3d Ph.D. School on Mathematical Modeling of Complex Systems

Heraklion, Crete, Greece 15 - 27 July, 2013

Hurry and apply !!! → ***http://nls.physics.uoc.gr***

- **Foundations of Complexity Science**
  Steven Bishop (London), Tassos Bountis (Patras), David K. Campbell (Boston), Gregoire Nicolis (Brussels), Constantino Tsallis (Rio)
- **Complex Networks: Theory and Applications**
  Panos Argyrakis (Thessaloniki), Barouch Barzel (Boston), Jeff Johnson (London), Moses Boudourides (Patras)
- **Complex Quantum Systems**
  Jurgen Lisenfeld (Karlsruhe), Tsampikos Kottos (Wesleyan), Stelios Tzortzakis , Giorgos Tsironis , Xenophon Zotos (Heraklion)
- **Complexity in Social Sciences**
  Rosaria Conte (Rome), Dirk Helbing (Zurich), Klaus Mainzer (Munich)

Complex Hamiltonian Dynamics

Tassos Bountis and Haris Skokos

Springer Synergetics series, April 2012

Recent Progress in Complex Hamiltonian Systems

Tassos Bountis and Yannis Kominis

Department of Mathematics and Center for Research and Applications of Nonlinear Systems

http://www.math.upatras.gr/~crans

University of Patras, Patras GREECE

Lecture at INCT-CS, 5th Workshop on Complex Systems, CBPF, Rio de Janeiro. 22-24 April 2013

Spatial Solitons in Nonlinear Optics

Mathematical Model: Nonlinear Schrödinger Equation (NLSE)

$$i\frac{\partial \psi}{\partial z}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}\pm|\psi|^2\psi=0$$

As a dynamical system:

- It has infinitely many degrees of freedom (P.DE.)

- It has infinitely many integrals of the motion (unphysical)

Stationary solution Bright Soliton (+): $\psi(x,z)=n\,\mathrm{sech}\left[n(x-x_0)\right]e^{in^2z/2}$

Stationary solution Dark Soliton (-): $\psi(x,z)=n\tanh\left[n(x-x_0)\right]e^{in^2z/2}$

Galilean transformation:

(traveling waves)

$$\psi'\left(x,z;\kappa\right)=\psi\left(x-\kappa z,z\right)e^{i\kappa x-i\kappa^2z/2}$$

Spatial Solitons in Nonlinear Optics

Spatially localized waves:

- **Do not change under propagation**
- **Balance beteen diffraction and nonlinearity**

Dark soliton

Bright soliton

They conserve their shape and velocity during propagation and interactions (collisions) with other solitons

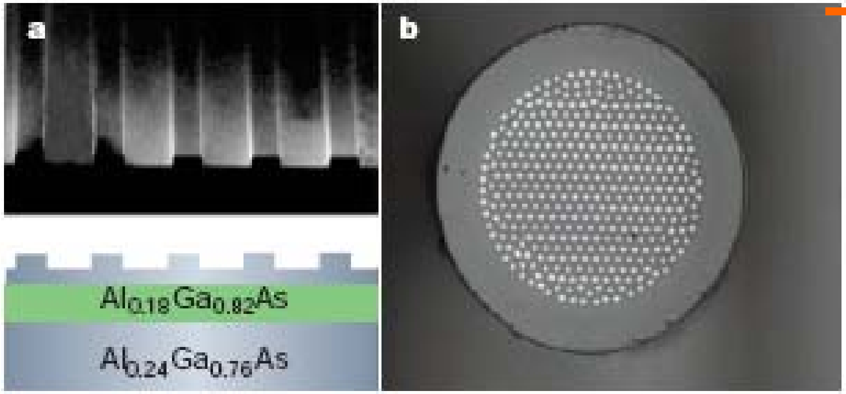
Elastic interaction between two solitons

