

$$u_f - u_i = Q + W = \frac{3^{2/3}}{10} (k\theta\lambda)^{2/3} \left({\alpha_f}^{1/3} - {\alpha_i}^{1/3} \right)$$

- adiabatic process

$$\frac{s}{k} = 1 - \frac{3^{2/3}}{5} \left(\frac{\alpha\lambda^2}{k\theta} \right)^{1/3} \rightarrow \frac{\alpha}{\theta} = cte$$ (adiabático)

$$u_f - u_i = W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \sigma(\alpha_f - \alpha_i)$$



0.5
0.4
0.3
0.2
0
1
2
3
4
c
b
d
a
Q2
Q1
W
σ/λ2
αλ2

efficiency

$$\eta = \frac{\mathcal{W}}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{\theta_2}{\theta_1} \qquad (0 \leq \eta \leq 1)$$

- ciclo de Carnot é válido com a temperatura efetiva (T = 0)

Obrigado

- EMFC, F. D. Nobre, PRE 67 (2003) 021107
- F. D. Nobre, EMFC, G. Rowlands, Physica A (2004) 109
- V. Schwammle, EMFC, F. D. Nobre, Eur. Phys. J. B 58 (2007) 159
- V. Schwammle, F. D. Nobre, EMFC, PRE 76 (2007) 041123
- V. Schwammle, EMFC, F. D. Nobre, Eur. Phys. J. B 70 (2009) 107
- J. S. Andrade, G. F. T. Silva, A. A. Moreira, F. D. Nobre, EMFC, PRL 105 (2010) 260601
- M. S. Ribeiro, F. D. Nobre, EMFC, Entropy 13 (2011) 1928
- M. S. Ribeiro, F. D. Nobre, EMFC, EPJB 85 (2012) 399
- M. S. Ribeiro, F. D. Nobre, EMFC, PRE 85 (2012) 021146
- F. D. Nobre, A. M. C. Souza, EMFC, PRE 86 (2012) 061113