Spatial solitons viewed as lattice solitons

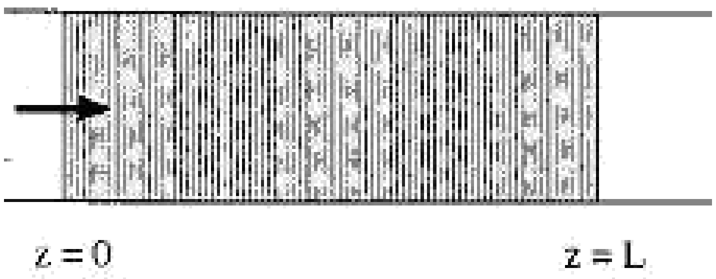
- **Applications to Noninear Optics**

Nonlinear Waveguide arrays

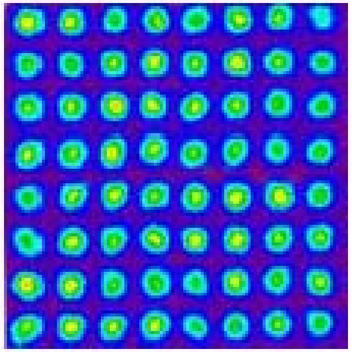
(α) 1D AlGaAs, (b) 2D silica glass



Important in optical fibers



and nonlinear photonic crystals



Review papers:

D.N. Christodoulides et al, “Discretizing light behaviour in linear and nonlinear waveguide lattices”, Nature **424**, 817 (2003)

A.A. Sukhorukov et al, “Spatial Optical Solitons in Waveguide Arrays”, IEEE J. Quant. Electron. **39**, 31 (2003)

J.W. Fleischer et al, “Spatial photonics in nonlinear waveguide arrays” **,** Opt. Express **13**, 1780 (2005)

Physically relevant mathematical model

Nonlinear Schrödinger equation with spatially dependent coefficients

$$\boxed{i\frac{\partial \psi}{\partial z}+\frac{\partial^2 \psi}{\partial x^2}+\varepsilon(x)\psi+g(x)|\psi|^2\psi=0}$$

Transversally: $\varepsilon(x)$, **linear refraction index,** $g(x)$, **nonlinear refraction index**

- Non-integrable system: No rigorous soliton solutions

- No translational symmetry): No traveling wave solutions

Stationary solutions:

$$\psi(x,z)=u(x;\beta)e^{i\beta z}$$

lead to a dynamical system:

$$\boxed{\frac{d^2u}{dx^2}+[\varepsilon(x)-\beta]u+g(x)u^3=0}$$

Which is: - Non-autonomous (x- horizontal inhomogeneity)

- Chaotic (near the saddle point at the origin, associated with the soliton solution)

Give up? Not yet! Try a perturbative approach!

Soliton existence in complex photonic structures

In realistic photonic structures the linear and nonlinear properties are transversally inhomogeneous:

$$i\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial x^2}+2|\psi|^2\psi+\varepsilon\left[n_0(x)\psi+n_2(x)|\psi|^2\psi\right]=0$$

***n*$_0$(*x*), linear refractive index** **$n_2(x)$, non-linear refractive index**

ε, perturbative parameter

Stationary solutions: $$\psi(x,z)=u(x)e^{i\beta z}$$

Dynamical system: $$\frac{d^2u}{dx^2}-\beta u+2u^3+\varepsilon\left[n_0(x)u+n_2(x)u^3\right]=0$$

Hamiltonian 1+1/2 degrees of freedom:

$$H=\frac{p^2}{2}-\beta\frac{q^2}{2}+\frac{q^4}{2}+\varepsilon\left[n_0(x)\frac{q^2}{2}+n_2(x)\frac{q^4}{4}\right]$$

$$(q,p)=(u,du/dx)$$

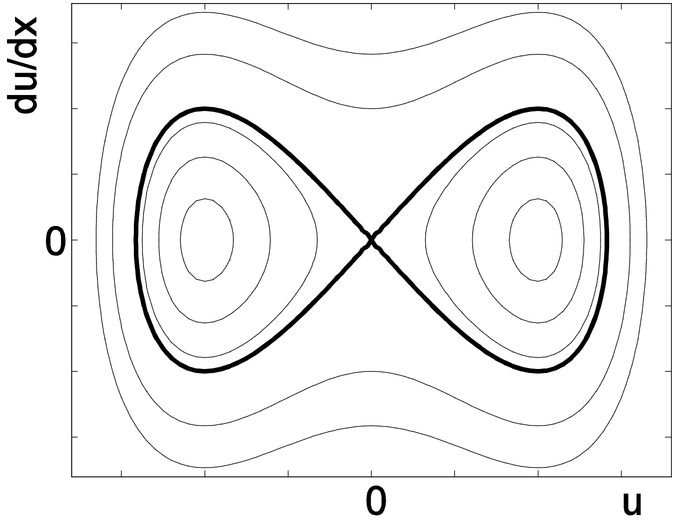
Soliton existence in complex photonic structures

The unperturbed system $\varepsilon = 0$ for $\beta > 0$ has a homoclinic solution:

$$q_0(x) = \pm\sqrt{\beta}\,\mathrm{sech}\left[\sqrt{\beta}(x - x_0)\right],$$

$$p_0(x) = \mp\beta\,\mathrm{sech}\left[\sqrt{\beta}(x - x_0)\right]\tanh\left[\sqrt{\beta}(x - x_0)\right]$$

which corresponds to the stationary soliton of NLSE <u>for every x_0</u>.



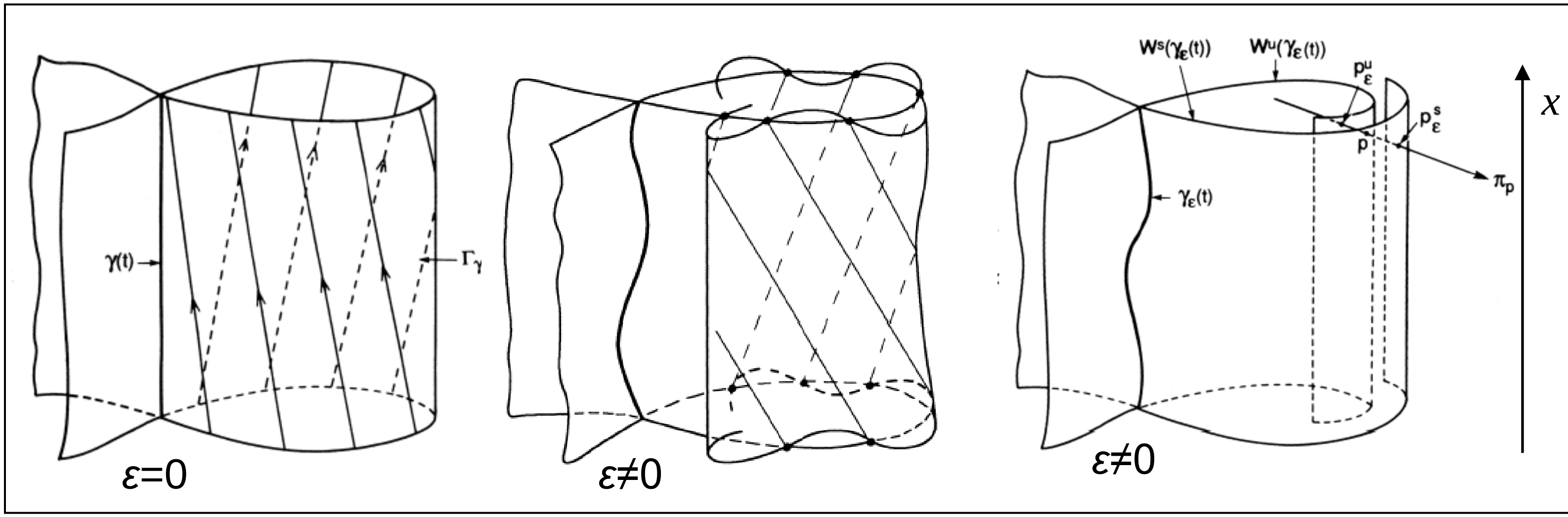
The homoclinic orbit:

- is formed by the smooth connection of the stable and unstable manifolds of the saddle fixed point at the origin

- when $\varepsilon \neq 0$ this smooth curve breaks into an infinite number of points where the stable and unstable manifolds intersect

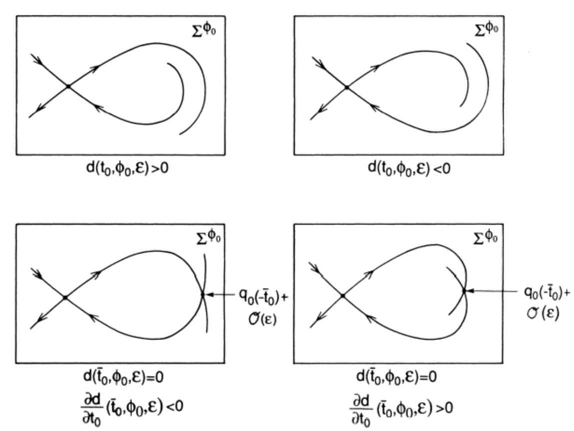
Soliton existence in complex photonic structures

Perturbations ($\varepsilon \neq 0$) bring about a dramatic change of the phase space portrait and generically break this curve in to an infinity of possible homoclinic orbits lying at the intersections of the stable and unstable manifolds



Solitons occur at the zeros of the Melnikov function $M(x_0)$, which is proportional to the distance between the manifolds $d(x_0)$ and is a periodic function of x_0 !!!

Poincaré Maps



S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer (2003)

Nonlinear Kronig-Penney Model: Analytical solutions

A method for constructing analytical solutions (Y. Kominis, 2005)

Suppose we could choose our efractive indices in such a way that we could write our NLS equation in the form

.

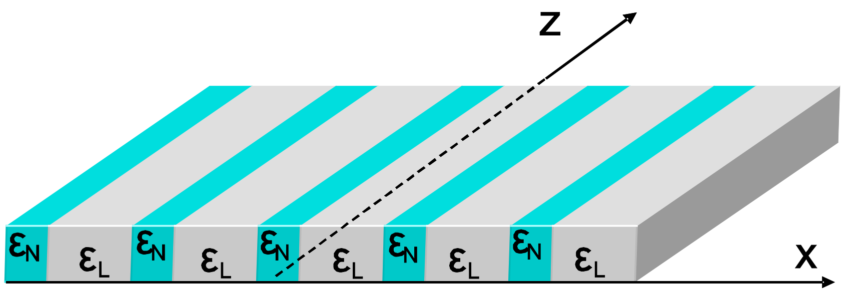
$$\boxed{i\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial x^2}+\varepsilon(x)\psi+g(x)|\psi|^2\psi=0}$$

where $\varepsilon(x)$, $g(x)$: **piecewise constant functions**

$$[\varepsilon(x),g(x)]=\begin{cases}(\varepsilon_N,\ \pm 1), & x\in U_N\\(\varepsilon_L,\ \ 0), & x\in U_L\end{cases}$$

$$U_N=\bigcup_k\left(kT-N/2,kT+N/2\right)$$

$$U_L=\bigcup_k\left(kT+N/2,(k+1)T-N/2\right)$$



L, **length of linear regime**
N, **length of nonlinear regime**
$T=L+N$, **spatial period**

Nonlinear Kronig-Penney Model: Self - Focusing

Stationary solutions: $\psi(x,z) = u(x;\beta)e^{i\beta z}$

$$\frac{d^2u}{dx^2} + [\varepsilon(x) - \beta]u + g\left(x, u^2\right)u = 0 \longrightarrow$$

NONLINEAR: $\frac{d^2u}{dx^2} + (\varepsilon_N - \beta)u \pm u^3 = 0, \qquad x \in U_N$

LINEAR: $\frac{d^2u}{dx^2} + (\varepsilon_L - \beta)u = 0, \qquad x \in U_L$

Our method

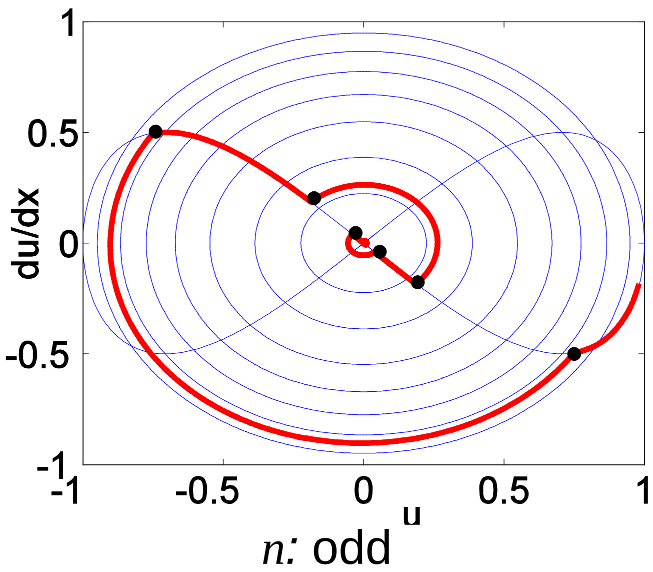
Exploits the phase space dynamics to compose exact solutions by matching analytical soliton solutions *u(x)and du(x)/dx* of the nonlinear problem at the boundaries of the linear regimes

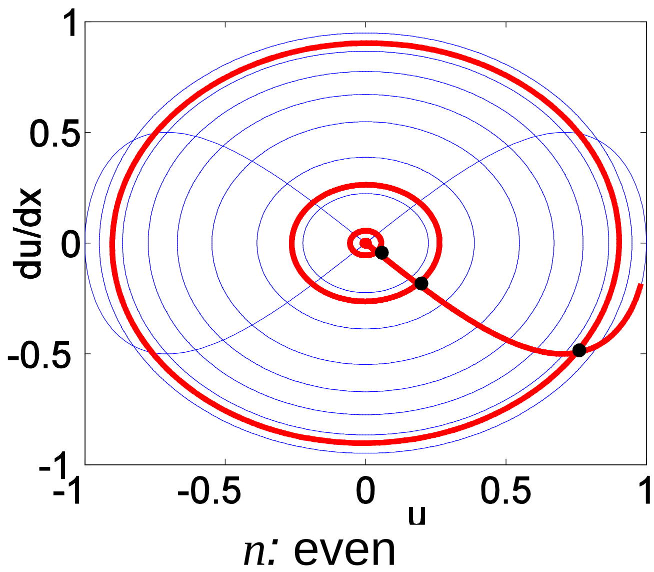
Assumptions:

- **The linear subsystem has periodic (sinusoidal) solutions** $\longrightarrow$ $\boxed{\beta < \varepsilon_L}$

- **The nonlinear subsystem has an orbit homoclinic to (0,0)** $\longrightarrow$ $\boxed{\beta > \varepsilon_N}$

Nonlinear Kronig-Penney Model: Self - Focusing





Construction of soliton solutions in the composite phase space. The dark dots are located at the boundaries between the linear and nonlinear regimes.

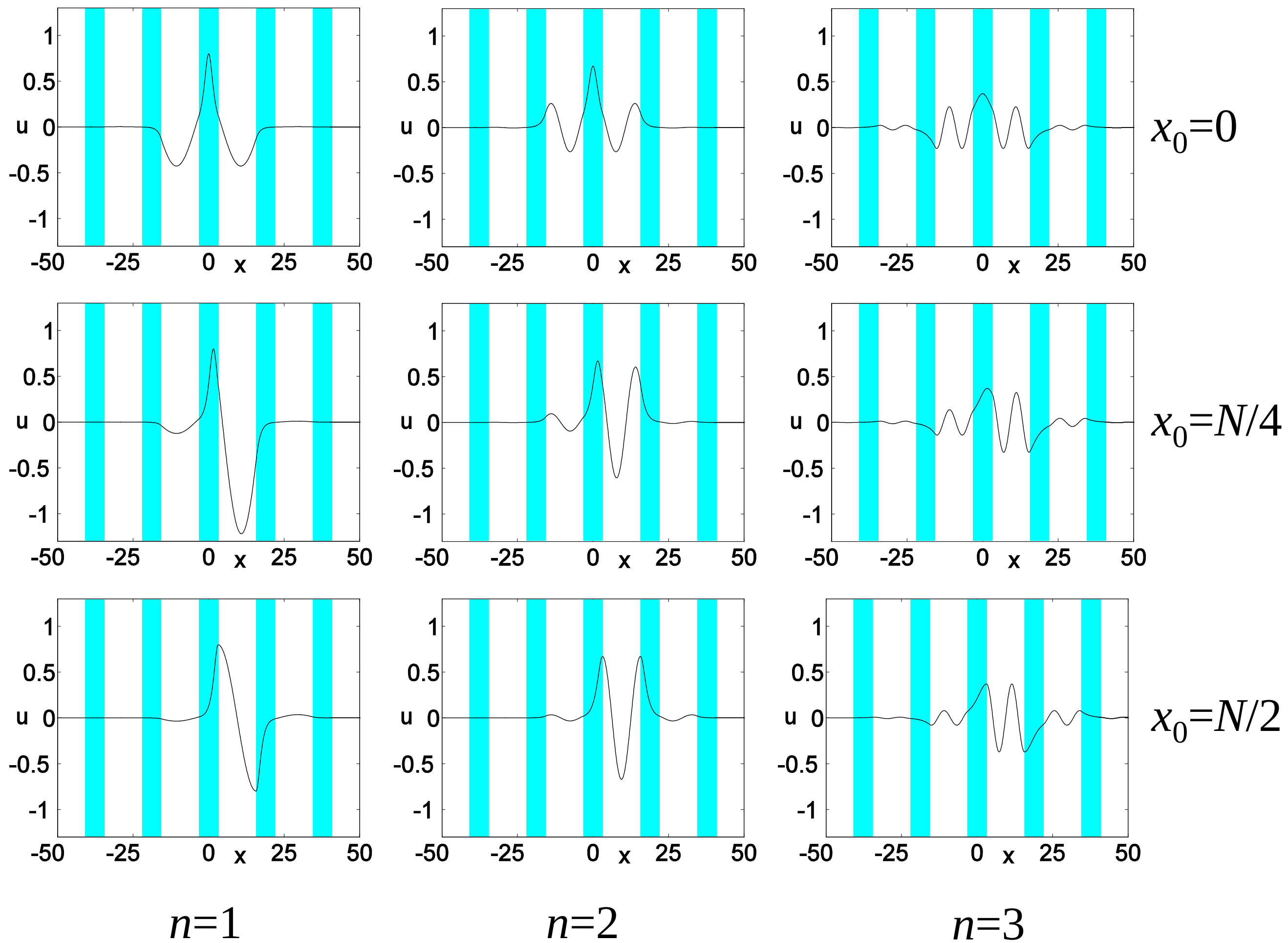
We choose values of the parameter *β*, for which an integer number (*n*) of half periods is contained in the linear regime of length *L*.

Resonance Condition

$$\boxed{\beta_n = \varepsilon_L - \left(\frac{n\pi}{L}\right)^2}$$

For these $\beta = \beta_n$ all boundary conditions are satisfied simultaneously!

Nonlinear Kronig-Penney Model: Self - Focusing



n is the number of zeros within the linear part